MAPS BETWEEN CLASSIFYING SPACES OF KAC-MOODY GROUPS

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1. Introduction

The study of the maps between classifying spaces of compact Lie groups from a homotopy point of view has been one of the highlights of algebraic topology in the final quarter of the XXth century. The project was started by Sullivan's construction of unstable Adams operations in his deeply influential manuscript *Geometric Topology, part I.* Along the seventies, Hubbuck, Adams-Mahmud, Wilkerson and Friedlander developed a deep analysis of the maps between classifying spaces using localization theory, Steenrod and K-theory operations and étale homotopy. In that same manuscript of Sullivan, he pointed out that the first obstruction to understand the maps between classifying spaces consists in understanding the maps from $B\mathbb{Z}/p$ to a compact space. This problem was called the Sullivan conjecture and the way to its solution (Miller, Carlsson) crossed some of the most beautiful landscapes of homotopy theory at the end of the century: the unstable Adams spectral sequence and the structure of injective objects in the category of unstable modules over the Steenrod algebra. Once this very strong weapon was available, Dwyer-Zabrodsky and Notbohm were able to understand the mod p homotopy type of the space of maps from the classifying space of a p-toral group to the classifying space of a compact Lie group. Then, the homotopy decompositions of Jackowski-McClure-Oliver made possible an inductive approach to compact Lie groups which culminated in the description by these three authors of the set of homotopy classes of self maps of BG for all compact connected simple Lie groups G. All these ideas and theorems, together with their many ramifications and generalizations (homotopy uniqueness of BG, p-compact groups, polynomial cohomology algebras, the mod p homotopy type of classifying spaces of finite groups...) pervades modern homotopy theory and is quickly becoming classic.[1]

The simply connected compact connected Lie groups are associated to the complex semi-simple (finite dimensional) Lie algebras. These Lie algebras can be described by their Cartan matrix $A = (a_{ij})$ which codifies its root system and satisfies these two conditions:

a) a_{ij} are non-positive integers for $i \neq j$, $a_{ii} = 2$ and $a_{ij} = 0$ implies $a_{ji} = 0$;
b) all principal minors of A are positive.

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[1] The interested reader is invited to read the surveys [13] and [21] as well as the introduction to [11] for a more detailed history of these topics.

Here, condition b is the one needed to prove that the Lie algebra $\mathfrak{g}(A)$ that we construct from A is finite dimensional. If we drop it, one can still construct a Lie algebra out from the data in A, but in general it turns out to be infinite dimensional. These infinite dimensional Lie algebras are called Kac-Moody algebras and they are important objects in many areas of mathematics, from finite group theory to physics (see for instance [14] and [15]).

The Kac-Moody groups are (infinite dimensional) topological groups which are obtained "integrating" the Kac-Moody algebras $\mathfrak{g}(A)$ in a way that generalizes the construction of the Lie groups from their Lie algebras. The construction of these groups $G(A)$ as well as an important geometric and topologic study of their properties was done by Kac in [15]. Inside $G(A)$ there is a "unitary form" $K(A)$ and these topological groups $K(A)$ are the groups that we call Kac-Moody groups through this paper. They behave, in many aspects, very much like their finite dimensional analogues. $K(A)$ has a maximal torus T of finite rank and a Weyl group W which is a reflection group (of infinite order); the flag variety $K(A)/T$ has a geometric structure like in the finite dimensional case, and so on.

The thesis of Kitchloo ([18]) investigated the cohomology and some other topological properties of the Kac-Moody groups of rank two (the rank of a Kac-Moody group is the rank of its maximal torus) and their classifying spaces. This work gave a first hint of the possibility of applying homotopy theory to investigate Kac-Moody groups through their classifying spaces, as an extension of the work on classifying spaces of compact Lie groups that we have summarized in the first paragraph above. Influenced by the work of Kitchloo, Aguadé, Broto, Ruíz and Saumell started a joint project with Kitchloo to study Kac-Moody groups from a homotopy point of view. The present paper, as well as [2], [5] and [6] are some of the results of this project. It should be noticed that most of the classical work on maps between classifying spaces is based on the Sullivan conjecture, i.e. on the compactness of the groups involved. Hence, extending the homotopy theoretical investigation of compact Lie groups to Kac-Moody groups is far from being a straightforward path.

Our goal here is to study the set of homotopy classes of self-maps of BK for any *rank two* Kac-Moody group K. At the same time, we will find some other results on the homotopy theory of BK and also some examples of results that are true for compact Lie groups but fail for Kac-Moody groups. In some cases, our results have not been stated in its maximum generality, since we are more interested in discovering the main lines of the homotopy theory of BK in the rank two case, as a first step to the general case. To go beyond rank two would be an interesting trip and this paper can give some hints of the phenomena that one could find there.

Let us summarize now the main results of this paper. K denotes a rank two (infinite dimensional, non-affine) Kac-Moody group.

Our most important result is a complete description of the monoid $[BK, BK]$ (section 10). To reach this result we obtain some other results that are interesting by themselves. First, we generalize the Adams maps to the maps that we call *generic Adams maps* $BK \to BK$ (section 7), we prove that any map $BK \to BK$ is homotopic to a generic Adams map (section 7) and we obtain a classification of generic

Adams maps up to homotopy on BT_K (theorem 7.6). Then we prove that any map $f : BK \to BK$ is determined, up to homotopy, by its restriction to BT_K (sections 8 and 9). We also obtain a characterization of the integers that can be the degree of some self-map of BK (theorem 7.3). All these results are parallel to the corresponding results for compact Lie groups, except that there are more Adams maps in the Kac-Moody case than in the Lie case.

We also find examples of some phenomena which do not happen in the case of compact Lie groups. For instance, we show that there are maps $BT \to BK$ which are not induced by any group homomorphism $T \to K$ (remark 3.4). We show that there are non-isomorphic Kac-Moody groups K, K' such that BK and BK' are homotopically equivalent and we obtain necessary and sufficient conditions for this to happen (theorem 6.2). We show that rational cohomology and integral cohomology are not enough to classify self maps of BK (remark 7.7).

As said before, this work is part of a more general project in collaboration with C. Broto, N. Kitchloo and L. Saumell. The authors want to thank them for all the helpful discussions that we have had all together. We are also grateful to the University of Aberdeen and the Université Paris 13 for their hospitality during the preparation of this paper.

2. Rank two Kac-Moody groups

We choose positive integers a, b such that $ab > 4$. Along this paper K will always denote the unitary form of the Kac-Moody group associated to the generalized Cartan matrix

$$\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix},$$

while we use mainly the letter L to denote a generic unitary form of a Kac-Moody group. Sometimes we write $K(a, b)$ instead of K when we want to make explicit the values of a and b used to construct K. The integers a and b can be interchanged, since the group associated to (a, b) is isomorphic to the group associated to (b, a). The case $ab < 4$ gives rise to compact Lie groups while the case $ab = 4$ is called the affine case and will be left aside. These infinite dimensional topological groups and their classifying spaces BK have been studied from a homotopical point of view in [18] and [2]. We recall here some properties of K and BK which we will use along this work.

K has a maximal torus of rank two T_K which is a maximal connected abelian subgroup of K. Any two such subgroups are conjugated. The Weyl group W of K is infinite dihedral group acting on the Lie algebra of T_K through reflections ω_1 and ω_2 given by the integral matrices:

$$w_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}.$$

The matrices of determinant $+1$ in W form a subgroup W^+ of index two which is infinite cyclic generated by $\tau = \omega_1\omega_2$.

A fundamental result in the homotopy theory of the classifying spaces of Kac-Moody groups is the following result of Kitchloo ([18]). If L is any Kac-Moody group with *infinite* Weyl group and $\{P_I\}$ are the parabolic subgroups of L indexed by proper subsets I of $\{1,\dots,\mathrm{rank}(L)\}$ then there is a homotopy equivalence

$$BL \simeq \operatorname*{hocolim}_I BP_I.$$

In the rank two case, this allows us to construct BK as a push out

$$\begin{array}{ccc} BT_K & \longrightarrow & BH_1 \\ \downarrow & & \downarrow \\ BH_2 & \longrightarrow & BK \end{array}$$

where H_1 and H_2 are some compact Lie groups of rank two which depend on the parity of the integers a and b. More precisely (see [2]) we have the following push out diagrams:

- $a \equiv b \equiv 0 \pmod 2$. Then

$$BK \simeq \operatorname{hocolim}\left\{BS^3\times BS^1 \xleftarrow{\begin{pmatrix} -\frac{a}{2} & 1 \\ 1 & 0\end{pmatrix}} BT \xrightarrow{\begin{pmatrix} 1 & -\frac{b}{2} \\ 0 & 1\end{pmatrix}} BS^3\times BS^1\right\}.$$

- $a \equiv b \equiv 1 \pmod 2$. Then

$$BK \simeq \operatorname{hocolim}\left\{BU(2) \xleftarrow{\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1\end{pmatrix}} BT \xrightarrow{\begin{pmatrix} 1 & \frac{1-b}{2} \\ -1 & \frac{1+b}{2}\end{pmatrix}} BU(2)\right\}.$$

- $a \equiv 1$, $b \equiv 0 \pmod 2$. Then

$$BK \simeq \operatorname{hocolim}\left\{BU(2) \xleftarrow{\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1\end{pmatrix}} BT \xrightarrow{\begin{pmatrix} 1 & -\frac{b}{2} \\ 0 & 1\end{pmatrix}} BS^3\times BS^1\right\}.$$

Here each matrix M written above an arrow means a map $B(i\circ\rho)$ where $i : T_K \hookrightarrow K$ is the inclusion and $\rho : T_K \to T_K$ is the homomorphism inducing M on the Lie algebra level.

The mod p cohomology of BK is known (see [18], [2]). We have (the subscripts denote the degrees and E denotes an exterior algebra):

- If $a \equiv b \equiv 0 \pmod 2$ then $H^*(BK;\mathbb{F}_2) \cong \mathbb{F}_2[x_4, y_4] \otimes E[z_5]$ with $\beta_r(y_4) = z_5$ where $2^r \| \gcd(a,b)$.
- If $a \equiv b \equiv 1 \pmod 2$ then $H^*(BK;\mathbb{F}_2) \cong \mathbb{F}_2[x_4, y_6] \otimes E[z_7]$ with $\beta_r(y_6) = z_7$ where $2^r \| (ab-1)$.
- If $a \equiv 1 \pmod 2$ and $b \equiv 0 \pmod 2$ then $H^*(BK;\mathbb{F}_2) \cong \mathbb{F}_2[x_4, y_8] \otimes E[z_9]$ with $\beta_r(y_8) = z_9$ where $2^r \| (ab-2)$.
- If $p > 2$ then $H^*(BK;\mathbb{F}_p) \cong \mathbb{F}_p[x_4, y_{2k}] \otimes E[z_{2k+1}]$ with $\beta_r(y_{2k}) = z_{2k+1}$ and the integers k and r are determined in the following way. The action of the Weyl

group W on T_K gives a representation $W \hookrightarrow \mathrm{GL}_2\mathbb{Z}$. Let W_{p^n} be the image of W in $\mathrm{GL}_2(\mathbb{Z}/p^n)$. Then $k = |W_p|/2$ and $r = \min\{n : |W_{p^n}| < |W_{p^{n+1}}|\}$.

We introduce now the group $\widehat{W}_p \subset GL_2(\widehat{\mathbb{Z}}_p)$ defined in the following way. We have the groups $W_{p^n} \subset GL_2(\mathbb{Z}/p^n)$ which are the reductions mod p^n of the Weyl group $W \subset GL_2(\mathbb{Z})$. Then we define $\widehat{W}_p$ as the inverse limit of $\{W_{p^n}\}$. We call $\widehat{W}_p$ the *p-completed Weyl group*. Notice that this group contains W as a subgroup. The structure of the group $\widehat{W}_p$ can be described as follows. We define $\widehat{W}_p^+ = \widehat{W}_p \cap SL_2(\widehat{\mathbb{Z}}_p)$. Then we have an extension $\widehat{W}_p^+ \rightarrowtail \widehat{W}_p \twoheadrightarrow \mathbb{Z}/2$. Consider the element $\tau = \omega_1\omega_2 \in W$ of infinite order and let $N(n)$ be the order of τ in $GL_2(\mathbb{Z}/p^n)$. Then $\widehat{W}_p^+ = \varprojlim \mathbb{Z}/N(n)$. We can write $N(n) = lp^{r(n)}$ where $r(n)$ is an increasing sequence. Then,

$$\widehat{W}_p^+ \cong \widehat{\mathbb{Z}}_p \times \mathbb{Z}/l.$$

The value of l is known (see [18]): for $p > 2$ we have $l = 1$ if $p|ab-4$, $l = 2$ if p divides a or b but not both, and l is the multiplicative order of the roots of $x^2 - (ab-2)x + 1$ in $\mathbb{F}_{p^2}$ in all other cases; if $p = 2$ then $l = 3$ if a and b are odd, while $l = 1$ otherwise. This function $l = l(K, p)$ will be used several times in this work. We will introduce another function $y = y(K, p)$ which is defined for all odd primes such that l is odd and which will be used in our investigation of Adams maps.

