FOUR LINES IN SPACE

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Abstract. We introduce an invariant of sets of four lines in three-dimensional projective space which classifies these sets up to linear equivalence, and we relate this invariant to the representation theory of the infinite dihedral group.
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1. Introduction

The modern development of computer vision has produced an increased interest in some problems in classic geometry related to joint invariants (see, for instance, [11] and [14]). For example, the invariants of n points in space are the volume cross-ratio, as noted in the classic book by Veblen and Young ([15], section 27; see [14] for a modern point of view): to determine if two sets of n points are equivalent under a linear transformation we just need to compute their volume cross-ratios and see if they agree for the two sets of points. Here we address a similar problem, namely to obtain the invariants of four lines in space. Given four lines in space we define a cross-ratio-like invariant —which can be easily computed— such that two sets of four lines are equivalent under a linear transformation if and only if the invariant has the same value for the two sets of lines. After some preliminaries to fix the notation, the invariant is defined in section 3 (see definition 2). The work in this section is inspired in the theory of moving frames as developed by Fels and Olver ([8]). Then, the next two sections are devoted to generalizations of the problem to lines which may intersect each other and to the case in which we add a fifth line. This second problem is related to the two-matrix problem (see [6]) and we provide an explicit solution in our particular case. In a final section, we use our invariant to investigate representations of the infinite dihedral group in dimension 4 and we see how this invariant provides what can be considered as a better parametrization of the results in [3] and [5].

We should mention that the problem of classifying sets of four lines in space is a particular case of what is called the four subspace problem consisting of classifying sets of four linear subspaces of a vector space. It is also a particular case of the classification of pairs of involutions in the general linear group or representations of the infinite dihedral group. The four subspace problem was solved by Gelfand

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and Ponomarev (over an algebraically closed field) and by Nazarova ([9], [12], [13], see also [10]) and the representations of the infinite dihedral group are known since the work of Berman and Kuzási ([3], see also [5]). Hence, one could say that the problem that we are dealing with in this paper is essentially solved. However, our approach here may be of independent interest, since it is elementary, geometric and computational. In contrast to the two approaches that we have just mentioned, we do not just provide a list of canonical forms, but an effective invariant which can be used to decide if two sets of four lines are equivalent or not.

2. Background and notation

Here *space* means three-dimensional projective space $P^3(K)$ over a fixed commutative field K. A line l in space can be given by its Plücker coordinates

$$l = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$$

or by giving two different points $\mathbf{x} = \{x_0, x_1, x_2, x_3\}$, $\mathbf{y} = \{y_0, y_1, y_2, y_3\}$ in $P^3(K)$. More precisely, we can determine l by a 4×2 matrix of rank two,

$$M = \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}$$

which is well determined up to the right action of $GL_2(K)$. This amounts to describe the set of lines in $P^2(K)$ —which is the same as the Grassmannian $G^{2,4}(K)$— as the cosets $\{M_{4\times 2}(K) \,|\, \mathrm{rank} = 2\}/GL_2(K)$. Through this paper we write a 4×2 matrix as $\left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$ where P, Q are 2×2 matrices. Then, a line is an equivalence class of rank two matrices $\left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$ under the equivalence relation $\left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} PA \\ QA \end{smallmatrix}\right)$ for $A \in GL_2(K)$.

The Plücker coordinates of a line l given by a matrix M as above are the minors of M:

$$\{\left|\begin{smallmatrix} x_0 & y_0 \\ x_1 & y_1 \end{smallmatrix}\right|, \left|\begin{smallmatrix} x_0 & y_0 \\ x_2 & y_2 \end{smallmatrix}\right|, \left|\begin{smallmatrix} x_0 & y_0 \\ x_3 & y_3 \end{smallmatrix}\right|, \left|\begin{smallmatrix} x_2 & y_2 \\ x_3 & y_3 \end{smallmatrix}\right|, \left|\begin{smallmatrix} x_3 & y_3 \\ x_1 & y_1 \end{smallmatrix}\right|, \left|\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix}\right|\}$$

These coordinates are well defined up to the product by a unit in K and they allow us to identify the set of lines in $P^3(K)$ to the variety in $P^5(K)$ given by the equation $\lambda_0\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_5 = 0$.

We will say that some lines are in *general position* if they are pairwise disjoint. Notice that two lines given by matrices M and M' are disjoint if and only if the 4×4 matrix (MM') has rank 4. Here we are using a notation that will be used often in this paper to express matrices out of smaller blocks. In this sense, (MM') means the matrix whose two first columns are the columns of M and the next two columns are the columns of M'.

We denote by ℓ_0, ℓ_1, ℓ_2 the lines with Plücker coordinates

$$\begin{aligned}\ell_0 &= \{1,0,0,0,0,0\},\\ \ell_1 &= \{0,0,0,1,0,0\},\\ \ell_2 &= \{1,0,1,1,0,-1\}.\end{aligned}$$

These lines are in general position and are given by the matrices

$$\ell_0 = \left(\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right), \quad \ell_1 = \left(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right), \quad \ell_2 = \left(\begin{smallmatrix} I \\ I \end{smallmatrix}\right)$$

The standard action of the linear group $GL_4(K)$ on $P^3(K)$ gives an action of $GL_4(K)$ on the set of lines. If $M \in GL_4(K)$, we will often write M as $\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right)$ where each M_i, $i = 1,2,3,4$, is a 2×2 matrix. Then, M transforms the line $l = \left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$ into the line $\left(\begin{smallmatrix} M_1P+M_2Q \\ M_3P+M_4Q \end{smallmatrix}\right)$.

3. The invariant

Let $l = \left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$, $l' = \left(\begin{smallmatrix} P' \\ Q' \end{smallmatrix}\right)$, $l'' = \left(\begin{smallmatrix} P'' \\ Q'' \end{smallmatrix}\right)$ be three lines in space in general position and let $\mathcal{M}_{l,l',l''} \subset GL_4(K)$ be the set of all matrices $M = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right) \in GL_4(K)$ which transform l, l' and l'' in ℓ_0, ℓ_1, ℓ_2. The following result shows that $\mathcal{M}_{l,l',l''}$ is nonempty and provides a complete description of $\mathcal{M}_{l,l',l''}$.

