HOMOTOPY CLASSIFICATION OF SPACES WITH INTERESTING COHOMOLOGY AND A CONJECTURE OF COOKE, PART I

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1. Introduction. The title of this paper is reminiscent of the title of one of the last papers by George Cooke ([**7**]). In that paper, Cooke observes that if ΩX is a p-complete loop space then there is an action of $[S^1\hat{}_p, S^1\hat{}_p] \cong \hat{\mathbb{Z}}_p$ on ΩX. In particular, there is an action of the $p-1$ roots of unity on ΩX and by taking the quotients of appropriate loop spaces by this action he obtains spaces with "interesting" cohomology, i.e. spaces whose cohomology algebras have quite few generators and relations and whose attaching maps represent interesting elements in the stable homotopy of spheres. By applying this technique to $S^3\langle 3\rangle$, the 3-connective covering of S^3, and to the fibre of the map $S^3 \to K(\mathbb{Z}, 3)$ of degree p, Cooke constructs spaces realizing the cohomology algebras (subscripts denote degrees)

$$\mathbb{F}_p[x_{2n}] \otimes E(\beta x_{2n}), \tag{1}$$

where n is any divisor of $p(p-1)$, β is the Bockstein homomorphism and $E(y_k)$ denotes an exterior algebra on one generator y_k of degree k. This method was generalized in [**5**] to construct spaces whose mod p cohomology has the form $P \otimes E$ where P is a polynomial algebra and E is an exterior algebra on the Bocksteins of the generators of P. Cooke ends his paper by saying "I expect that the condition $n|p(p-1)$ is necessary as well" and this is the Cooke conjecture that we mention in the title of this paper. The conjecture was proved to hold true in some particular cases in [**2**] where cohomology algebras of the form $\mathbb{F}_p[x] \otimes E(y)$ where studied by completely different methods to those used in the present paper.

Our purpose is to develop a study of spaces whose mod p cohomology has the form (1). In this part I of our work we consider the case of p odd and we prove the Cooke conjecture in full generality but we go further than that for we obtain a classification up to p-completion of all homotopy types with mod p cohomology of the form (1) above. When $p = 2$ both the results and the techniques involved in the proofs are significantly different and deserve a separate discusion which we plan to work out in part II of this work ([**4**]). In particular, the $p = 2$ version of the Cooke conjecture (which was certainly stated with only the case of p odd in mind) turns out to be wrong and additional fascinating families of spaces with "interesting cohomology" appear.

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We present in this introduction a rough overview of the main results of the paper. We start with a classification of unstable algebras over the Steenrod algebra of the form (1). We see that there are exactly two families of such algebras which we call $\mathbf{B}_{i,r}$ and $\mathbf{A}_r$ for $i \geq 0$ and r dividing $p-1$. As graded algebras, we have (subscripts denote degrees)

$$\mathbf{B}_{i,r} \cong \mathbb{F}_p[x_{2p^ir}] \otimes E(y_{2p^ir+1})$$

and the Steenrod algebra action is determined by $\beta(x) = y$ and $P^{p^i}(y) = (r-1)x^s y$. $\mathbf{A}_r$ is isomorphic to $\mathbf{B}_{0,r}$ as graded algebras but the action of the Steenrod algebra is different in $\mathbf{B}_{0,r}$ and $\mathbf{A}_r$ for in $\mathbf{A}_r$ we have the relation $P^1(y) = rx^s y$. The algebras realized by Cooke in [**7**] are $\mathbf{B}_{0,r}$ and $\mathbf{B}_{1,r}$ while the algebras $\mathbf{A}_r$ seem to have remained unnoticed although their study will be fundamental in our classification of spaces realizing $\mathbf{B}_{i,r}$. In this context, the conjecture of Cooke is stated as follows:

Theorem A. *If $H^*(X;\mathbb{F}_p) \cong \mathbf{B}_{i,r}$ as algebras over the Steenrod algebra, then $i \leq 1$.*