Lemma 2.1. *Assume $p > 2$ and $l = l(K, p)$ odd. Then there is a unique element $y = y(K, p) \in \widehat{\mathbb{Z}}_p$ such that $ay^2 = b$ and $\begin{pmatrix} 0 & y \\ 1/y & 0 \end{pmatrix} \in W_p$.*

Proof. If p and l are odd then $\widehat{W}_p^+$ is uniquely 2-divisible and there is a unique matrix $A \in \widehat{W}_p^+$ such that $A^2 = \omega_1\omega_2$. Let us investigate what are the possible values of A. $\omega_1\omega_2$ has two different eigenvalues ζ, ζ^{-1} which are the roots of the characteristic polynomial $x^2 - (ab-2)x + 1$ and an easy direct calculation shows that the square roots of $\omega_1\omega_2$ in $SL_2\widehat{\mathbb{Z}}_p$ are the matrices $\begin{pmatrix} ay & -y \\ 1/y & 0 \end{pmatrix}$ with $y = \pm(1+\zeta)/a\sqrt{\zeta}$. Only one of these square roots is in $\widehat{W}_p^+$. This gives a choice for y. Consider now $A\omega_2 = \begin{pmatrix} 0 & y \\ 1/y & 0 \end{pmatrix} \in \widehat{W}_p$. We see easily that y satisfies the lemma. Uniqueness is clear since $ay^2 = b$ determines y up to a sign and the fact that W_p^+ has odd order settles the question. $\square$

The center of K is also well understood ([18]):

$$ZK = \begin{cases} 2\mathbb{Z}/(ab-4) \times \mathbb{Z}/2, & a \equiv b \equiv 0 \pmod 2 \\ \mathbb{Z}/(ab-4), & \text{otherwise.} \end{cases}$$

We have that $H^4(BK;\mathbb{Z}) \cong \mathbb{Z}$ and we can fix a generator in the following way. $H^4(BK;\mathbb{Z})$ injects in $H^4(BT_K;\mathbb{Z})$. Let d be the g.c.d. of a and b and let $a' = a/d$, $b' = b/d$. Let $\{t_1, t_2\}$ be the simple roots basis of $H^2(BT_K;\mathbb{Z})$. Then a generator of

$H^4(BK;\mathbb{Z})$ is given by

$$q = a't_1^2 + b't_2^2 - a'bt_1t_2.$$

Notice that this is a binary quadratic form with integral coefficients and discriminant $\Delta(q) = {a'}^2b^2 - 4a'b' > 0$. Such objects are classically called primitive indefinite binary forms and constitute an important topic in number theory since Gauss (see [7]). In particular, the form q is an *ambiguous* one. (A form is called ambiguous if there is an integral matrix of determinant -1 which leaves it invariant.) An integral matrix of determinant $+1$ which leaves q invariant is called an *automorph* of q. The automorphs of q form a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which is closely related to the Weyl group W. We will discuss this relation elsewhere.

The class q produces a map $q : BK \to K(\mathbb{Z}, 4)$ which is a rational equivalence. This class q allows us to define the *degree* of a self map $f : BK \to BK$ as the integer g such that the equality $f^*(q) = gq$ holds in integral cohomology. The degree gives a map

$$\deg : [BK, BK] \to \mathbb{Z}$$

which is a monoid homomorphism and will play an important role in our description of $[BK, BK]$.

The outer automorphisms of a Kac-Moody group are computed by Kac and Wang in [17]. Their results give the following description of $\mathrm{Out}(K)$:

$$\mathrm{Out}(K) = \begin{cases} \mathbb{Z}/2 \cdot \Psi^{-1}, & a \neq b \\ \mathbb{Z}/2 \cdot \Psi^{-1} \times \mathbb{Z}/2 \cdot \Psi^{1,1}, & a = b. \end{cases}$$

Here $\Psi^{-1}, \Psi^{1,1} : K \to K$ are automorphisms which induce $-I$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the Lie algebra of T_K, respectively. They induce self maps $BK \to BK$ which we denote by ψ^{-1} and $\psi^{1,1}$, respectively, by analogy to the Adams maps between classifying spaces of compact Lie groups.

Let N_K denote the normalizer in K of the maximal torus T_K. Then, if p is odd, a theorem of Kitchloo ([18]) proves that the inclusion $N_K \hookrightarrow K$ induces a homotopy equivalence $(BN_K)^\wedge_p \simeq BK^\wedge_p$.

3. Relations between global and local maps

In general, it is easier to study the maps $X \to Y$ one prime at a time. Then, the set $[X, Y]$ can be recovered from the sets $[X, Y^\wedge_p]$ by using the arithmetic square of [4], p. 192. In this section we want to investigate this situation when the target space is BK.

Recall that there is a class $q : BK \to K(\mathbb{Z}, 4)$ which induces a rational equivalence. Hence, $BK_{\mathbb{Q}} \simeq K(\mathbb{Q}, 4)$ and $(BK^\wedge_p)_{\mathbb{Q}} \simeq K(\widehat{\mathbb{Q}}_p, 4)$. We denote by $q \otimes \widehat{\mathbb{Q}}_p$ the class $BK^\wedge_p \to K(\widehat{\mathbb{Z}}_p, 4) \to K(\widehat{\mathbb{Q}}_p, 4)$.

Proposition 3.1. *Let X be a space such that $H^5(X;\mathbb{Q}) = 0$. Then the map l : $[X, BK] \to \prod_p [X, BK^\wedge_p]$ is injective. The image consists of those families of maps*

$\{f_p : X \to BK_p^\wedge\}$ such that there is $x \in H^4(X;\mathbb{Q})$ such that $f_p^*(q \otimes \widehat{\mathbb{Q}}_p) = x \otimes \widehat{\mathbb{Q}}_p$ for all p.

Proof. The proof of the injectivity of l is essentially analogous to the proof of theorem 3.1 in [11]. The arithmetic square ([4], p. 192) gives a pull back diagram

$$\begin{tikzcd} BK \arrow[r] \arrow[d] & \prod_p BK_p^\wedge \arrow[d] \\ K(\mathbb{Q},4) \arrow[r] & K(\widehat{\mathbb{Q}},4) \end{tikzcd}$$

Here $\widehat{\mathbb{Q}} = (\prod_p \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q}$ and the hypothesis on the vanishing of $H^5(X;\mathbb{Q})$ ensures that each component of the spaces $\mathrm{Map}(X, K(\mathbb{Q},4))$ and $\mathrm{Map}(X, K(\widehat{\mathbb{Q}},4))$ is simply connected.

The coherence condition on $\{f_p\}$ is clearly necessary. Let us show that it is sufficient too. The class $x \in H^4(X;\mathbb{Q})$ gives a map $X \to K(\mathbb{Q},4)$ while $\{f_p\}$ gives $\prod_p f_p : X \to \prod_p BK_p^\wedge$. Because of the pullback diagram above, the global map $f : X \to BK$ will exist if we prove that the two maps $X \to \prod_p BK_p^\wedge \to K(\widehat{\mathbb{Q}},4)$ and $X \to K(\mathbb{Q},4) \to K(\widehat{\mathbb{Q}},4)$ are homotopic. It is enough to check that both maps coincide on $H^4(-;\widehat{\mathbb{Q}})$. The inclusion of divisible abelian groups $\widehat{\mathbb{Q}} \subset \prod_p \widehat{\mathbb{Q}}_p$ reduces the problem to check equality on $H^4(-;\widehat{\mathbb{Q}}_p)$ for all primes p. The commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f_p"'] & \prod_p BK_p^\wedge \arrow[r] \arrow[d, "\pi_p"] & K(\widehat{\mathbb{Q}},4) \arrow[d, "(\pi_p)_{\mathbb{Q}}"] \\ & BK_p^\wedge \arrow[r] & K(\widehat{\mathbb{Q}}_p,4) \end{tikzcd}$$

together with the coherence hypothesis on $\{f_p\}$ solves the problem. $\square$

We want to use this result when $X = BT$ and $X = BK$. Let us denote by T_{p^∞} the p-torsion subgroup of a torus T. We have $H^*(BT_{p^\infty};\mathbb{Z}) \cong H^*(BT;\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_p$ and the inclusion $T_{p^\infty} \hookrightarrow T$ induces a mod p equivalence $BT_{p^\infty} \to BT$. Hence $[BT, BK_p^\wedge] \cong [BT_{p^\infty}, BK_p^\wedge]$ and 3.1 can be written in this form:

Proposition 3.2. *The map $l : [BT, BK] \to \prod_p [BT_{p^\infty}, BK]$ is injective. The image consists of those families of maps $\{f_p : BT_{p^\infty} \to BK\}$ such that $f_p^*(q)$ lies in $H^4(BT;\mathbb{Z}) \subset H^4(BT_{p^\infty};\mathbb{Z})$ and is independent of p.* $\square$

In the case $X = BK$ 3.1 can be written in this form:

Proposition 3.3. *The map $l : [BK, BK] \to \prod_p [BK_p^\wedge, BK_p^\wedge]$ is injective. The image consists of those families of maps $\{f_p : BK_p^\wedge \to BK_p^\wedge\}$ such that $f_p^*(q \otimes \widehat{\mathbb{Q}}_p) = \lambda(q \otimes \widehat{\mathbb{Q}}_p)$ and λ is a rational number independent of p.* $\square$

From these rather elementary facts it follows easily that one of the classical results on maps between classifying spaces of compact Lie groups fails for infinite dimensional Kac-Moody groups. Notbohm proved ([19]) that any map from the classifying space of a torus T to the classifying space of a compact Lie group G is homotopic to a map induced by a homomorphism from T to G. This is not true for Kac-Moody groups:

Remark 3.4. In general, the map $\operatorname{Hom}(T,K) \to [BT,BK]$ is not surjective.

Proof. Chose a and b such that the binary form q has non trivial genus set (see [7]) and take $K = K(a,b)$, $T = T_K$. Let q' be a binary form in the genus of q but not equivalent to q. (There are plenty of examples of this phenomenon.) This means that q and q' are equivalent over the ring $\widehat{\mathbb{Z}}_p$ for all p, but not equivalent over $\mathbb{Z}$. Hence, we have for each prime p an isomorphism $\phi_p : T_{p^\infty} \to T_{p^\infty}$ which is an equivalence between q and q' and can be used to produce a coherent family of maps $\{f_p : BT_{p^\infty} \to BK\}$ and, by 3.2, a map $f : BT \to BK$ with $f^*(q) = q'$. Assume that $f \simeq B\rho$ for some group homomorphism $\rho : T_K \to K$. Then ρ factors through T_K up to an inner automorphism of K and so ρ is given by an integral two-by-two matrix M. Then $q' = f^*(q) = M^t q M$. Since forms in the same genus have the same discriminant, M gives an integral equivalence between q and q', and this is a contradiction. $\square$

4. Maps into $BK^\wedge_p$ and representations

Let us fix a prime p. In this section we want to study the relation between homotopy classes of maps $BT \to BK^\wedge_p$ and representations $T_{p^\infty} \to K$. The starting point for this research is the crucial result on maps from the classifying space of a p-group into the p-completed classifying space of a Kac-Moody group that were obtained by Broto-Kitchloo in [5] and [6]. We recall here these results.