Theorem 1. *The function $\phi(M) = M_1P + M_2Q$ gives a bijection from $\mathcal{M}_{l,l',l''}$ to $GL_2(K)$.*

Proof. The identities $M(l) = \ell_0$, $M(l') = \ell_1$, $M(l'') = \ell_2$ translate immediately into the following conditions on M_1, M_2, M_3, M_4:

$$\begin{aligned}
&(1) & M_3P + M_4Q &= 0\\
&(2) & M_1P' + M_2Q' &= 0\\
&(3) & M_1P'' + M_2Q'' &= M_3P'' + M_4Q'' \in GL_2(K)\\
&(4) & M_1P + M_2Q &\in GL_2(K)\\
&(5) & M_3P' + M_4Q' &\in GL_2(K)
\end{aligned}$$

and $\mathcal{M}_{l,l',l''}$ is thus the set of matrices in $GL_4(K)$ which satisfy these five conditions. Let us define four 2×2 matrices R, S, R', S' in the following way. Notice that since l and l' are disjoint, the matrix $\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right)$ has rank four. Let

$$\begin{pmatrix} R & R' \\ S & S' \end{pmatrix} = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix}^{-1}.$$

Then, from

$$\begin{aligned}\begin{pmatrix} R & R' \\ S & S' \end{pmatrix}\begin{pmatrix} P & P'' \\ Q & Q'' \end{pmatrix} &= \begin{pmatrix} I & RP''+R'Q'' \\ 0 & SP''+S'Q'' \end{pmatrix} \\ \begin{pmatrix} R & R' \\ S & S' \end{pmatrix}\begin{pmatrix} P'' & P' \\ Q'' & Q' \end{pmatrix} &= \begin{pmatrix} RP''+R'Q'' & 0 \\ SP''+S'Q'' & I \end{pmatrix}\end{aligned}$$

we deduce that both $SP''+S'Q''$ and $RP''+R'Q''$ are invertible.

Let us define a function $\psi : GL_2(K) \to GL_4(K)$ as follows. Given $A \in GL_2(K)$, let $B \in GL_2(K)$ be the matrix defined by

$$B^{-1}A = (SP''+S'Q'')(RP''+R'Q'')^{-1}.$$

Then, let us define

$$\psi(A) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} R & R' \\ S & S' \end{pmatrix}.$$

It is straightforward to check that $\psi(A)$ satisfies the conditions (1) to (5) above and so $\psi(A) \in \mathcal{M}_{l,l',l''}$. We finish the proof by showing that ϕ and ψ are inverse to each other. We have

$$\phi\psi(A) = ARP + AR'Q = A(RP+R'Q) = A.$$

Let $M \in \mathcal{M}_{l,l',l''}$. Notice that using $\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right)\left(\begin{smallmatrix} R & R' \\ S & S' \end{smallmatrix}\right) = I$ and the equations (1), (2), (3) for M, we get

$$\begin{aligned}(M_1P+M_2Q)(RP''+R'Q'') &= M_1P''+M_2Q'' \\ (M_3P'+M_4Q')(SP''+S'Q'') &= M_3P''+M_4Q''\end{aligned}$$

and so, for $A = \phi(M) = M_1P+M_2Q$ we have

$$(M_3P'+M_4Q')^{-1}A = (SP''+S'Q'')(RP''+R'Q'')^{-1}$$

and we get $B = M_3P'+M_4Q'$. Then

$$\psi\phi(M) = \begin{pmatrix} M_1P+M_2Q & 0 \\ 0 & M_3P'+M_4Q' \end{pmatrix}\begin{pmatrix} R & R' \\ S & S' \end{pmatrix} = M$$

and the proof is complete. □

Let now $l''' = \left(\begin{smallmatrix} P''' \\ Q''' \end{smallmatrix}\right)$ be a fourth line in space, in general position with respect to the lines $l = \left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$, $l' = \left(\begin{smallmatrix} P' \\ Q' \end{smallmatrix}\right)$, $l'' = \left(\begin{smallmatrix} P'' \\ Q'' \end{smallmatrix}\right)$ that we have been considering. Let R, S, R', S' have the same meaning as in the proof of the previous theorem. Then, let us define a 2×2 matrix H which depends on these four lines as follows:

Definition 2.
$H(l,l',l'',l''') = (SP''+S'Q'')(RP''+R'Q'')^{-1}(RP'''+R'Q''')(SP'''+S'Q''')^{-1}$. □

This matrix $H(l,l',l'',l'')$ has the following properties:

(1) Since the four lines l, l', l'', l''' are in general position, all four matrices which appear in the definition of H are invertible and so $H \in GL_2(K)$.
(2) $H(l, l', l'', l''')$ is not well defined as a function of the lines l, l', l'', l'''. Remember that $\left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} P' \\ Q' \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} P'' \\ Q'' \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} P''' \\ Q''' \end{smallmatrix}\right)$ are defined up to the right action of $GL_2(K)$. However, if we choose other matrices to represent the same lines, it is easy to see that the new value that we obtain for H is a matrix which is conjugated to the original matrix H. This implies that $H(l, l', l'', l''')$ *is well defined as a conjugacy class in* $GL_2(K)$.
(3) A straightforward computation shows that for $l = \left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$ we have
$$H(\ell_0, \ell_1, \ell_2, l) = PQ^{-1}.$$
(4) For $M = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right) \in GL_4(K)$ we have
$$M(\ell_0) = \left(\begin{smallmatrix} M_1 \\ M_3 \end{smallmatrix}\right), \quad M(\ell_1) = \left(\begin{smallmatrix} M_2 \\ M_4 \end{smallmatrix}\right), \quad M(\ell_2) = \left(\begin{smallmatrix} M_1+M_2 \\ M_3+M_4 \end{smallmatrix}\right).$$
Then the computation of $H(M(\ell_0), M(\ell_1), M(\ell_2), M(l))$ can be done as follows. We have
$$\begin{pmatrix} R & R' \\ S & S' \end{pmatrix} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}^{-1} = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$$
with $L_1M_2 + L_2M_4 = L_3M_1 + L_4M_3 = 0$ and $L_1M_1 + L_2M_3 = L_3M_2 + L_4M_4 = I$. Thus,
$$\begin{aligned} H &= (L_3(M_1 + M_2) + L_4(M_3 + M_4)) \\ &\quad (L_1(M_1 + M_2) + L_2(M_3 + M_4))^{-1} \\ &\quad (L_1(M_1P + M_2Q) + L_2(M_3P + M_4Q)) \\ &\quad (L_3(M_1P + M_2Q) + L_4(M_3P + M_4Q))^{-1} \\ &= PQ^{-1}. \end{aligned}$$
(5) We have seen in theorem 1 that three lines in general position can always be transformed into the standard lines ℓ_0, ℓ_1, ℓ_2 by a linear transformation $M \in GL_4(K)$. This fact, together with the computations in (3) and (4) above show that H *is invariant under the action of* $GL_4(K)$. We mean that for any four lines l, l', l'', l''' in general position and any $M \in GL_2(K)$, the matrices $H(l, l', l'', l''')$ and $H(M(l), M(l'), M(l''), M(l'''))$ are conjugated in $GL_2(K)$.

Theorem 3. *H is the only invariant of four lines in space in general position.*

We mean that if two (ordered) sets of four lines in general position have the same value of the invariant H, then there is a linear transformation of space which transforms the lines in the first set into the corresponding lines in the second set.

Proof. From the remarks about the invariant H that we have made above, it follows that we can consider, without loss of generality, that one of the two sets of lines consists of the three standard lines ℓ_0, ℓ_1, ℓ_2 plus a fourth line in general position $\ell = \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$. In other words, it is enough to prove that given four lines in general position l, l', l'', l''' such that $H(l, l', l'', l''')$ is conjugated to XY^{-1}, then there is a linear transformation $M \in GL_4(K)$ such that $M(l) = \ell_0$, $M(l') = \ell_1$, $M(l'') = \ell_2$, $M(l''') = \ell$.

Theorem 1 gives a description of *all* matrices $M \in GL_4(K)$ such that $M(l) = \ell_0$, $M(l') = \ell_1$, $M(l'') = \ell_2$. These matrices are parametrized by $GL_2(K)$. For any $A \in GL_2(K)$ we have

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} R & R' \\ S & S' \end{pmatrix}$$

with $B^{-1}A = (SP'' + S'Q'')(RP'' + R'Q'')^{-1}$. Now we want to show that there exists a suitable matrix A such that $M(l''') = \ell$. $M \begin{pmatrix} P''' \\ Q''' \end{pmatrix} = \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ translates into these two equations in $GL_2(K)$:

$$\begin{aligned} A(RP''' + R'Q''') &= XC \\ A(RP'' + R'Q'')(SP'' + S'Q'')^{-1}(SP''' + S'Q''') &= YC \end{aligned}$$

where $A, C \in GL_2(K)$ are unknown matrices. A system of two equations like this one has a solution if and only if $(SP'' + S'Q'')(RP'' + R'Q'')^{-1}(RP''' + R'Q''')(SP''' + S'Q''')^{-1}$ is conjugated to $XY^{-1} = H(\ell_0, \ell_1, \ell_2, \ell)$. □

Notice that the entries of the matrix $H(l, l', l'', l''')$ can be written as rational functions of the coordinates of pairs of points defining each of the four lines, in a similar way as how the cross ratio of four points in a line can be written as a rational function of the homogeneous coordinates of these four points. An algebraic computer language like Magma ([4]) can do this in an efficient way. However, if we want to write explicitly these four rational functions in 4×8 variables in plain text, we need about 50 megabytes of space, which amounts to more than 10.000 printed pages.

The invariant H adopts a much simpler form, which is very much reminiscent of the standard formula for the cross ratio, if we assume that all four lines have non-vanishing first Plücker coordinate. If this is the case, then we can take $P = P' = P'' = P''' = I$. Then

$$I = \begin{pmatrix} R & R' \\ S & S' \end{pmatrix} \begin{pmatrix} I & I \\ Q & Q' \end{pmatrix} = \begin{pmatrix} R + R'Q & R + R'Q' \\ S + S'Q & S + S'Q' \end{pmatrix}$$

which implies

$$\begin{aligned} R' &= (Q - Q')^{-1} \\ S' &= (Q' - Q)^{-1} \\ R &= (Q' - Q)^{-1}Q' \\ S &= (Q - Q')^{-1}Q \end{aligned}$$

and the invariant is given by a much simpler, easily computable formula

$$H(l,l',l'',l''') = (Q''-Q)(Q''-Q')^{-1}(Q'''-Q')(Q'''-Q)^{-1}.$$

We refer to this as the *affine* formula for H.

We finish this section by studying the range of the invariant H. Recall that H is defined on sets of four lines in general position and takes values in $GL_2(K)$ modulo conjugacy. It is easy to determine which matrices can appear as values of H.

