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ON COHOMOLOGY ALGEBRAS

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When I met Guido Mislin at the ETH in Zurich for the first time — in the late seventies of the past century — I was fascinated by the many subtle and mysterious conditions that a graded commutative $\mathbf{F}_p$-algebra has to satisfy in order to be the mod p cohomology algebra of some space. Back in those days, the Adams–Wilkerson embedding ([1]) was hot news, the Sullivan conjecture was not yet a theorem and new ideas were appearing quickly. Since then, we have learned a lot about the big questions of yore, like polynomial algebras which are cohomology algebras, the injective objects in the category of unstable modules over the Steenrod algebra, the type of finite loop spaces, the mod p homotopy uniqueness of classifying spaces of finite loop spaces and so on.

It is not unusual that concentration on the major problems of a certain subject leaves unsolved many questions which just fail to be in the mainstream of mathematical progress. I wish to mention here two problems in realizability of algebras as cohomology rings that were considered around 1980 and, to the best of my knowledge, were forgotten and have remained unanswered. I do not think they have ever appeared in print but I clearly remember how Alex Zabrodsky and myself spent many long afternoons discussing these matters.

If you wonder if, say, $K := \mathbf{F}_p[x]$ ($\deg(x) = 2n$, p odd) can be the cohomology of some space, then you notice that $x^p \neq 0$ implies that the mod p Steenrod algebra acts non trivially on K. The existence of this non trivial action puts severe restrictions on the admissible values of n and opens the door to the use of secondary operations and the many tools invented by topologists since Hopf asked the question of realizability. All this is well known. But, if you start with an algebra *where all p-th powers vanish*, there seems to be no way to start any such argument.

Does this mean that all commutative graded $\mathbf{F}_p$-algebras with trivial p-th powers are realizable?

There is also a related question about realizability of graded $\mathbf{Z}$-algebras. Here the idea is that the integral cohomology of a generic space has to have quite a lot of divisibility (cf. [2]) and if we artificially inject enough divisibility in an algebra then it eventually becomes realizable. We know that the realizability problem is trivial over the rational field but it may be that p-th powers just need to be divisible by p, and so on. More precisely, let A be a (commutative graded) $\mathbf{Z}$-algebra and define a new product $x \cdot y$ in A by the formula

$$x \cdot y := \binom{|x| + |y|}{|x|} xy \,.$$

Is then the resulting new algebra always realizable as the integral *cohomology of some space ?*

This question may be related to Bill Dwyer's *tame homotopy theory* ([3]), another beautiful topic that I learned during my days at the ETH.

REFERENCES

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