Theorem 4.1 ([5],[6])**.** *If L is a Kac-Moody group and π is a finite p-group then*

a) *There is a homotopy equivalence*

$$\coprod_{\rho \in \operatorname{Rep}(\pi,L)} (BC_L(\rho))^\wedge_p \xrightarrow{\simeq} \operatorname{Map}(B\pi, BL^\wedge_p)$$

where $C_L(\rho)$ denotes the centralizer in L of the image of ρ. In particular, $[B\pi, BL^\wedge_p] \cong \operatorname{Rep}(\pi, L)$.

b) *If $\{P_I\}$ denotes the poset of parabolic subgroups of L which are Lie groups, then there is a homotopy equivalence*

$$\big(\operatorname*{hocolim}_I \operatorname{Map}(B\pi, B{P_I}^\wedge_p)\big)^\wedge_p \xrightarrow{\simeq} \operatorname{Map}(B\pi, BL^\wedge_p). \quad \square$$

We want to apply this result to the p-groups $T_{p^n} \subset T_K$ consisting of the elements of T_K of order dividing p^n. We introduce the following notation. $\operatorname{Map}(BT_{p^n}, BK^\wedge_p)_{(s)}$ is the subspace of $\operatorname{Map}(BT_{p^n}, BK^\wedge_p)$ which contains all maps which are homotopic to some $B\rho$ where $\rho : T_{p^n} \to K$ is a homomorphism with kernel of order $\leq p^s$. We denote by $\operatorname{End}^{(s)}(T_{p^n})$ the set of endomorphisms of T_{p^n} with kernel of order $\leq p^s$. The elements of $\operatorname{End}^{(s)}(T_{p^n})$ are represented by integral two-by-two matrices $M = (a_{ij})$

and for $n > s$ these matrices have the property that $\nu_p(\det(M)) \leq s$. To prove this, consider the abelian group T_{p^n}/MT_{p^n} and apply to it the classification theorem of finitely generated abelian groups. It turns out to be isomorphic to a group of the form $\mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z}$ with $d_1|d_2$ and $d_1d_2 = gcd(p^{2n}, p^n a_{ij}, \det M)$. Then the size of the kernel of M in T_{p^n} is the same as the size of T_{p^n}/MT_{p^n} which is equal to d_1d_2. If $d_1d_2|p^s < p^n$ then $\nu_p(\det(M)) \leq s$.

The inclusion $T_{p^n} \subset T_K$ gives an action of the Weyl group of K on T_{p^n}. Hence, the group W acts on the left on the sets $\operatorname{End}^{(s)}(T_{p^n})$. The next theorems incorporate some ideas of Dwyer and Kitchloo.

Theorem 4.2. *If $n \gg s$ then each component of $\operatorname{Map}(BT_{p^n}, BK_p^\wedge)_{(s)}$ is homotopy equivalent to $(BT_K)_p^\wedge \times K(\widehat{\mathbb{Z}}_p, 1)$.*

Proof. When we apply theorem 4.1 b) to our case we obtain a push out diagram up to p-completion

$$\operatorname{Map}(BT_{p^n}, BK_p^\wedge)_{(s)} \xleftarrow{\simeq^p}$$

$$\operatorname{hocolim}\big(\operatorname{Map}(BT_{p^n}, (BH_1)_p^\wedge)_{(s)} \leftarrow \operatorname{Map}(BT_{p^n}, (BT_K)_p^\wedge)_{(s)} \to \operatorname{Map}(BT_{p^n}, (BH_2)_p^\wedge)_{(s)}\big).$$

Each of the mapping spaces in this push out diagram can be computed using theorem 4.1a which in the case of Lie groups reduces to a fundamental theorem of Dwyer-Zabrodsky (see [8]). If P denotes any of the parabolics of K (i.e. P is either T_K or H_1 or H_2), then

$$\operatorname{Map}(BT_{p^n}, BP_p^\wedge)_{(s)} \simeq \coprod (BC_P(\rho))_p^\wedge$$

where the disjoint union ranges over all representations ρ of T_{p^n} in P with $|\operatorname{Ker}\rho| \leq p^s$. Now, if n is big enough then all these representations factor through the maximal torus of P, which is the same as the maximal torus of K, and their centralizer is T_K. Moreover, for n big enough the representations of T_{p^n} in P with small kernel are in one-to-one correspondence to the cosets in $W_P \backslash \operatorname{End}^{(s)}(T_{p^n})$. Hence,

$$\operatorname{Map}(BT_{p^n}, BP_p^\wedge)_{(s)} \simeq \coprod_{W_P\backslash \operatorname{End}^{(s)}(T_{p^n})} (BT_K)_p^\wedge.$$

This reduces the problem to the computation of a homotopy push out of finite sets

$$(*)\qquad \begin{tikzcd} \operatorname{End}^{(s)}(T_{p^n}) \arrow[r] \arrow[d] & \langle\omega_1\rangle\backslash \operatorname{End}^{(s)}(T_{p^n}) \\ \langle\omega_2\rangle\backslash \operatorname{End}^{(s)}(T_{p^n}) & \end{tikzcd}$$

Notice that ω_1 and ω_2 are matrices or order two. Hence, the components of this push out of sets are either circles or points. To show that no isolated points occur, we need to check that ω_1 and ω_2 act freely on the set $\operatorname{End}^{(s)}(T_{p^n})$. Let M be an integral two-by-two matrix such that $\omega_i M \equiv M\,(p^n)$ for some $i = 1, 2$. Since ω_i has determinant -1, we have $\det M \equiv 0\,(p^{n-1})$ and so such a matrix M does not exist in $\operatorname{End}^{(s)}(T_{p^n})$

if n is big enough. Hence, the homotopy push out above has the homotopy type of a disjoint union of circles and the theorem follows. □

Corollary 4.3. *Let $\rho : T_{p^\infty} \to K$ be a homomorphism with finite kernel. Then $\mathrm{Map}(BT_K, BK_p^\wedge)_{B\rho}$ is homotopically equivalent to $(BT_K)_p^\wedge$.*

Proof. We can use 4.2 in the following way:

$$\mathrm{Map}(BT_K, BK_p^\wedge)_{B\rho} \simeq \operatorname*{holim}_n \mathrm{Map}(BT_{p^n}, BK_p^\wedge)_{B\rho} \simeq \mathrm{holim}\{(BT_K)_p^\wedge \times K(\widehat{\mathbb{Z}}_p, 1)\}.$$

To prove the corollary, we see that the inverse system of p-completed circles has trivial homotopy limit. To do this, let us look more carefully at the diagram $(*)$ above. A circle in the push out is obtained from a matrix M with $(\omega_1\omega_2)^m M \equiv M\ (p^n)$. Fix M with kernel of size $\leq p^s$ and define $m(n)$ as the minimum positive integer such that $(\omega_1\omega_2)^{m(n)} M \equiv M\ (p^n)$. If we prove that $\{\nu_p(m(n))\}$ is an increasing sequence then $\mathrm{holim}\{K(\widehat{\mathbb{Z}}_p, 1)\}$ will be a point. If we multiply the congruence $(\omega_1\omega_2)^{m(n)} M \equiv M\ (p^n)$ by the adjoint matrix to M we see that $m(n)$ has to be a multiple of the order of $\omega_1\omega_2$ in $GL_2(\mathbb{Z}/p^{n-s})$. Since the p-adic valuation of this order tends to infinity, so does the p-adic valuation of $m(n)$. □

As said before, the Weyl group W acts on T_{p^∞}. Also, the integral representation $W \hookrightarrow GL_2\mathbb{Z}$ gives an action of W on $(\widehat{\mathbb{Z}}_p)^2$. These two actions are essentially the same in the sense that the action of W on the character group of T_{p^∞} $\mathrm{Hom}(T_{p^\infty}, \mathbb{Z}_{p^\infty}) \simeq (\widehat{\mathbb{Z}}_p)^2$ is the dual to the natural action of W on $(\widehat{\mathbb{Z}}_p)^2$.

Proposition 4.4. *Let A be a proper subgroup of T_{p^∞} such that $WA \subset A$. Then A is finite.*

Proof. First, one proves easily that the hypothesis $ab > 4$ implies that $(\widehat{\mathbb{Z}}_p)^2$ has no W-invariant sub-$\widehat{\mathbb{Z}}_p$-modules of rank one. Then, if we apply the functor $\mathrm{Hom}(-, \mathbb{Z}_{p^\infty})$ to the exact sequence of abelian groups $0 \to A \to T_{p^\infty} \to T_{p^\infty}/A \to 0$ we obtain an exact sequence of character groups

$$0 \to (T_{p^\infty}/A)^* \to (\widehat{\mathbb{Z}}_p)^2 \to A^* \to 0$$

and we obtain a non trivial W-invariant sub-$\widehat{\mathbb{Z}}_p$-module of $(\widehat{\mathbb{Z}}_p)^2$. Hence, $(T_{p^\infty}/A)^*$ has $\widehat{\mathbb{Z}}_p$-rank two and A is finite. □

Proposition 4.5. *Let $f : BK \to BK_p^\wedge$. There is a homomorphism $\rho : T_{p^\infty} \to K$ such that $f|_{BT_{p^\infty}} \simeq B\rho$. If $\rho \neq 1$ then ρ has finite kernel.*

Proof. For each n, theorem 4.1 shows that $f|_{BT_{p^n}} \simeq B\rho_n$ for some homomorphism $\rho_n : T_{p^n} \to K$ uniquely determined up to an inner automorphism of K. It is clear that we can turn $\{\rho_n\}$ into a compatible family which gives a homomorphism $\rho : T_{p^\infty} \to K$ with the property that $B\rho|_{BT_{p^n}} \simeq f|_{BT_{p^n}}$. To conclude that $B\rho \simeq f|_{BT_{p^\infty}}$ we need to prove that some obstructions vanish. These obstructions live in

$$\varprojlim_n{}^1 \, \pi_1 \, \mathrm{Map}(BT_{p^n}, BK_p^\wedge)_{B\rho_n}.$$

If $\rho = 1$ then $\operatorname{Map}(BT_{p^n}, BK_p^\wedge)_{B\rho_n} \simeq BK_p^\wedge$ by theorem 4.1 and the obstructions vanish since BK is simply connected.

If $\rho \neq 1$ then the vanishing of the $\varprojlim^1$ follows from the computation of the homotopy type of $\operatorname{Map}(BT_{p^n}, BK_p^\wedge)$ done in 4.2, as soon as we can prove that the condition on the size of the kernel of ρ_n holds. We will prove that $\operatorname{Ker}\rho$ is finite. Notice that if $\omega \in W$ and $i_n : T_{p^n} \hookrightarrow K$ then

$$B\rho_n \simeq f|_{BT_{p^n}} \simeq f \cdot Bi_n \simeq f \cdot Bi_n \cdot Bc_{\bar{\omega}} \simeq B(\rho_n \cdot \omega)$$

($c_{\bar{\omega}}$ is conjugation by some $\bar{\omega} \in N_K(T)$ which projects onto $\omega \in W$). Hence, there is $g_n \in K$ such that

$$\rho_n \cdot \omega|_{T_{p^n}} = c_{g_n} \cdot \rho|_{T_{p^n}}.$$

This shows that $\operatorname{Ker}\rho$ is W-invariant. Then, proposition 4.4 implies that $\operatorname{Ker}\rho$ is finite and the theorem is proved. □

Next, we want to describe the isotropy of the homomorphism ρ that appears in the preceding proposition. It is given by the *p-completed Weyl group* $\widehat{W}_p$ that we discussed in section 2.