Proposition 4. *The range of H consists of all matrices in $GL_2(K)$ for which 1 is not an eigenvalue.*

Proof. Without loss of generality, it is enough to consider $H(\ell_0,\ell_1,\ell_2,\ell) = PQ^{-1}$ for $\ell = \left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$. Since the lines are supposed to be in general position, $\left(\begin{smallmatrix} I & P \\ I & Q \end{smallmatrix}\right)$ has rank four. Then, if $v \neq 0$ is an eigenvector of eigenvalue 1 we obtain

$$\begin{pmatrix} I & P \\ I & Q \end{pmatrix} \begin{pmatrix} -v \\ Q^{-1}v \end{pmatrix} = 0$$

which is absurd. On the other side, if $H \in GL_2(K)$ and we take $\ell = \left(\begin{smallmatrix} H \\ I \end{smallmatrix}\right)$ we obtain $H(\ell_0,\ell_1,\ell_2,\ell) = H$. It is clear that ℓ does not intersect ℓ_0, ℓ_1 and ℓ intersects ℓ_2 if and only if $\left(\begin{smallmatrix} I & H \\ I & I \end{smallmatrix}\right)$ has rank < 4 which is easily seen to be equivalent to $H(v) = v$ for some non-trivial vector v. $\square$

By permuting the four lines l, l', l'', l''' the matrix $H(l,l',l'',l''')$ changes in the same way as the classic cross ratio does. The following identities hold

$$\begin{aligned} H(l,l',l''',l'') &= H(l,l',l'',l''')^{-1} \\ H(l,l'',l',l''') &= I - H(l,l',l'',l''') \\ H(l',l,l'',l''') &= H(l,l',l'',l''')^{-1}. \end{aligned}$$

Notice that these formulas make sense in the set of conjucacy classes of matrices in $GL_2(K)$ with no eigenvalue equal to 1. To check these formulas we first see that they hold if $l = \ell_0$, $l' = \ell_1$, $l'' = \ell_2$, which is straightforward, and then we notice that this implies that the formulas hold in general because H is invariant under linear transformations and we can always map three lines in general position into the three standard lines ℓ_0, ℓ_1, ℓ_2.

This striking similarity between the invariant H of four lines in space and the classic cross ratio of four points in a line may suggest that H is indeed a true cross ratio after some suitable interpretation. However, we must observe that H is not a scalar, but a conjugacy class of 2×2 matrices.

4. Degenerate cases

To complete the picture, we will briefly discuss the case in which the four lines have some intersections. Assume that l, l', l'' are three lines in general position and let l''' be a fourth line which intersects some or all of the first three lines. We need to distinguish between three cases, depending if l''' intersects one, two or all three lines l, l', l''. We use the same notation as in the preceding section.

If l''' does not intersect l, l', then the matrices $SP''+S'Q''$, $RP''+R'Q''$, $SP'''+S'Q'''$ and $RP'''+R'Q'''$ are invertible and the invariant $H(l,l',l'',l''')$ is well defined as a conjugacy class in $GL_2(K)$. One can easily check that the methods and results in the previous section can be applied to this case, without any modification, even if l''' intersects l''. If this happens, then $H(l,l',l'',l''')$ has an eigenvalue equal to 1 and we cannot freely permute the lines l, l', l'', l''' since H is only defined for $\{l,l',l''\}$ and $\{l,l',l'''\}$ in general position.

Hence, we can extend the results of the previous section in the following way.

Proposition 5. *Let $\mathcal{L}$ be the set of ordered 4-plas of lines in space (l,l',l'',l''') such that $\{l,l',l''\}$ and $\{l,l',l'''\}$ are in general position. Then, H gives a bijection between the orbit set $GL_4(K)\backslash\mathcal{L}$ and the set of conjugacy classes in $GL_2(K)$.* $\square$

Let us consider now the case in which the fourth line l''' intersects l' and l'', but not l and let us assume also that l, l', l'' are in general position. In this case the matrices $SP''+S'Q''$, $RP''+R'Q''$ and $SP'''+S'Q'''$ are invertible while the matrix $RP'''+R'Q'''$ has rank one. The invariant H can be defined and it is an invariant, but now it has rank one and has an eigenvalue equal to 1. There is only one conjugacy class with these properties and the equations

$$\begin{aligned} A(RP'''+R'Q''') &= XC \\ A(RP''+R'Q'')(SP''+S'Q'')^{-1}(SP'''+S'Q''') &= YC \end{aligned}$$

that we considered in the previous section can always be solved. This proves that, in this situation, all 4-plas of lines are equivalent under $GL_4(K)$.

The same is true in the remaining case in which l, l', l'' are in general position and l''' intersects l, l' and l''. To prove this it is not restrictive to assume that the three lines in general position are the standard lines ℓ_0, ℓ_1, ℓ_2 and we just need to prove that given a line l which intersects ℓ_0, ℓ_1 and ℓ_2, there is a linear transformation M which fixes ℓ_0, ℓ_1, ℓ_2 and transforms some suitable line, like $\ell_3=\langle(1,0,0,0),(0,0,1,0)\rangle$, into l.

We can take as a basis for l the intersection points with ℓ_0 and ℓ_1, i.e. $l=\binom{P}{Q}$ with $P=\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$, $Q=\left(\begin{smallmatrix} 0 & c \\ 0 & d \end{smallmatrix}\right)$. Let $u,v\in K$ be any choice of scalars such that $A=\left(\begin{smallmatrix} a & u \\ b & v \end{smallmatrix}\right)\in GL_2(K)$. Then it is easy to check that the fact that l intersects ℓ_2 implies that we can choose $c=a$ and $d=b$ and so $M=\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ has the desired properties.

We can summarize all this in the following statement. We write $|l \cap l'| = 0, 1$ if the lines l and l' are disjoint or have a unique common point, respectively.

Theorem 6. *Let (l_1, l_2, l_3, l_4) and (s_1, s_2, s_3, s_4) be two 4-plas of lines in space ordered such that $|l_4 \cap l_1| \leq |l_4 \cap l_2| \leq |l_4 \cap l_3|$ and $|s_4 \cap s_1| \leq |s_4 \cap s_2| \leq |s_4 \cap s_3|$ and assume that $\{l_1, l_2, l_3\}$ and $\{s_1, s_2, s_3\}$ are in general position. There is $M \in GL_4(K)$ such that $M(l_i) = s_i$, $i = 1, \dots, 4$ if and only if $|l_4 \cap l_i| = |s_4 \cap s_i|$, $i = 1, \dots, 4$ and one of the following two cases occur:*

(1) $|l_4 \cap l_2| = 1$;
(2) $|l_4 \cap l_2| = 0$ *and* $H(l_1, l_2, l_3, l_4) = H(s_1, s_2, s_3, s_4)$. □

5. The fifth line

If we add now a fifth line to our problem, we have to deal with the existence or non existence of a linear transformation mapping five lines l, l', l'', l''', h to five lines s, s', s'', s''', k. Of course, a necessary condition is that $H(l, l', l'', l''')$ has to be conjugate to $H(s, s', s'', s''')$ and $H(l, l', l'', h)$ has to be conjugate to $H(s, s', s'', k)$, but these two conditions are not sufficient. The analysis in the preceding sections shows that the sought linear transformation exists if and only if there is a matrix which conjugates $H(l, l', l'', l''')$ to $H(s, s', s'', s''')$ *and* $H(l, l', l'', h)$ to $H(s, s', s'', k)$ simultaneously.

The problem of classifying two matrices over a field up to simultaneous conjugation is an unsolved problem which is considered as the prototype of the so-called *wild* problems in representation theory. The literature about this questions is very large and the reader may read, for instance, [9], [6] or section 7 of [7] for an overview of this theory. However, in the case of 2×2 matrices, it is rather elementary to provide an explicit solution of the two-matrix problem.

We use the following notation:

$$C = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}; \quad D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; \quad J = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

for any $a \neq b \in K$. We write $A \sim A'$ to denote that $A = P^{-1}A'P$ for some $P \in GL_2(K)$ and we write $(A, B) \sim (A', B')$ to denote that there is $P \in GL_2(K)$ such that $A = P^{-1}A'P$ and $B = P^{-1}B'P$. Let $B = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$, $B' = \left(\begin{smallmatrix} x' & y' \\ z' & t' \end{smallmatrix}\right)$ be two generic 2×2 matrices with entries in the field K.

Proposition 7. *Assume $B \sim B'$, $B \neq B'$. Then*

(1) $(C, B) \sim (C, B')$.
(2) $(D, B) \sim (D, B')$ *if and only if there is a unit $\tau \in K$ such that one of the following holds: (a) $x = x'$, $t = t'$, $y = \tau y'$, $\tau z = z'$; (b) $x = t'$, $x' = t$, $z = \tau y'$, $\tau y = z'$.*
(3) $(J, B) \sim (J, B')$ *if and only if $z = z'$ and one of the following holds: (a) $z \neq 0$; (b) $z = 0$ and $x = x' \neq t = t'$.*

Proof. It is straightforward and can be left as an exercise to the reader. Let us just mention that in the case (3a) the matrix $\left(\begin{smallmatrix} 1 & -t/z \\ 0 & 1 \end{smallmatrix}\right)$ conjugates (J, B) to $(J, \left(\begin{smallmatrix} u & v \\ z & 0 \end{smallmatrix}\right))$ and u, v are determined by the conjugacy class of B. $\square$

This result allows us to classify pairs of 2×2 matrices over a field up to simultaneous conjugation, assuming that at least one of the matrices has an eigenvector, and it is possible to give a complete and non-overlapping family of *normal forms* for such pairs of matrices. The reader can easily check that the following list (which was pointed out to me by Y. A. Drozd) is complete and without repetitions, except for the trivial equivalences $(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right), \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)) \sim (\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} v & 0 \\ 0 & u \end{smallmatrix}\right))$ and $(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right), \left(\begin{smallmatrix} u & 1 \\ w & v \end{smallmatrix}\right)) \sim (\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} v & 1 \\ w & u \end{smallmatrix}\right))$ in the second row of the list.

$$
\begin{array}{l}
\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right)\right), \ \left(\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & b_1 \\ 1 & b_2 \end{smallmatrix}\right)\right) \\
\left(\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} b_1 & 0 \\ 0 & b_2 \end{smallmatrix}\right)\right), \ \left(\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} b_1 & 1 \\ b_3 & b_2 \end{smallmatrix}\right)\right), \ (a_1 \neq a_2) \\
\left(\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} b_1 & 0 \\ 0 & b_2 \end{smallmatrix}\right)\right), \ (b_1 \neq b_2), \\
\left(\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} b_1 & b_2 \\ 0 & b_1 \end{smallmatrix}\right)\right), \ \left(\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & 0 \end{smallmatrix}\right)\right), \ (b_3 \neq 0).
\end{array}
$$

It remains the case in which none of the matrices has any eigenvector. In this case, we cannot provide a list of representatives. However, the problem of deciding if $(A, B) \sim (A', B')$ for some given matrices can also be easily solved in this case in the following way. We can conjugate A to $\left(\begin{smallmatrix} 0 & -b \\ 1 & -a \end{smallmatrix}\right)$. Then, the centralizer of A is the set of matrices of the form $\left(\begin{smallmatrix} x & -by \\ y & x-ay \end{smallmatrix}\right)$ for all x, y with $x^2 - axy + by^2 \neq 0$. Then, to determine if B and B' are conjugate through a matrix of this type we just need to decide if a linear system of the form $\alpha_i x + \beta_i y = 0$, $i = 1, \ldots, 4$, has a solution $\neq (0, 0)$, and this is trivial.

Then, this analysis, together with the invariant H, provides a solution to the problem of the equivalence between two sets of five (or more) lines.

6. Pairs of involutions in $GL_4(K)$

In this section we assume that K has characteristic different from two and we relate the results of the previous sections to the representation theory of the infinite dihedral group in $GL_4(K)$.

A pair of non-intersecting lines in space determines an involution $\sigma \in GL_4(K)$ whose ± 1-eigenspaces σ^+ and σ^- coincide with the two given lines. Hence, four lines in space in general position determine a pair of involutions in $GL_4(K)$ —which is the same as a well defined representation of the infinite dihedral group D_∞ in $GL_4(K)$. Obviously, this representation is invariant under linear transformations of the four lines and so it is a function of the invariant H.