Proposition 4.6. *Let $\rho, \rho' : T_{p^\infty} \to T_{p^\infty}$ be homomorphisms with finite kernel. Then $B\rho \simeq B\rho'$ in $[BT_{p^\infty}, BK_p^\wedge]$ if and only if there is $\alpha \in \widehat{W}_p$ such that $\rho = \alpha\rho'$.*

Proof. Choose s such that $B\rho, B\rho' \in \operatorname{Map}(BT_{p^\infty}, BK_p^\wedge)_{(s)}$. Clearly, this proposition is about $\pi_0 \operatorname{Map}(BT_{p^\infty}, BK_p^\wedge)_{(s)}$. The equivalence $BT_{p^\infty} = \operatorname{hocolim} BT_{p^n}$ reduces the problem to the study of $\operatorname{Map}(BT_{p^n}, BK_p^\wedge)_{(s)}$ for large n. This space has been analyzed in theorem 4.2. From the arguments there, it follows that the set of components of $\operatorname{Map}(BT_{p^n}, BK_p^\wedge)_{(s)}$ is in one-to-one correspondence to $W_{p^n}\backslash \operatorname{End}^{(s)}(T_{p^n})$. Then, we have a surjection

$$\pi_0 \operatorname{Map}(BT_{p^\infty}, BK_p^\wedge)_{(s)} \twoheadrightarrow \varprojlim_n{}^0\, \pi_0 \operatorname{Map}(BT_{p^n}, BK_p^\wedge)_{(s)}$$

and each fiber of this surjection is in one-to-one correspondence to

$$\varprojlim_n{}^1\, \pi_1(\operatorname{Map}(BT_{p^n}, BK_p^\wedge)_{(s)}, *)$$

for some choice of the base point $*$. This $\varprojlim^1$ vanishes since each group in the tower is compact and so we obtain a one-to-one correspondence

$$\pi_0 \operatorname{Map}(BT_{p^\infty}, BK_p^\wedge)_{(s)} \cong \widehat{W}_p \backslash \operatorname{End}^{(s)}(T_{p^\infty})$$

which proves the theorem. □

Now we can display the complete picture:

Theorem 4.7. *Let $f : BK \to BK_p^\wedge$. There is an endomorphism ρ of T_{p^∞} such that $f|_{BT_{p^\infty}} \simeq B\rho$. ρ is unique up to the left action of $\widehat{W}_p$ on* $\operatorname{End}(T_{p^\infty})$.

Proof. Proposition 4.5 shows that there is a homomorphism $\rho : T_{p^\infty} \to K$ with $f|_{BT_{p^\infty}} \simeq B\rho$. If $\rho \neq 1$ then ρ has finite kernel and so

$$f|_{BT_{p^\infty}} \in \mathrm{Map}(BT_{p^\infty}, BK_p^\wedge)_{(s)} \cong \widehat{W}_p \backslash \mathrm{End}^{(s)}(T_{p^\infty})$$

for some s. It follows that ρ can be replaced by $\rho' : T_{p^\infty} \to T_{p^\infty}$ with finite kernel. Also, if ρ' is another endomorphism of T_{p^∞} such that $B\rho' \simeq B\rho''$ in $[BT_{p^\infty}, BK_p^\wedge]$ then $\rho'|_{T_{p^n}}$ and $\rho''|_{T_{p^n}}$ are conjugated in K for any n. Hence, ρ' and ρ'' have the same kernel. Hence, proposition 4.6 applies and ρ' and ρ'' differ by an element of $\widehat{W}_p$. The case $\rho = 1$ is clear. □

5. Admissible matrices

Let now $K = K(a, b)$, $K' = K(a', b')$ be two Kac-Moody groups of rank two with Weyl groups W, W'. As always, we assume $ab, a'b' > 4$. We have then two integral representations of the infinite dihedral group D_∞ given by the actions of W and W' on the Lie algebras of the maximal tori T_K, $T_{K'}$.

We say that a two-by-two matrix M with entries in $\widehat{\mathbb{Z}}_p$ is *p-admissible* if for each $\omega \in W$ there is $\omega' \in \widehat{W}'_p$ (the p-completed Weyl group of K') such that $M\omega = \omega' M$. (We should say that M is $W{-}W'{-}p$-admissible, but we allow ourselves this simplified notation.) The next result classifies all admissible matrices.

Proposition 5.1. *Assume $ab = a'b'$. Then the p-admissible matrices are those of the form $\omega' A$ with $\omega' \in \widehat{W}'_p$ and with A of the form either $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $a'\lambda = a\mu$ and $\lambda, \mu \in \widehat{\mathbb{Z}}_p$ or $\begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$ with $a\lambda = b'\mu$ and $\lambda, \mu \in \widehat{\mathbb{Z}}_p$.*

Proof. First, one checks easily that the matrices A that appear in the proposition are exactly the ones with the property that $A\omega_i = \omega'_{\sigma(i)} A$, $i = 1, 2$, where σ is a permutation of $\{1, 2\}$ and ω_i, ω'_i are the generating reflections of W and W', respectively.

Assume $M \neq 0$ is a p-admissible matrix. Then the kernel of M is a sub-$\widehat{\mathbb{Z}}_p$-module of $(\widehat{\mathbb{Z}}_p)^2$ which is W-invariant. As said before, this implies that this kernel is trivial and so M^{-1} exists in $GL_2(\widehat{\mathbb{Q}}_p)$. This shows that we can write $M\omega = \beta(\omega) M$ for some group homomorphism $\beta : W \to \widehat{W}'_p$. Let us investigate the behavior of β.

Let us consider the matrix $\tau = \omega_1\omega_2 = \begin{pmatrix} ab-1 & -b \\ a & -1 \end{pmatrix}$ which generates an infinite cyclic subgroup of W. This matrix has two different eigenvalues $\zeta^{\pm 1}$ which are not roots of unity and depend only on the product ab (cf. the proof of lemma 2.1). The matrices τ and $\beta(\tau)$ are conjugated in $GL_2(\widehat{\mathbb{Q}}_p)$. Let $\tau' = \omega'_1\omega'_2 = \begin{pmatrix} a'b'-1 & -b' \\ a' & -1 \end{pmatrix}$. Then, τ' and $\beta(\tau)$ have the same eigenvalues and both belong to the abelian group $\widehat{W}'^{+}_p$. Hence, there is a basis (in some extension field) such that τ' and $\beta(\tau)$ both diagonalize simultaneously. This implies that $\beta(\tau) = \tau'^{\pm 1}$. Composing with an inner automorphism of $\widehat{W}'^{+}_p$ we can assume that $\beta : W \to \widehat{W}'_p$ is a monomorphism such that

$\beta(\omega_1\omega_2) = \omega_1'\omega_2'$. Recall now the structure of the group $\widehat{W}_p'$ as discussed in section 2. $\widehat{W}_p'$ is an extension of $\mathbb{Z}/2$ by the abelian group $\widehat{W}_p'^{+} \cong \widehat{\mathbb{Z}}_p \times \mathbb{Z}/l$ and W' sits inside $\widehat{W}_p'$ sending τ' to the element $(1,1) \in \widehat{\mathbb{Z}}_p \times \mathbb{Z}/l$. Recall also that l is odd if $p = 2$. From this description of the group $\widehat{W}_p'^{+}$ it follows easily that given $\omega \in \widehat{W}_p'^{+}$ either $\omega = \gamma^2$ or $\omega = \gamma^2\tau'$ for some $\gamma \in \widehat{W}_p'^{+}$. Now, from $\beta(\omega_1\omega_2) = \omega_1'\omega_2'$ it follows that there is $\omega \in \widehat{W}_p'^{+}$ such that $\beta(\omega_i) = \omega\omega_i'$. If $\omega = \gamma^2$ then $\beta(\omega_i) = \gamma\omega_i'\gamma^{-1}$, $i = 1, 2$, while if $\omega = \gamma^2\tau'$ then $\beta(\omega_i) = \gamma\omega_1'\omega_{\sigma(i)}'\omega_1'\gamma^{-1}$, σ the non trivial permutation of $\{1, 2\}$. In any case, it follows that β is, up to conjugation, either the natural inclusion or the homomorphism which interchanges ω_1 and ω_2. Put $\beta(\omega_i) = \omega'\omega_{\sigma(i)}'\omega'^{-1}$ for some $\omega' \in \widehat{W}_p'$, $i = 1, 2$, σ a permutation of $\{1, 2\}$. Then, the matrix $A = \omega'^{-1}M$ satisfies $A\omega_i = \omega_{\sigma(i)}'A$, $i = 1, 2$ and so, as said before, this implies that A has the form stated in the proposition. $\square$

The importance of p-admissible matrices in our work comes from the fact that they appear naturally in the investigation of maps between classifying spaces, as shown by the following proposition.

Proposition 5.2. *Let $f : BK \to BK'^{\wedge}_p$ be a map. There is a p-admissible matrix M such that $f|_{BT_{p^\infty}}$ is homotopic to the map induced by M.*

Proof. By 4.7, $f|_{BT_{p^\infty}}$ is homotopic to the map induced by a homomorphism $\rho : T_{p^\infty} \to T'_{p^\infty}$ which is unique up to the left action of $\widehat{W}_p'$. (Actually, 4.7 was stated for $K = K'$ but it is also valid in our more general situation, without any change at all.) If $\omega \in W$ then $\rho\omega$ is another choice for ρ and so $\rho\omega = \omega'\rho$ for some $\omega' \in \widehat{W}_p'$. $\square$

6. Groups with the same classifying space

An special feature of (non-Lie) Kac-Moody groups is the fact that the classifying space functor is not faithful, i.e. there are non-isomorphic Kac-Moody groups $K \ncong K'$ with the same classifying space up to homotopy $BK \simeq BK'$. If this happens then K and K' have, in particular, the same homotopy type. While there are non isomorphic semi simple Lie groups of the same homotopy type (see [3]), there is a theorem of Notbohm ([20]) which shows that two compact Lie groups are isomorphic if and only if their classifying spaces are homotopy equivalent. This is not true for Kac-Moody groups.

In this section we will classify the rank two Kac-Moody groups with classifying space of the same homotopy type. First, we recall the classification of these groups up to continuous isomorphism. To simplify the notation, we denote by $K = K(a, b)$, $K' = K(a', b')$ rank two Kac-Moody groups with Weyl groups W and W' respectively.

Proposition 6.1. *$K \cong K'$ if and only if $\{a, b\} = \{a', b'\}$.*

Proof. Let $\phi : K \to K'$ be a continuous isomorphism. We know ([16]) that a maximal torus in a Kac-Moody group is a maximal connected abelian subgroup and two such subgroups are conjugated. Hence, we can assume that ϕ sends T_K to $T_{K'}$. Then, ϕ is represented by a matrix $M \in GL_2(\mathbb{Z})$ which is admissible in an obvious sense. Then, an argument similar to the one in 5.1 (but easier) shows that this matrix M can exist only if $\{a,b\} = \{a',b'\}$. $\square$

Next we obtain a characterization of rank two Kac-Moody groups K, K' with $BK \simeq BK'$. (Recall that we always assume $ab > 4$.)

Theorem 6.2. *$BK \simeq BK'$ if and only if the following conditions hold:*

a) $ab = a'b'$ and $gcd(a,b) = gcd(a',b')$.
b) One can order a', b' in such a way that aa' is a square (in $\mathbb{Z}$) and ab' is a square in $\widehat{\mathbb{Z}}_p$ for all primes p such that $\nu_p(a) \neq \nu_p(a')$.

Proof. Assume we have a homotopy equivalence $f : BK \to BK'$. Then there is also a homotopy equivalence $K \simeq K'$. The integral cohomology of these spaces has been computed in [18] and it turns out that $H^4(K(a,b);\mathbb{Z}) \cong \mathbb{Z}/\mathrm{gcd}(a,b)\mathbb{Z}$ and $H^6(K(a,b);\mathbb{Z}) \cong \mathbb{Z}/(ab-1)\mathbb{Z}$. This shows that condition a) is necessary.

For each prime p the homotopy equivalence f produces, by proposition 5.2, a p-admissible matrix M_p such that the composition $BT_{p^\infty} \to BK \to BK'^{\wedge}_p$ is induced by M_p. By 5.1, $M_p = \omega' A$ with $\omega' \in \widehat{W}'_p$ and A is of one of the two types that appear in 5.1.

Let q, q' be the quadratic forms associated to K and K', respectively. They are invariant by the corresponding Weyl groups and it follows easily that they are also invariant by the p-completed Weyl groups $\widehat{W}_p$ and $\widehat{W}'_p$. Hence, if λ, μ are the non-zero entries of A, we have $f^*(q') = \lambda\mu q$ and since f is a homotopy equivalence, we obtain $\lambda\mu = \pm 1$. Now there are two possibilities depending on the actual type of A for the prime p. In the first case, $\pm a'/a$ is a square in $\widehat{\mathbb{Z}}^*_p$ and in the second case $\pm a/b'$ is a square in $\widehat{\mathbb{Z}}^*_p$.

The existence of a homotopy equivalence $BK \simeq BK'$ has led us to the fact that for each prime p at least one of the rational numbers $\pm a'/a$ or $\pm a/b'$ is a square in $\widehat{\mathbb{Z}}^*_p$. Now, it is a well known arithmetic fact that given non-squares $x, y \in \mathbb{Q}$ there are infinitely many primes p such that x, y are non-squares in $\widehat{\mathbb{Z}}_p$. This shows that the minus sign cannot happen, i.e. f must have degree $+1$ and it implies also that either a/a' or a/b' are squares in $\mathbb{Q}$. Let us order $\{a', b'\}$ in such a way that a/a' is a square in $\mathbb{Q}$. Then aa' is also a square. Moreover, if $\nu_p(a) \neq \nu_p(a')$ then a/a' is not a unit in $\widehat{\mathbb{Z}}_p$ and so a/b' has to be a square in $\widehat{\mathbb{Z}}_p$ and so does ab' as well. This shows that condition b) is necessary.

To prove the converse we have to construct a homotopy equivalence $f : BK \simeq BK'$ assuming conditions a) and b). We do it by constructing maps $f_p : BK \to BK'^{\wedge}_p$ one prime at a time and then applying 3.3 to glue together these maps and get a global map.

First we notice that a) and b) imply that for each prime p either a/a' or a/b' are squares in $\widehat{\mathbb{Z}}^*_p$. Let us call a prime p *plain* if a/a' is a square in $\widehat{\mathbb{Z}}^*_p$ and *twisted* if it is not (and so a/b' is a square in $\widehat{\mathbb{Z}}_p^*$). Recall that we have push out diagrams

$$BK_p^\wedge \simeq \Big(\operatorname{hocolim}\big\{(BH_1)_p^\wedge \longleftarrow (BT_K)_p^\wedge \longrightarrow (BH_2)_p^\wedge\big\}\Big)_p^\wedge$$

$$BK'^\wedge_p \simeq \Big(\operatorname{hocolim}\big\{(BH'_1)_p^\wedge \longleftarrow (BT_{K'})_p^\wedge \longrightarrow (BH'_2)_p^\wedge\big\}\Big)_p^\wedge$$

where H_i, H'_i are compact Lie groups of rank two which depend on the parity of a, b, a', b'. We will define f_p on each node of these diagrams. Roughly speaking, at the plain primes, we map each node of the push out diagram for K to the corresponding node of the push out diagram of K' while at the twisted primes we permute the "exterior" nodes of the push out before mapping them.

We introduce the following maps. For any prime p and any p-adic unit λ define

$$\Phi^\lambda : BU(2)_p^\wedge \to BU(2)_p^\wedge$$

$$\Phi^\lambda : (BS^3 \times BS^1)_p^\wedge \to (BS^3 \times BS^1)_p^\wedge$$

as the maps extending the self maps of the maximal torus given by the p-adic matrices

$$\begin{pmatrix} \frac{\lambda+\lambda^{-1}}{2} & \frac{\lambda^{-1}-\lambda}{2} \\ \frac{\lambda^{-1}-\lambda}{2} & \frac{\lambda+\lambda^{-1}}{2} \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

respectively. Existence and uniqueness (up to homotopy) of these maps follow from corollary 3.5 in [12]. For odd primes p we consider also the map $\varphi : (BS^3 \times BS^1)_p^\wedge \to BU(2)_p^\wedge$ which is given on the maximal torus by the matrix $\begin{pmatrix} 1 & 1/2 \\ -1 & 1/2 \end{pmatrix}$. φ is a homotopy equivalence and its existence follows easily from the fact that the Weyl groups of the two Lie groups involved have order two.

Choose now a *plain* prime p and let $a/a' = \lambda^2$, λ a p-adic unit. We will define a map $f_p : BK_p^\wedge \to BK'^\wedge_p$. We distinguish several cases according to the parities of a and b.

Case 1: a and b even.

In this case a' and b' are even too and the Lie groups H_i, H'_i, $i = 1, 2$ are all isomorphic to $S^3 \times S^1$. A direct computation shows that the following diagram commutes up to homotopy:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(BS^3 \times BS^1)_p^\wedge \arrow[d, "\Phi^{\lambda^{-1}}"'] & (BT_K)_p^\wedge \arrow[l, "\begin{pmatrix} -\frac{a}{2} & 1 \\ 1 & 0 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}"'] & (BS^3 \times BS^1)_p^\wedge \arrow[d, "\Phi^\lambda"'] \\
(BS^3 \times BS^1)_p^\wedge & (BT_K)_p^\wedge \arrow[l, "\begin{pmatrix} -\frac{a'}{2} & 1 \\ 1 & 0 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 1 & -\frac{b'}{2} \\ 0 & 1 \end{pmatrix}"'] & (BS^3 \times BS^1)_p^\wedge
\end{tikzcd}$$

and we get a map $f_p : BK \to BK'^\wedge_p$.

Case 2: a and b odd.

In this case a' and b' are also odd and the Lie groups H_i, H'_i, $i=1,2$ are all isomorphic to $U(2)$. We get also a map $f_p : BK \to BK'^{\wedge}_p$ from the commutative diagram (up to homotopy)

$$\begin{tikzcd}
BU(2)^{\wedge}_p \arrow[d, "\Phi^{\lambda^{-1}}"'] & (BT_K)^{\wedge}_p \arrow[l, "\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1 & \frac{1-b}{2} \\ -1 & \frac{1+b}{2} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}"'] & BU(2)^{\wedge}_p \arrow[d, "\Phi^{\lambda}"'] \\
BU(2)^{\wedge}_p & (BT_K)^{\wedge}_p \arrow[l, "\begin{pmatrix} \frac{1-a'}{2} & 1 \\ \frac{1+a}{2} & -1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 1 & \frac{1-b'}{2} \\ -1 & \frac{1+b'}{2} \end{pmatrix}"'] & BU(2)^{\wedge}_p
\end{tikzcd}$$

Case 3: a odd and b even.

In this case a' and b' have also different parity and we need to distinguish between two cases. If a' is odd and b' is even then the map $f_p : BK \to BK'^{\wedge}_p$ comes from the commutative diagram (up to homotopy)

$$\begin{tikzcd}
BU(2)^{\wedge}_p \arrow[d, "\Phi^{\lambda^{-1}}"'] & (BT_K)^{\wedge}_p \arrow[l, "\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}"'] & (BS^3 \times BS^1)^{\wedge}_p \arrow[d, "\Phi^{\lambda}"'] \\
BU(2)^{\wedge}_p & (BT_K)^{\wedge}_p \arrow[l, "\begin{pmatrix} \frac{1-a'}{2} & 1 \\ \frac{1+a'}{2} & -1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 1 & -\frac{b'}{2} \\ 0 & 1 \end{pmatrix}"'] & (BS^3 \times BS^1)^{\wedge}_p
\end{tikzcd}$$

If a' is even and b' is odd then p is odd and the map $f_p : BK \to BK'^{\wedge}_p$ comes from the commutative diagram (up to homotopy)

$$\begin{tikzcd}
BU(2)^{\wedge}_p \arrow[d, "\varphi^{-1}\Phi^{\lambda^{-1}}"'] & (BT_K)^{\wedge}_p \arrow[l, "\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}"'] & (BS^3 \times BS^1)^{\wedge}_p \arrow[d, "\Phi^{\lambda}\varphi"'] \\
(BS^3 \times BS^1)^{\wedge}_p & (BT_K)^{\wedge}_p \arrow[l, "\begin{pmatrix} -\frac{a'}{2} & 1 \\ 1 & 0 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 1 & \frac{1-b'}{2} \\ -1 & \frac{1+b'}{2} \end{pmatrix}"'] & BU(2)^{\wedge}_p
\end{tikzcd}$$

Now we have $f_p : BK \to BK'^{\wedge}_p$ for any plain prime p and we need to deal now with the *twisted* primes. If p is twisted then p is plain for $BK(b', a')$ and we construct the map $f'_p : BK \to BK(b', a')^\wedge_p$ as above; then we obtain f_p composing with the equivalence $BK(b', a') \cong BK'$.

All these maps $f_p : BK \to BK'^\wedge_p$ are rationally compatible and they produce a map $f : BK \to BK'$. To see that f is a homotopy equivalence notice that the even mod p cohomology of BK injects in the mod p cohomology of BT_K and the effect of each f_p on $H^*(BT_K; \mathbb{F}_p)$ shows that each f_p is a mod p equivalence. Hence, $f : BK \simeq BK'$. □

In spite of this result, which shows that the classifying space functor is not faithful among Kac-Moody groups, we believe that the results in this paper show that the homotopy type of BK is, nevertheless, rich enough to reflect a good deal of the properties of the group K.

7. Adams maps

As it happens in the case of compact connected Lie groups, the Adams maps will play a fundamental role in the study of self maps of classifying spaces of Kac-Moody groups.

If L is a Kac-Moody group and λ is an integer, we define an *Adams map* ψ^λ as any map $\psi^\lambda : BL \to BL$ such that ψ^λ extends (up to homotopy) the map $B\rho : BT_L \to BT_L$ where $\rho : T_L \to T_L$ is the homomorphism $t \mapsto t^\lambda$. Notice that we are not claiming (yet) homotopy uniqueness of Adams maps: any map extending the λ-power self-map from BT_L will be called an Adams map ψ^λ. When L is a compact connected Lie group this definition is equivalent to the classical definition of unstable Adams operations. The Adams maps form a monoid under composition, isomorphic to the multiplicative monoid of the integers, since $\psi^\lambda\psi^\mu = \psi^{\lambda\mu}$.

Proposition 7.1. *There is an Adams map $\psi^\lambda : BK \to BK$ if and only if λ is an odd integer or $\lambda = 0$.*

Proof. Recall that K has an outer automorphism Ψ^{-1} extending the automorphism $t \mapsto t^{-1}$ of T_K (see section 2). Hence, we only need to consider Adams maps ψ^λ with $\lambda \geq 0$. For ψ^0 we take the constant map. Now we use the existence of Adams maps in compact connected Lie groups G (see [23]). There is a map $\psi^\lambda : BG \to BG$ if (and only if, see [9]) λ is prime to the order of the Weyl group of G. This implies that we have Adams maps $\psi^\lambda : BH_i \to BH_i$, $i = 1, 2$, for any odd integer λ, where H_i are the proper parabolic subgroups of K. Then the push out diagram for BK gives an Adams map $\psi^\lambda : BK \to BK$.

Now we have to prove that ψ^λ does not exist if λ is even. Essentially, the same proof as in the Lie group case ([9]) works here. Let ω be one of the generating reflections of the Weyl group and let $\tilde{\omega}$ be a representative of ω in the normalizer of T_K. It is known ([22]) that we can choose $\tilde{\omega}$ such that $\tilde{\omega}^4 = 1$. Assume that there is $\psi^\lambda : BK \to BK$ with λ even. Since any element of finite order of K can be conjugated to T_K we see that the map ψ^{λ^2} is nullhomotopic when restricted to $B\langle\tilde{\omega}\rangle$. Choose $n \geq 1$ and let N be the subgroup of the normalizer of T_K generated by T_{2^n} (the elements of T_K of order dividing 2^n) and $\tilde{\omega}$. N is a finite 2-group and so (theorem 4.1) there is a homomorphism $\rho : N \to K$ such that $\psi^{\lambda^2}|_{BN} \simeq B\rho$. Now ρ is trivial on $\tilde{\omega}$ and ρ is $t \mapsto t^{\lambda^2}$ on T_K. Hence, we obtain that ω acts trivially on t^{λ^2} for any $t \in T_{2^n}$. Taking n big enough we get a contradiction. $\square$

A special feature of the non-Lie Kac-Moody groups is the existence of another family of self maps different from the Adams maps. For simplicity, we define these maps only in the case of the rank two groups K. Given (non zero) integers λ, μ we define a *twisted Adams map* $\psi^{\lambda,\mu}$ as any map $\psi^{\lambda,\mu} : BK \to BK$ such that it extends up to homotopy the self map of BT_K given by the matrix $\begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$.

Proposition 7.2. *There is a twisted Adams map $\psi^{\lambda,\mu} : BK \to BK$ if and only if λ and μ are odd integers such that $a\lambda = b\mu$.*

Proof. The proof is similar to the proof of 6.2 and uses the idea of mapping the push out diagram for BK to itself in a twisted way.