The representations of D_∞ have been classified in [3] and [5]. The purpose of this section is to make explicit the correspondence between the representations in rank four as listed in [5] and the values of the invariant H and in this way we think that the reader can see that the invariant H gives a simpler parametrization of these representations which has also the advantage of being easily computable. The reader may also compare with section 6 of [2] and section 3 of [1]. (It may be worth mentioning that the arguments in section 3 of [1] provided the motivation for the research in the present paper.)

Let us begin by extracting from [5] a listing of all representations of D_∞ in $GL_4(K)$ such that (a) for each of the two involutions both the $+1$-eigenspace and the -1-eigenspace have dimension two, and (b) the four lines associated to the representation are in general position. Notice that these conditions are not very restrictive, since the only representations that do not fit into them are those with a direct summand of rank one and those which have as a direct summand one of the four irreducible representations in theorem 1-(iii) in [5]. We say that representations with these properties are *in general position*.

For any representation ρ of D_∞ in general position we denote by σ_1, σ_2 the two involutions determined by ρ and by σ_1^+, σ_1^-, σ_2^+, σ_2^- the four lines in space corresponding to the eigenspaces of the two involutions.

Let us consider the following representations of D_∞:

Type I For any $\alpha \in K^*$, $\alpha \neq \pm 1$, let $\nu(\alpha)$ be the representation in rank two given by the matrices $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & \alpha^{-1} \\ \alpha & 0 \end{smallmatrix}\right)$.

For any $\beta \in K$ such that $\beta^2 - 1$ is not a square in K, let $\tau(\beta)$ be the representation in rank two given by the matrices $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} -1 & 0 \\ 2\beta & 1 \end{smallmatrix}\right)$.

A representation of type I is then the direct sum of two representations of rank two as above.

Type II For each $a, b \in K$, $b \neq 0, 1$, consider the polynomial $p = x^2 + ax + b$ and assume that either p is irreducible over K or p has a double root in K. Then, we denote by $\phi(a, b)$ the irreducible representation in rank four given by the matrices $\sigma_1 = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & A \\ A^{-1} & 0 \end{smallmatrix}\right)$ where $A = \left(\begin{smallmatrix} 0 & -b \\ 1 & -a \end{smallmatrix}\right)$.

Type III For each $a, b \in K$, let $p = x^4 + ax^3 + bx^2 + ax + 1$ and assume that either p is irreducible over K or $p = (x^2 + ex + 1)^2$ with $x^2 + ex + 1$ irreducible over K. Then, we denote by $\gamma(a, b)$ the irreducible representation in rank four given by the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -a & 0 & 0 & 1 \\ -b & 0 & 1 & 0 \\ -a & 1 & 0 & 0 \end{pmatrix}.$$

The representations above are all different except for the trivial equivalence $\rho_1 \oplus \rho_2 \sim \rho_2 \oplus \rho_1$ in type I and the equivalences[1]

$$\nu(\alpha) \sim \nu(\alpha^{-1}), \quad \phi(a,b) \sim \phi(a/b, 1/b)$$

in types I and II, respectively.

It is easy to check that all these representations are in general position. Then, the results of [5] imply that this is a complete list without repetitions of the representations of D_∞ in $GL_4(K)$ in general position. If ρ is one of these representations, we define

$$H(\rho) = H(\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-).$$

The remainder of this section will be devoted to compute the invariant H for all representations of types I, II, III.

Representations of type I. Consider $\rho = \nu(\alpha) \oplus \nu(\beta)$. To simplify the computations, it is useful to realize a direct sum $\rho_1 \oplus \rho_2$ by letting ρ_1 act on the first and third coordinates and ρ_2 act on the second and fourth coordinates. Then, the four lines associated to ρ are

$$\sigma_1^+ = \left(\begin{smallmatrix} I \\ I \end{smallmatrix}\right), \sigma_1^- = \left(\begin{smallmatrix} I \\ -I \end{smallmatrix}\right), \sigma_2^+ = \left(\begin{smallmatrix} I \\ A \end{smallmatrix}\right), \sigma_2^- = \left(\begin{smallmatrix} I \\ -A \end{smallmatrix}\right),$$

with $A = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, and we can use the affine formula to compute H. We obtain $H(\nu(\alpha) \oplus \nu(\beta)) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ with $\lambda = (\alpha-1)^2/(\alpha+1)^2$, $\mu = (\beta-1)^2/(\beta+1)^2$.

For $\rho = \tau(\alpha) \oplus \tau(\beta)$ we obtain the four lines

$$\sigma_1^+ = \left(\begin{smallmatrix} I \\ I \end{smallmatrix}\right), \sigma_1^- = \left(\begin{smallmatrix} I \\ -I \end{smallmatrix}\right), \sigma_2^+ = \left(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right), \sigma_2^- = \left(\begin{smallmatrix} I \\ -A \end{smallmatrix}\right),$$

and we cannot use the affine formula for H. Nevertheless, the computation of $H(\rho)$ in this case has no difficulty. We obtain $H(\tau(\alpha) \oplus \tau(\beta)) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ with $\lambda = (\alpha-1)/(\alpha+1)$, $\mu = (\beta-1)/(\beta+1)$.

In a similar way, we have $H(\nu(\alpha) \oplus \tau(\beta)) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ with $\lambda = (\alpha-1)^2/(\alpha+1)^2$, $\mu = (\beta-1)/(\beta+1)$.

This analysis shows that the invariant H gives a one-to-one correspondence between the representations of type I and the conjugacy classes in $GL_2(K)$ containing all diagonal matrices (without eigenvalues equal to 1). More explicitly, let $H = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$, $\lambda_1, \lambda_2 \neq 0, -1$. If λ_i is a square in K, let $\lambda = \epsilon^2$, $\epsilon \neq 1$, and let $\rho_i = \nu(\alpha)$ with $\alpha = (1+\epsilon)/(1-\epsilon)$, which does not depend on the choice of ϵ. If λ_i is not a square, define $\rho_i = \tau(\beta)$ with $\beta = (1+\lambda)/(1-\lambda)$. Notice that this makes sense because $\beta^2 - 1$ is a square in K if and only if λ is a square in K. Then, $H = H(\rho_1 \oplus \rho_2)$.