By hypothesis, a and b have the same parity. If they are both even then we consider the homotopy commutative diagram

$$\begin{tikzcd}
BS^3 \times BS^1 \arrow[d, "\psi^\lambda \times \psi^\mu"'] & BT_K \arrow[l, "\begin{pmatrix} -\frac{a}{2} & 1 \\ 1 & 0 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}"'] & BS^3 \times BS^1 \arrow[d, "\psi^\mu \times \psi^\lambda"'] \\
BS^3 \times BS^1 & BT_K \arrow[l, "\begin{pmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} -\frac{a}{2} & 1 \\ 1 & 0 \end{pmatrix}"'] & BS^3 \times BS^1
\end{tikzcd}$$

which produces the map $\psi^{\lambda,\mu} : BK \to BK$. If a and b are odd then we consider the diagram

$$\begin{tikzcd}
BU(2) \arrow[d, "\gamma(\frac{\lambda+\mu}{2}, \frac{\mu-\lambda}{2})"'] & BT_K \arrow[l, "\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1 & \frac{1-b}{2} \\ -1 & \frac{1+b}{2} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}"'] & BU(2) \arrow[d, "\gamma(\frac{\lambda+\mu}{2}, \frac{\lambda-\mu}{2})"'] \\
BU(2) & BT_K \arrow[l, "\begin{pmatrix} 1 & \frac{1-b}{2} \\ -1 & \frac{1+b'}{2} \end{pmatrix}"] \arrow[r, "\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1 \end{pmatrix}"'] & BU(2)
\end{tikzcd}$$

Here $\gamma(x, y) : BU(2) \to BU(2)$ denotes a map which extends the map $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ on the maximal torus. This map exists if $x \not\equiv y\,(2)$. This follows easily from the results in [12].

We have to prove now that the conditions on λ and μ are necessary for the existence of a map $\psi^{\lambda,\mu}$. The condition $a\lambda = b\mu$ follows from the study of admissible matrices done in 6.2. Since the composition $\psi^{\lambda,\mu}\psi^{\lambda,\mu}$ is an Adams map $\psi^{\lambda\mu}$ it follows from 7.1 that λ and μ must be both odd. □

If we complete at a prime p then we can generalize the Adams maps above (both ordinary and twisted) to p-adic degrees. If $\lambda, \alpha, \beta \in \widehat{\mathbb{Z}}_p$ with $a\alpha = b\beta$ then there are maps $\psi^{\lambda}, \psi^{\alpha,\beta} : BK_p^{\wedge} \to BK_p^{\wedge}$ for any odd p. If $p = 2$ then ψ^{λ}, $\psi^{\alpha,\beta}$ exist if and only if $\lambda, \alpha, \beta \not\equiv 0\,(2)$. The proofs of these facts are completely analogous to the proofs of 7.1 and 7.2. Notice that proposition 3.2 implies that this concept of a p-adic Adams map is compatible with the integral concept introduced before.

For each prime p choose a map $f_p : BK_p^{\wedge} \to BK_p^{\wedge}$ which is an Adams map (ordinary or twisted). If these maps f_p are rationally compatible then they lift to a global map $f : BK \to BK$. The maps constructed in this way will be called *generic Adams maps*. As seen in 3.3, the rational compatibility reduces to the existence of an *integral degree* g such that $f^*(q) = gq$ where $q \in H^4(BK;\mathbb{Z})$ is the quadratic form introduced in section 2. More precisely, a generic Adams map f is described by its *type* $\{(\epsilon_p, \lambda_p)\} \in \prod_p \left(\{0,1\} \times \widehat{\mathbb{Z}}_p\right)$. $\epsilon_p = 0$ means that f_p is an ordinary Adams map ψ^{λ_p} while $\epsilon_p = 1$ means that f_p is a twisted Adams map ψ^{λ_p,μ_p} with $\mu_p = a\lambda_p/b \in \widehat{\mathbb{Z}}_p$.

We want to determine now the set of degrees of generic Adams maps. The following result gives a complete description of these degrees in an effective way, in the sense that given an integer g one can decide in a finite number of steps if g is the degree of some generic Adams map or not. Put $d = \gcd(a, b)$, $a' = a/d$, $b' = b/d$.

Proposition 7.3. *Let $g \neq 0$ be an integer.*

a) *If g is a square then g is the degree of some generic Adams map if and only if g is odd;*

b) *If g is not a square then g is the degree of some generic Adams map if and only if g is odd and there exist integers A, B, r such that*
   1) *$g = A^2 r$, $a'b' = B^2 r$;*
   2) *If B is even then $r \equiv 1\,(8)$;*
   3) *If p is odd and $\nu_p B > \nu_p A$ then r is a square in $\widehat{\mathbb{Z}}_p$.*

Proof. Clearly, g is the degree of some generic Adams map if and only if, for each prime p, g can be written either as $g = \lambda_p^2$ for some $\lambda_p \in \widehat{\mathbb{Z}}_p$ ($\lambda_2 \equiv 1\,(2)$) or $g = \lambda_p\mu_p$ for some $\lambda_p, \mu_p \in \widehat{\mathbb{Z}}_p$ with $a'\lambda_p = b'\mu_p$ ($\lambda_2, \mu_2 \equiv 1\,(2)$). In particular, g is odd and part a is clear.

Assume that g is not a square but we have $g = A^2 r$, $a'b' = B^2 r$ with A, B, r satisfying the conditions of the proposition. We have to prove that g is the degree of some generic Adams map. Choose a prime p. If g is a square in $\widehat{\mathbb{Z}}_p$, we can realize g at the prime p by a p-adic Adams map. If g is not a square in $\widehat{\mathbb{Z}}_p$ then we realize g at the prime p by a twisted Adams map ψ^{λ_p,μ_p} with $\lambda_p = b'A/B$ and $\mu_p = a'A/B$. One checks immediately that $g = \lambda_p\mu_p$ and $a'\lambda_p = b'\mu_p$ but we also need to check that $A/B \in \widehat{\mathbb{Z}}_p$ and that $\lambda_2, \mu_2 \equiv 1\,(2)$. Notice that r is not a square in $\widehat{\mathbb{Z}}_p$ (otherwise

g would be a square in $\widehat{\mathbb{Z}}_p$ and it is not). Then, if p is odd condition 3 implies that $A/B \in \widehat{\mathbb{Z}}_p$. If $p = 2$ and B were even then by condition 2 we would have $r \equiv 1\,(8)$ and r would be a square in $\widehat{\mathbb{Z}}_2$, which is not. Since g is odd so are A and r. Hence, a' and b' are also odd and $\lambda_2, \mu_2 \equiv 1\,(2)$.

Next, we will prove that the conditions on g are necessary. We need a little lemma which is an easy exercise on quadratic residues. We are grateful to Enric Nart for his advise on these topics.

Lemma 7.4. *Let $x, y \in \mathbb{Z}$, x an odd non-square, $y > 0$. The following two conditions are equivalent:*

a) $\left(\frac{y}{p}\right) = -1$ *for all p prime to y such that* $\left(\frac{x}{p}\right) = -1$.
b) *There is $r \in \mathbb{Z}$ such that $x = A^2 r$ and $y = B^2 r$.*

We apply this lemma to $x = g$ and $y = a'b'$. If g is a non-square mod p for some p prime to $a'b'$ then g is a non-square in $\widehat{\mathbb{Z}}_p$ and so $g = \lambda\mu$ with $a\lambda = b\mu$. Hence, $a'b'\lambda^2 = b'^2 g$ and $a'b'$ is a non-square in $\widehat{\mathbb{Z}}_p$. Hence, condition a in the lemma is satisfied and this implies that we can write $g = A^2 r$, $a'b' = B^2 r$ for some integers A, B, r. If B is even then $a'b'$ is even too and so g has to be a square in $\widehat{\mathbb{Z}}_2$. Hence, $r \equiv 1\,(8)$. If r is a non-square in $\widehat{\mathbb{Z}}_p$ then g is a non-square in $\widehat{\mathbb{Z}}_p$ and so $g = \lambda\mu$ with $a'\lambda = b'\mu$. This implies that $g/a'b' = (\mu/a')^2 \in \widehat{\mathbb{Z}}_p$ and so $\nu_p(B) \leq \nu_p(A)$. The theorem is proved. $\square$

It is an easy matter to use an algebraic computer language to write a short program which, given the values of a, b and g, decides if $BK(a, b)$ admits a generic Adams map of degree g.

Proof of 7.4: Recall the following well known fact from the theory of quadratic residues: given different primes $p_1, \dots, p_r$, $q_1, \dots, q_s$ and given $\epsilon \in \{\pm 1\}$, there are infinitely many primes ℓ such that $\left(\frac{p_i}{\ell}\right) = 1$, $1 \leq i \leq r$, $\left(\frac{q_i}{\ell}\right) = -1$, $1 \leq i \leq s$ and $\left(\frac{-1}{\ell}\right) = \epsilon$.

The implication b$\Rightarrow$a is obvious. Write $x = \pm p_1^{a_1} \cdots p_m^{a_m}$ with p_i odd and a_1 odd. Write $y = 2^k p_1^{b_1} \cdots p_m^{b_m} \cdots p_n^{b_n}$ with $b_i \geq 0$. If $x < 0$ then choose a prime ℓ such that $\left(\frac{-1}{\ell}\right) = -1$ while $\left(\frac{p_i}{\ell}\right) = \left(\frac{2}{\ell}\right) = 1$. Then we would have $\left(\frac{x}{\ell}\right) = -1$ but $\left(\frac{y}{\ell}\right) = 1$ in contradiction to condition a. Hence, $x > 0$. If some a_j is odd, then choose a prime ℓ such that $\left(\frac{2}{\ell}\right) = \left(\frac{p_i}{\ell}\right) = 1$ for $i \neq j$ while $\left(\frac{p_j}{\ell}\right) = -1$. Then, condition a implies that b_j is odd too. If some a_j is even, choose a prime ℓ such that $\left(\frac{2}{\ell}\right) = \left(\frac{p_i}{\ell}\right) = 1$ for $i \neq 1, j$ while $\left(\frac{p_1}{\ell}\right) = \left(\frac{p_j}{\ell}\right) = -1$. Then, condition a implies that b_j is even too. We deduce that x and y can be written as $x = A^2 r$, $y = 2^k C^2 r$ where r is an odd non-square. Let ℓ be such that $\left(\frac{r}{\ell}\right) = \left(\frac{2}{\ell}\right) = -1$. Then $-1 = \left(\frac{y}{\ell}\right) = \left(\frac{r}{\ell}\right)\left(\frac{2}{\ell}\right)^k$ and so k is even and $y = B^2 r$. $\square$

For each integer g and each prime p let us consider the set

$$S_{g,p} = \left\{ M \in \mathfrak{gl}_2(\widehat{\mathbb{Z}}_p) \,\middle|\, M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},\ \lambda^2 = g \ \text{or}\ M = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},\ \alpha\beta = g,\ a\alpha = b\beta \right\}.$$

Let us consider the equivalence relation in $S_{g,p}$ given by

$$M \sim M' \Leftrightarrow M = \omega M', \quad \omega \in \widehat{W}_p.$$

The following lemma describes the cosets under this relation. Let us recall that in section 2 we introduced integers $l = l(K,p)$ and $y = y(K,p)$ which are related to the structure of the group $\widehat{W}_p$.

Lemma 7.5. *Let $M, M' \in S_{g,p}$. Then*

1. *If l is even then $M \sim M'$ if and only if $M = \pm M'$.*
2. *If $p = 2$ then $M \sim M'$ if and only if $M = M'$.*
3. *If l is odd and $p \neq 2$ then $M \sim M'$ if and only if $M = M'$ or $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $M' = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ and $\lambda = \beta y$.*

Proof. Recall that $\widehat{W}_p^+ \cong \widehat{\mathbb{Z}}_p \times \mathbb{Z}/l$ and so $\widehat{W}_p^+$ has an element of order two if and only if l is even. This matrix of order two has to be $-I$. Hence, $M \sim -M$ if and only if l is even.

On the other hand, if p and l are odd then by lemma 2.1 we can find a matrix $\omega = \begin{pmatrix} 0 & y \\ 1/y & 0 \end{pmatrix}$ in $\widehat{W}_p$ with $\omega \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$. Conversely, if $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \sim \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ then there is a matrix $\omega \in \widehat{W}_p$ with the property that $(\omega\omega_2)^2 = \omega_1\omega_2$ and this implies that l and p are odd. □

This lemma can be used to decide when two generic Adams maps are homotopic on BT_K.

Proposition 7.6. *Let $f, f' : BK \to BK$ be generic Adams maps of types $\{(\epsilon_p, \lambda_p)\}$, $\{(\epsilon'_p, \lambda'_p)\}$ and degrees g, g', respectively. $f|_{BT_K} \simeq f'|_{BT_K}$ if and only if the following holds:*

1. *$g = g'$;*
2. *$\epsilon_p = \epsilon'_p$ for $p = 2$ and for any p such that $l(K,p)$ is even;*
3. *$\lambda_p = y(K,p)^{\epsilon_p - \epsilon'_p}\lambda'_p$ for any p such that $l(K,p)$ is odd.*

Proof. It is clear that $g = g'$ is a necessary condition for the existence of a homotopy $f|_{BT_K} \simeq f'|_{BT_K}$.

Now f and f' are homotopic on BT_K if and only if they are homotopic on BT_{p^∞} for all primes p (by 3.2). By proposition 4.6 this holds if and only if at each prime p the matrices used to define f and f' are related by the equivalence relation considered above. Then 7.5 yields the result stated. □

We would like to put the results of this proposition in a more transparent way. For instance, given a prime p and an integer g, one would like to have a set of representatives of p-adic Adams maps $\psi : BK_p^\wedge \to BK_p^\wedge$ (up to homotopy on BT_{p^∞}). The following table displays this information:

g a square in $\widehat{\mathbb{Z}}_p$?	ag/b a square in $\widehat{\mathbb{Z}}_p$?	p	$l(K,p)$	representatives of p-adic Adams maps of degree g
no	no	any	any	none
yes: $g=\lambda^2$	no	2	odd	ψ^{λ}, $\psi^{-\lambda}$
		odd	even	ψ^{λ}
no	yes: $ag=b\beta^2$, $\alpha\beta=g$	2	odd	$\psi^{\alpha,\beta}$, $\psi^{-\alpha,-\beta}$
		odd	even	$\psi^{\alpha,\beta}$
yes: $g=\lambda^2$	yes: $ag=b\beta^2$, $\alpha\beta=g$	2	odd	ψ^{λ}, $\psi^{-\lambda}$, $\psi^{\alpha,\beta}$, $\psi^{-\alpha,-\beta}$
		odd	even	ψ^{λ}, $\psi^{\alpha,\beta}$
			odd	ψ^{λ}, $\psi^{-\lambda}$

Remark 7.7. In the case of compact connected simple Lie groups, rational cohomology classifies self maps. More precisely, one of the main results of [11] is that for $f, f' : BG \to BG$, G as above, f is homotopic to f' if and only if $H^*(f;\mathbb{Q}) = H^*(f';\mathbb{Q})$. One could wonder if there is a result of this type for Kac-Moody groups. In particular, one would like to replace a cohomological classification by the characterization in proposition 7.6, which is precise but aesthetically not so pleasant as the corresponding result for Lie groups.

First of all, it is clear that rational cohomology is not enough, since two self maps of BK induce the same homomorphism in rational cohomology if and only if they have the same degree. Unfortunately, integral cohomology does not suffice either. Let us discuss this in more detail. For any odd prime p we have (see section 2) $H^*(BK;\mathbb{F}_p) \cong \mathbb{F}_p[x_4, y_{2k}] \otimes E[z_{2k+1}]$ with $\beta_r(y_{2k}) = z_{2k+1}$. Here x_4 is the mod p reduction of the integral class q. Hence, $H^{\mathrm{even}}(BK;\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[q]$ while $H^{\mathrm{odd}}(BK;\mathbb{Z}_{(p)})$ is torsion. Then, the even integral cohomology does not carry any more information on self maps than the degree, but some additional information could be read on $H^{2k+1}(BK;\mathbb{Z}_{(p)}) \cong \mathbb{Z}/p^r$. The action of a generic Adams map $f : BK \to BK$ on $H^{2k+1}(BK;\mathbb{Z}_{(p)})$ can be determined from the transgression τ in the fibration $K/T \to BT \to BK$

$$\tau : H^{2k}(K/T;\mathbb{Z}_{(p)}) \twoheadrightarrow H^{2k+1}(BK;\mathbb{Z}_{(p)})$$

and the knowledge on $H^*(K/T)$ given by its Schubert calculus (see [18]).

Given a, b, choose an odd prime ℓ such that $l(K,\ell)$ is odd. Then, let $f = \psi^{\ell^n}$ and let f' be the generic Adams map given by $f'_p = \psi^{\ell^n}$ for $p \neq \ell$ and $f_\ell = \psi^{-\ell^n}$. By proposition 7.6, f and f' are not homotopic but if we choose n big enough then f and f' both induce the same endomorphism on $H^*(BK;\mathbb{Z}_{(p)})$ for all p.

We finish our discussion of Adams maps with the important result which claims that *all self maps are Adams maps*:

Theorem 7.8. *Let $f : BK \to BK$ be a map. Then f is a generic Adams map.*

Proof. This is an immediate consequence of 4.7, 5.2 and 5.1. □

8. Homotopically trivial self maps

In this section we want to prove the following result:

Theorem 8.1. *Let $f : BK \to BK$ be a map. The following are equivalent:*
1. *f is nullhomotopic.*
2. *$f|_{BT_K}$ is nullhomotopic.*
3. *$deg(f) = 0$*

We prove first a couple of lemmas. Recall that a p-toral group is a group which is an extension of a finite p-group by torus.

Lemma 8.2. *Let P be a p-toral group and let L be a Kac-Moody group. Then the map $BL^\wedge_p \to \mathrm{Map}(BP, BL^\wedge_p)_0$ (whose image is the space of constant maps) is a homotopy equivalence.*

Proof. We approximate BP (up to p-completion) by classifying spaces of finite p-groups. Now, according to 4.1, for a p-group π we have $\mathrm{Map}(B\pi, BL^\wedge_p) \simeq \coprod BC_L(\rho)^\wedge_p$ where the disjoint union is over all representations $\rho : \pi \to L$. If we consider nullhomotopic maps, then each of these centralizers is equal to L and the result follows. □

Lemma 8.3. *Let $f : BK \to BK^\wedge_p$ be a map such that $f|_{BT_K}$ is nullhomotopic. Let P be a subgroup of K which is an extension of a finite p-group by T_K. Then $f|_{BP}$ is nullhomotopic.*

Proof. For each n, let P_n be the subgroup of P which is an extension of P/T_K by T_{p^n}. Since P_n is a finite p-group, by 4.1 there is a homomorphism $\rho_n : P_n \to K$ such that $f|_{BP_n} \simeq B\rho_n$. For each $x \in K$ consider the map

$$B(xT_Kx^{-1} \cap P_n) \to BP_n \to BK \xrightarrow{f} BK^\wedge_p.$$

Since $f|_{BT_K}$ is nullhomotopic and since conjugation by x induces a self map of BK which is homotopic to the identity, we deduce that the above composition is nullhomotopic. Hence, ρ_n is trivial on $xT_Kx^{-1} \cap P_n$. But P_n is covered by the conjugates xT_Kx^{-1} (each element of finite order in a Kac-Moody group is in a maximal torus and two maximal torus are conjugated, see [16]) and so $\rho_n = 1$ for all n and $f|_{BP_n}$ is nullhomotopic for all n. Since $BP^\wedge_p \simeq (\mathrm{hocolim}\, BP_n)^\wedge_p$, to conclude that $f|_{BP}$ is nullhomotopic it is enough to check the vanishing of some obstructions which live in $\varprojlim^1 \pi_1 \mathrm{Map}(BP_n, BK^\wedge_p)_0$. Each of these mapping spaces is homotopically equivalent to $BK^\wedge_p$, which is simply connected. Hence, the obstructions vanish and the result is proven. □

Lemma 8.4. *Let $\mathcal{M}$ be a $\mathbb{Z}_{(p)}[W]$-module with p odd. Then $H^j(W;\mathcal{M}) = 0$ for $j \geq 2$. If $\mathcal{M}$ has trivial W-action then $H^j(W;\mathcal{M}) = 0$ for $j \geq 1$.*

Proof. The Weyl group W is infinite dihedral and so it is an extension of an infinite cyclic group by a group of order two with non-trivial action. Since $\mathcal{M}$ is a $\mathbb{Z}_{(p)}$-module with p odd, the Serre spectral sequence of this group extension collapses to an isomorphism $H^*(W;\mathcal{M}) \simeq H^*(\mathbb{Z};\mathcal{M})^{\mathbb{Z}/2}$ and the lemma follows easily from this. □

Proof of theorem 8.1: The implications $1 \Rightarrow 2 \Rightarrow 3$ are obvious. Assume f has degree zero. By 7.8, f should be an Adams map ψ^0. Hence, $f|_{BT_K}$ is nullhomotopic.

It remains to prove the implication $2 \Rightarrow 1$. Let us denote by $\mathrm{Map}(BK, BK_p^\wedge)_{\bar{0}}$ the space of maps which are nullhomotopic on BT_K. We want to prove that this space is connected. Let us first consider the case of $p > 2$. In this case $BK_p^\wedge \simeq BN_p^\wedge$, where N is the normalizer of T_K in K and we have

$$\mathrm{Map}(BK, BK_p^\wedge)_{\bar{0}} \simeq \mathrm{Map}(BT_K, BK_p^\wedge)_0^{hW} \simeq (BK_p^\wedge)^{hW} \simeq \mathrm{Map}(BW, BK_p^\wedge).$$

By lemma 8.4, the space BW is p-acyclic and $\mathrm{Map}(BK, BK_p^\wedge)_{\bar{0}} \simeq BK_p^\wedge$ is connected.

When $p = 2$ we proceed in the following way. Let H denote any of the standard parabolic subgroups of K. H is a compact Lie group isomorphic to either $S^1 \times S^3$ or $U(2)$. There is an important bibliography on colimit decompositions of classifying spaces of compact Lie groups. In our case, it is known (see, for instance, [10]) that H is 2-equivalent to the homotopy colimit of a small diagram of the form

$$\Sigma_3 \circlearrowright BP_1 \overset{\Sigma_2\backslash\Sigma_3}{\begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix}} BP_2$$

where P_1 and P_2 are 2-toral groups. The derived functors $\lim^i$ over this category are well understood (see [1]) and are know to vanish in the category of $\mathbb{Z}_{(2)}$-modules except for $\lim^0$ and $\lim^1$.

We prove now that $f|BH \simeq *$. Lemma 8.3 implies that $f|BP \simeq *$ for any P appearing in the diagram for BH. Hence, we just need to show that some obstructions vanish. These obstructions live in $\lim^1 \pi_1 \mathrm{Map}(BP_i, BK_2^\wedge)_0$ but this is trivial because of 8.2.

We observe now that BK is the push out of $BH_1 \leftarrow BT_K \rightarrow BH_2$. Hence, to conclude that f is nullhomotopic we check the vanishing of the obstruction set in $\lim^1 \pi_1 \mathrm{Map}(BH_i, BK_2^\wedge)_0$. This higher limit is a quotient of $\pi_1 \mathrm{Map}(BT_K, BK_2^\wedge)_0 = \pi_1 BK_2^\wedge$ (by lemma 8.2) and so it vanishes because BK is 3-connected.

Now the triviality of $f : BK \rightarrow BK_p^\wedge$ for all primes p implies the triviality of $f : BK \rightarrow BK$ by 3.3. □

9. Detecting maps on the maximal torus

In this section we will prove the final result that we need in order to have a complete picture of the self maps of BK.

Theorem 9.1. *Let $f, f' : BK \rightarrow BK$. The following are equivalent:*

1. *f and f' are homotopic.*
2. *$f|_{BT_K}$ and $f'|_{BT_K}$ are homotopic.*

Proof. Of course, only the implication $2 \Rightarrow 1$ needs a proof. This theorem has already been proved in the case in which $f|_{BT_K}$ is nullhomotopic (see 8.1). Hence, we can assume that $f|_{BT_K}$ is not nullhomotopic. By 3.3, we have to prove that f and f' are homotopic after completion at each prime p. We treat the odd primes and the prime two separately.