[1] This corrects a mistake in the paper [5] which forgets to mention that in the representations in theorem 1-(i) of [5] the representation associated to p is equivalent to the representation associated to $\tilde{p}$ (notation as in [5]).

Representations of type II. Consider now a representation $\phi(a,b)$ of type II and let $A = \left(\begin{smallmatrix} 0 & -b \\ 1 & -a \end{smallmatrix}\right)$. The four lines associated to $\phi(a,b)$ are

$$\sigma_1^+ = \left(\begin{smallmatrix} I \\ I \end{smallmatrix}\right), \sigma_1^- = \left(\begin{smallmatrix} I \\ -I \end{smallmatrix}\right), \sigma_2^+ = \left(\begin{smallmatrix} I \\ A^{-1} \end{smallmatrix}\right), \sigma_2^- = \begin{pmatrix} I \\ -A^{-1} \end{pmatrix}$$

and we can use the affine formula to compute the invariant H of these four lines:

$$H(\phi(a,b)) = (A^{-1} - I)(A^{-1} + I)^{-1}(-A^{-1} + I)(-A^{-1} - I)^{-1}.$$

To obtain an explicit formula for $H(\phi(a,b))$ we proceed in the following way. The polynomial $p = x^2 + ax + b$ is either irreducible in K or $p = (x - \lambda)^2$ for some $\lambda \in K$. In the first case, we have that A is conjugated to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar\lambda \end{smallmatrix}\right)$ in a quadratic extension $K(\sqrt{d})$, $d = a^2 - 4b$, while in the second case A is conjugated in K to $\left(\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right)$.

The particular form of $H(\phi(a,b))$ allows us to perform the computations with any conjugate of A. If $A \sim \left(\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right)$ for $\lambda \in K$, we have $H(\phi(a,b)) = \left(\begin{smallmatrix} \alpha & 0 \\ 1 & \alpha \end{smallmatrix}\right)$ with $\alpha = (\lambda-1)^2/(\lambda+1)^2$.

Otherwise, if $A \sim \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar\lambda \end{smallmatrix}\right)$ for $\lambda \in K(\sqrt{d})$ (conjugation in $K(sqrtd)$, we obtain $H(\phi(a,b)) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \bar\alpha \end{smallmatrix}\right)$ with $\alpha = (\lambda-1)^2/(\lambda+1)^2$. Notice that $\alpha = \bar\alpha$ would imply $\lambda = \bar\lambda$ which is not possible. Going back from $K(\sqrt{d})$ to K, we obtain that in this case $H(\phi(a,b)) = \left(\begin{smallmatrix} 0 & -s \\ 1 & -r \end{smallmatrix}\right)$ with $x^2 + rx + s$ the irreducible polynomial over K whose roots in $K(\sqrt{d})$ are α, $\bar\alpha$ with $\alpha = (\lambda-1)^2/(\lambda+1)^2$.

Let us analyze now which conjugacy classes in $GL_2(K)$ can be obtained as H-invariants of representations of type II. If H is a Jordan block, $H = \left(\begin{smallmatrix} \alpha & 0 \\ 1 & \alpha \end{smallmatrix}\right)$ with $\alpha = \epsilon^2$ a square in K, then $H = H(\phi(-2\lambda, \lambda^2))$ for $\lambda = (1+\epsilon)/(1-\epsilon)$. Notice that the other election for ϵ gives an equivalent representation.

If the characteristic polynomial of H is irreducible over K, let d be its discriminant and let $\lambda, \bar\lambda$ be the (different) eigenvalues of H in $K(\sqrt{d})$. If $\lambda = \epsilon^2$ is a square in $K(\sqrt{d})$, then $H = H(\phi(a,b))$ with $a = -\alpha - \bar\alpha$, $b = \alpha\bar\alpha$, $\alpha = (1+\epsilon)/(1-\epsilon)$. Notice also that the other election of ϵ gives an equivalent representation.

Representations of type III. Consider now a representation $\gamma(a,b)$ of type III and let $A = \begin{pmatrix} b/2 & 0 \\ a & -1 \end{pmatrix}$, $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The four lines associated to $\gamma(a,b)$ are

$$\sigma_1^+ = \left(\begin{smallmatrix} I \\ T \end{smallmatrix}\right), \sigma_1^- = \left(\begin{smallmatrix} I \\ -T \end{smallmatrix}\right), \sigma_2^+ = \left(\begin{smallmatrix} M \\ I \end{smallmatrix}\right), \sigma_2^- = \left(\begin{smallmatrix} I \\ M \end{smallmatrix}\right).$$

which are in general position because $p = x^4 + ax^3 + bx^2 + ax + 1$ has no roots in K. To perform the computations needed in this case Magma ([4]) can be useful. We obtain

$$H(\gamma(a,b)) = \frac{1}{d}\begin{pmatrix} 2a - b + 2 & -4 \\ -4a + 4b - 4 & -2a - b + 10 \end{pmatrix}$$

with $d = 2a - b - 2$. Notice that $d \neq 0$ because -1 is not a root of p. To classify the conjugacy class of H we proceed in the following way. Let us denote by α, β,