The case of an odd prime: In this (easier) case, we have that $BK_p^\wedge \simeq BN_p^\wedge \simeq [(BT_K)_{hW}]_p^\wedge$ where N is the normalizer of T_K in K. Since f and f' coincide up to homotopy on BT_K, to prove that they coincide up to homotopy on $BK_p^\wedge$ we check that the obstructions to uniqueness for maps defined on a homotopy colimit vanish. In our case, these obstruction live in $H^j(W;\pi_j \operatorname{Map}(BT_K, BK_p^\wedge)_{f|_{BT_K}})$ for $j \geq 1$.

By 4.5, $f|_{BT_{p^\infty}}$ is induced by a homomorphism $\rho : T_{p^\infty} \to K$ with finite kernel. Hence, we can use 4.3 to conclude that $\operatorname{Map}(BT_K, BK_p^\wedge)_{f|_{BT_K}} \simeq (BT_K)_p^\wedge$. Using now lemma 8.4 we see that all obstructions vanish and f and f' are homotopic on $BK_p^\wedge$.

The case of the even prime: We know that f and f' are generic Adams maps of odd degree (see 7.8, 7.3). Then, by considering the action of these maps on mod 2 cohomology we see that they are mod 2 homotopy equivalences. This reduces our problem to the case in which we have $f : BK \to BK_2^\wedge$ with $f|_{BT_K}$ homotopic to the map induced by the inclusion and we need to prove that f is homotopic to the identity. To simplify the notation we call "identity" (id) the map induced by any natural inclusion which is clear from the context.

BK is a push out

$$BK \simeq \operatorname{hocolim}(BH_1 \leftarrow BT_K \to BH_2).$$

Assume we have proved that both $f|_{BH_1}$ and $f|_{BH_2}$ are homotopic to the identity. Then the obstruction to the existence of a homotopy between f and the identity is in $\varprojlim^1 \pi_1 \operatorname{Map}(B*, BK_2^\wedge)_{id}$ where $*$ ranges on $\{T_K, H_1, H_2\}$. This is a quotient of $\pi_1 \operatorname{Map}(BT_K, BK_2^\wedge)_{id}$ which is trivial by 4.3 and the theorem is proved.

Let us denote by H any of the maximal (proper) parabolic subgroups of K. We have just seen that it is enough to prove that $f|_{BH} \simeq \mathrm{id}$. H is a rank two compact Lie group with Weyl group W_H of order two and maximal torus T_K. The group H is isomorphic to $U(2)$ or to $S^1 \times S^3$, depending on the parity of a and b. Let us denote by N_H the normalizer in H of T_K. N_H is a 2-toral group.

Consider the map $h : BH_2^\wedge \to BK_2^\wedge$ induced by the inclusion of H on K and let

$$h^\sharp : \operatorname{Map}(BT_K, BH_2^\wedge)_{id} \to \operatorname{Map}(BT_K, BK_2^\wedge)_{id}$$

be the map induced between mapping spaces. By 4.3 and [8], this map is a homotopy equivalence. W_H acts on BT_K and W_H is a subgroup of the Weyl group of K. Hence, W_H acts on the source and target of $h^\sharp$ and $h^\sharp$ is equivariant with respect to this action. Hence, $h^\sharp$ induces a homotopy equivalence

$$(h^\sharp)^{hW_H} : \left[\operatorname{Map}(BT_K, BH_2^\wedge)_{id}\right]^{hW_H} \xrightarrow{\simeq} \left[\operatorname{Map}(BT_K, BK_2^\wedge)_{id}\right]^{hW_H}$$

and we conclude that the inclusion of H on K induces a homotopy equivalence

$$\operatorname{Map}(BN_H, BH_2^\wedge)_{\overline{id}} \simeq \operatorname{Map}(BN_H, BK_2^\wedge)_{\overline{id}}$$

where the subscript $\overline{id}$ means that we consider all components of maps which are homotopic to the identity on BT_K. Now, if $g : BN_H \to BH_2^\wedge$ is a map such that $g|_{BT_K} \simeq \mathrm{id}$ then by 3.1 in [11] we can lift g to a global map $\hat{g} : BN_H \to BH$ by taking the identity map at all odd primes. Then, by the results in [19], $\hat{g}$ comes from a representation $\rho : N_H \to H$ with $\rho|_{T_K} = \mathrm{id}$. Then, if $H \cong U(2)$ it is easy to check that $\rho = \mathrm{id}$ and so $\hat{g} \simeq \mathrm{id}$. In the case $H \cong S^1 \times S^3$ one has to be a bit more careful since besides the identity, there is another representation $\rho : N_H \to H$ which is the identity on T_K. To rule out this representation we use the following argument. Consider $\Gamma_1 = \{\pm 1\} \times Q_8$ as a subgroup of N_H and consider $B\rho : B\Gamma_1 \to BK^\wedge_2$. If $f|_{B\Gamma_1} \simeq B\rho$ then this map $B\rho$ should extend to BH. There is a conjugation in H which gives an automorphism σ of Q_8 of order 3. This implies that $B\rho(id \times \sigma) \simeq B\rho$ in $BK^\wedge_2$. Then, since Γ_1 is a finite 2-group and since ρ does not factor though T_K, we see that $B\rho(id \times \sigma) \simeq B\rho$ in BH. Hence, the representations $\rho(id \times \sigma)$ and ρ are conjugated in H and this is not true. Hence, we have proved that $f|_{BN_H} \simeq \mathrm{id}$.

One checks immediately that the centralizer of N_H in H is the center of H. Hence,

$$\mathrm{Map}(BN_H, BK^\wedge_2)_{id} \simeq BZH^\wedge_2 \simeq \begin{cases} K(\widehat{\mathbb{Z}}_2, 2), & H \cong U(2) \\ K(\widehat{\mathbb{Z}}_2, 2) \times K(\mathbb{Z}/2, 1), & H \cong S^1 \times S^3. \end{cases}$$

Let us recall now the homotopy colimit decomposition of BH (up to 2-completion) that we have used in the proof of 8.1. Let $\mathcal{C}$ be the small category

$$\mathcal{C} : \Sigma_3 \circlearrowright \mathbf{1} \overset{\Sigma_2\backslash\Sigma_3}{\underset{}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}} \mathbf{0}.$$

Then there is a diagram $F : \mathcal{C} \to \mathcal{T}$ with $F(\mathbf{0}) \simeq BN_H$, $F(\mathbf{1}) \simeq B\Gamma$ and such that $BH^\wedge_2 \simeq (\mathrm{hocolim}_{\mathcal{C}}\, F)^\wedge_2$. Here Γ is a certain 2-toral subgroup of N_H. If $H \cong U(2)$ then Γ is the subgroup of the matrices

$$\begin{pmatrix} \pm\alpha & 0 \\ 0 & \pm\alpha \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm\alpha \\ \pm\alpha & 0 \end{pmatrix}, \quad |\alpha| = 1.$$

It is an extension of an elementary abelian 2-group of rank two by S^1. In the case in which $H \cong S^1 \times S^3$, Γ is the product of S^1 by the quaternion subgroup Q_8 of S^3. In any case, Γ is 2-toral and the intersection $\Gamma \cap T_K$ is isomorphic to $S^1 \times \mathbb{Z}/2$. If we replace the toral part of Γ by 2^n-roots of unity, $n \geq 1$ we obtain subgroups Γ_n of Γ which are finite 2-groups with the property that $\mathrm{hocolim}_n\{B\Gamma_n\}$ approximates $B\Gamma$ up to 2-completion. One sees easily that the centralizer of Γ_n in H coincides with the center of H. Hence, by the main theorem in [8] we have

$$\mathrm{Map}(B\Gamma_n, BH^\wedge_2)_{id} \simeq \begin{cases} K(\widehat{\mathbb{Z}}_2, 2), & H \cong U(2) \\ K(\widehat{\mathbb{Z}}_2, 2) \times K(\mathbb{Z}/2, 1), & H \cong S^1 \times S^3. \end{cases}$$

Then, to compute the homotopy type of the mapping space $\mathrm{Map}(B\Gamma, BH^\wedge_2)_{id}$ we have to take the homotopy limit of a tower which is indeed constant. Hence, we obtain a homotopy equivalence $\mathrm{Map}(B\Gamma, BH^\wedge_2)_{id} \simeq \mathrm{Map}(B\Gamma_n, BH^\wedge_2)_{id}$.

We investigate now the mapping space $\mathrm{Map}(B\Gamma, BK_2^\wedge)_{id}$. By 4.1 we have a push out diagram up to 2-completion

$$\begin{tikzcd}
\mathrm{Map}(B\Gamma_n, BT_K{}_2^\wedge) \arrow[r] \arrow[d] & \mathrm{Map}(B\Gamma_n, BH_2^\wedge) \arrow[d] \\
\mathrm{Map}(B\Gamma_n, (BH')_2^\wedge) \arrow[r] & \mathrm{Map}(B\Gamma_n, BK_2^\wedge)
\end{tikzcd}$$

where H' denotes the maximal parabolic subgroup of K other than H. Notice that the representation $\Gamma_n \hookrightarrow H$ does not factor through any representation $\Gamma_n \to T_K$. Hence, the component $\mathrm{Map}(B\Gamma_n, BH_2^\wedge)_{id}$ cannot be amalgamated to any other component and so the above push out implies that

$$\mathrm{Map}(B\Gamma_n, BK_2^\wedge)_{id} \simeq \mathrm{Map}(B\Gamma_n, BH_2^\wedge)_{id}.$$

Also,

$$\mathrm{Map}(B\Gamma, BK_2^\wedge)_{id} \simeq \mathrm{Map}(B\Gamma, BH_2^\wedge)_{id}.$$

We are ready now to finish the proof of theorem 9.1. $f|_{BH}, \mathrm{id} : BH \to BK_2^\wedge$ are two extensions of $\mathrm{id} : BN_H \to BK_2^\wedge$ to BH which is 2-equivalent to a homotopy colimit. The obstructions to uniqueness of the extension are in $\varprojlim^i{}_{\mathcal{C}} \pi_i \mathrm{Map}(B*, BK_2^\wedge)_{id}$ for $i > 0$ where $*$ ranges over $\{B\Gamma, BN_H\}$. The proof is finished if we see that these obstruction sets are trivial.

To see this we need to recall the computation of higher limits over $\mathcal{C}$. A suitable reference for this is [1]. For a general functor F from $\mathcal{C}$ to the category of $\mathbb{Z}_{(2)}$-modules we have that $\varprojlim^i{}_{\mathcal{C}} F = 0$ for $i > 1$ while there is an exact sequence

$$0 \to \varprojlim_{\mathcal{C}}{}^0 F \to F(\mathbf{0}) \to F(\mathbf{1})^{\mathbb{Z}/2}/F(\mathbf{1})^{\Sigma_3} \to \varprojlim_{\mathcal{C}}{}^1 F \to 0.$$

In our case F is the functor $\pi_i \mathrm{Map}(B*, BK_2^\wedge)_{id}$ and the computations done above show that F is valued in the category of $\mathbb{Z}_{(2)}$-modules. Hence, the only obstruction is in $\varprojlim^1{}_{\mathcal{C}} \pi_1 \mathrm{Map}(B*, BK_2^\wedge)_{id}$ which is a quotient of

$$(\pi_1 \mathrm{Map}(B\Gamma, BK_2^\wedge)_{id})^{\Sigma_2}/(\pi_1 \mathrm{Map}(B\Gamma, BK_2^\wedge)_{id})^{\Sigma_3}$$

and this vanishes. Now the proof of 9.1 is complete. □

10. $[BK, BK]$

In this short final section of this paper we just notice that the results in the preceding sections give a complete description of the monoid $[BK, BK]$ of homotopy classes of self maps of BK.

There is a canonical map

$$\deg : [BK, BK] \to \mathrm{Hom}(H^4(BK; \mathbb{Z}), H^4(BK; \mathbb{Z})) \cong \mathbb{Z}$$

from the set $[BK, BK]$ to the group endomorphisms of $H^4(BK; \mathbb{Z})$. This map is a monoid homomorphism and sends each self map of BK to its degree.

Then, 7.3 gives a complete description of the image of deg, since it characterizes the integers that can appear as the degree of a generic Adams map and 7.8 shows that any self map of BK can be represented by a generic Adams map. Also, 7.6 describes when two generic Adams map are homotopic on BT_K. By 8.1 and 9.1, two maps are homotopic if and only if they are homotopic on BT_K. Hence, the table following 7.6 gives a complete description of the fibers of deg, together with their monoid structure. In this sense, we can say that we have a complete description of the structure of $[BK, BK]$.

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