$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = H(\rho_1 \oplus \rho_2)$	$\rho_i = \begin{cases} \nu\left(\dfrac{1+\epsilon}{1-\epsilon}\right), \lambda_i = \epsilon^2 \\ \tau\left(\dfrac{1+\lambda}{1-\lambda}\right), \ \lambda_i \text{ not a square} \end{cases} \qquad i = 1, 2$
$\begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix} = H(\rho)$	$\rho = \begin{cases} \phi\left(\dfrac{2+2\epsilon}{\epsilon-1}, \left(\dfrac{1+\epsilon}{1-\epsilon}\right)^2\right), \lambda = \epsilon^2 \\ \gamma\left(\dfrac{4+4\lambda}{\lambda-1}, \dfrac{6\lambda^2+4\lambda+6}{(\lambda-1)^2}\right), \ \lambda \text{ not a square} \end{cases}$
$\begin{pmatrix}0 & -s\\ 1 & -r\end{pmatrix} = H(\rho)$ $r^2 - 4s$ not a square	$\rho = \begin{cases} \phi(-\alpha-\bar{\alpha}, \alpha\bar{\alpha}), \lambda = \epsilon^2, \epsilon \in K(\sqrt{D}) \\ \gamma(-u-\bar{u}, 2+u\bar{u}), \ \lambda \text{ not a square in } K(\sqrt{D}) \end{cases}$ λ eigenvalue of H in $K(\sqrt{D})$, $D = r^2 - 4s$ $\alpha = \dfrac{1+\epsilon}{1-\epsilon}, \ u = \dfrac{2\lambda+2}{1-\lambda}$

Table 1

α^{-1}, β^{-1} the roots of p in an algebraic closure of K and let us write

$$p = (x-\alpha)(x-\alpha^{-1})(x-\beta)(x-\beta^{-1}) = (x^2-ux+1)(x^2-vx+1)$$

with $u = \alpha+\alpha^{-1}$, $v = \beta+\beta^{-1}$.

If $u \neq v$ then p is irreducible over K and $u, v \notin K$. If $u = v$ then $u \in K$ and p is the square of the irreducible polynomial x^2-ux+1. In any case, the symmetric functions on u, v are in K and $u, v \neq \pm 2$. The characteristic polynomial of $H(\gamma(a,b))$ is

$$X^2 - \frac{2}{d}(6-b)X + \frac{1}{d}(-2a-b-2) = \left(X - \frac{u-2}{u+2}\right)\left(X - \frac{v-2}{v+2}\right).$$

If p is irreducible over K, then the two eigenvalues of $H(\gamma(a,b))$ are different and are not in K. Hence, $H(\gamma(a,b))$ is conjugated to $\left(\begin{smallmatrix}0 & -\Delta\\ 1 & t\end{smallmatrix}\right)$, $t = 2(6-b)/d$, $\Delta = (-2a-b-2)/d$.

If p is reducible over K, $p = (x^2 - ux + 1)^2$, then $H(\gamma(a, b))$ is (conjugated to) a Jordan block $\left(\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right)$ with $\lambda = (u-2)/(u+2)$ and the computation of the invariant H for all representations of type III is complete.

Let us analyze now which conjugacy classes in $GL_2(K)$ can be realized as $H(\gamma(a, b))$. If $H = \left(\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right)$ is a Jordan block with $\lambda \in K^*$, $\lambda \neq 1$ and λ is not a square in K, then the polynomial $q = x^2 + ((2 + 2\lambda)/(\lambda - 1))x + 1$ is irreducible over K and $H(\gamma) = H$ if γ is the representation of type III associated to the polynomial q^2.

If H has no eigenvalues in K, then H is conjugated to some $\left(\begin{smallmatrix} 0 & -s \\ 1 & -r \end{smallmatrix}\right)$ and $D = r^2 - 4s$ is not a square in K. Then, in $K(\sqrt{D})$, H is conjugated to a diagonal matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar\lambda \end{smallmatrix}\right)$. Assume now λ is not a square in $K(\sqrt{D})$ and let $u = (2\lambda + 2)/(1 - \lambda)$, $a = -u - \bar{u}$, $b = 2 + u\bar{u}$. Notice that λ is not well defined because we cannot distinguish between λ and $\bar\lambda$, but a and b are well defined and they are in K. The computation above shows that $H(\gamma(a, b)) = H$ but it remains to prove that $\gamma(a, b)$ is a valid representation of type III. We will see that $p = x^4 + ax^3 + bx^2 + ax + 1 = (x^2 - ux + 1)(x^2 - \bar{u}x + 1)$ is irreducible over K.

The roots of p in some algebraic closure of K are α, β, α^{-1}, β^{-1} with $\alpha = (1 + \epsilon)/(1 - \epsilon)$, $\beta = (1 + \delta)/(1 - \delta)$, with ϵ a square root of λ and δ a square root of $\bar\lambda$ (conjugation in $K(\sqrt{D})$). None of these roots can be in K and so p cannot have linear factors. Since $u \notin K$, the only way in which p could decompose over K would be $p = p_1 p_2$ with $p_1 = (x - \alpha)(x - \beta)$, which implies $\alpha + \beta$, $\alpha\beta \in K$. But this would imply that $\epsilon + \delta$, $\epsilon\delta \in K$ and ϵ would be a root of a degree two polynomial over K. On the other side, $K(\lambda)$ is a quadratic extension of K and since we are assuming that λ is not a square in $K(\sqrt{D}) = K(\lambda)$, the extension $K \subset K(\epsilon)$ is of degree 4, a contradiction. Hence, p is irreducible and $\gamma(a, b)$ is a valid representation of type III.

Table 1 summarizes the correspondence given by the invariant H between the representations of D_∞ in $GL_4(K)$ (in general position) and the conjugacy classes of matrices in $GL_2(K)$. Thus, the results in this section provide a new, geometric proof for the classification of representations of D_∞ in dimension four, as well as a new parametrization for these representations which appears to us as being more natural and "linear" than the classification into types $\nu(\alpha)$, $\tau(\beta)$, $\phi(a, b)$ and $\gamma(a, b)$ provided by [5]. Moreover, the classification given by the invariant H is effective in the sense that it provides an easy computational method to determine if two given representations are equivalent or not. Also, the information contained in table 1 allows a straightforward translation between the classification as given by the invariant H and the list of representations as provided by [5].

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