It is a natural question to ask about the realizability of the algebras $\mathbf{A}_r$. It turns out that all algebras $\mathbf{A}_r$ are realizable as mod p cohomology of some appropriate spaces. More in general, for any $k \geq 0$ we introduce the notation $\mathbf{A}_r^{(k)}$ to denote a cohomology algebra which looks like $\mathbf{A}_r$ except for the fact that the relation $\beta(x) = y$ is replaced by the relation $\beta_{(k+1)}(x) = y$ where $\beta_{(k+1)}$ denotes the Bockstein homomorphism of order $k+1$. In particular, $\mathbf{A}_1^{(0)} = \mathbf{A}_1$. Of course, $\beta_{(k+1)}$ is not a Steenrod operation for $k > 0$ and so $\mathbf{A}_r^{(k)}$ is the same as $\mathbf{A}_r^{(k')}$ as algebras over the Steenrod algebra for any $k, k' > 0$. Nevertheless, it makes sense to say that the mod p cohomology of some space X is isomorphic to $\mathbf{A}_r^{(k)}$.

Theorem B. *For $k \geq 0$ and $r|(p-1)$ there is a p-complete space $X_k(r)$ such that $H^*(X_k(r);\mathbb{F}_p) \cong \mathbf{A}_r^{(k)}$.*

The spaces in theorem B are constructed by first taking the quotient of $(BS^1)^\wedge_p$ by some appropriate action of the p-adic integers and then killing the one dimensional skeleton.

Having established which algebras $\mathbf{B}_{i,r}$, $\mathbf{A}_r$ are realizable, we consider the problem of classifying up to p-completion all homotopy types which realize these algebras. In the case of the algebras $\mathbf{A}_r$ we obtain that the spaces of part (1) of theorem B form a complete list of p-complete homotopy types realizing the algebras $\mathbf{A}_r^{(k)}$.

Theorem C. *Let $H^*(X;\mathbb{F}_p) \cong \mathbf{A}_r^{(k)}$. Then $\hat{X}_p \simeq X_k(r)$.*

It is interesting to note that in proving this theorem we face the problem of computing the mod p cohomology of some component of $\mathrm{map}(B\mathbb{Z}/p, X)$ in a case in which the appropriate T functor does not vanish in degree 1.

Finally, we consider the problem of classifying up to p-completion all homotopy types realizing $\mathbf{B}_{i,r}$. Because of theorem A, we only need to deal with the cases of $\mathbf{B}_{0,r}$ and $\mathbf{B}_{1,r}$. We obtain the amazing result that for each of these algebras there are infinitely many different p-complete spaces realizing it.

Theorem D. *Let $r|(p-1)$. There are spaces $Y_{k,r}$ for $0 \leq k \leq \infty$ and $Z_{k,r}$ for $0 < k \leq \infty$ such that*

$$H^*(Y_{k,r}; \mathbb{F}_p) \cong \mathbf{B}_{1,r}$$
$$H^*(Z_{k,r}; \mathbb{F}_p) \cong \mathbf{B}_{0,r}$$

All these spaces are p-complete and have different homotopy type.

Here $Y_{\infty,r}$ and $Z_{\infty,r}$ are the p-completions of the spaces constructed by Cooke ([**7**]). In particular, $Y_{\infty,1} = S^3\langle 3\rangle\hat{}_p$ and theorem D shows that there is an infinite family of "fake" $S^3\langle 3\rangle$, i.e. spaces with the same mod p cohomology as $S^3\langle 3\rangle$ but not homotopy equivalent to $S^3\langle 3\rangle$ even after p-completion. Among these spaces the true $S^3\langle 3\rangle$ is distinguished by being the only one which can carry an H-space structure.

Our next result shows that the spaces of theorem D form a complete list of p-complete homotopy types realizing the algebras $\mathbf{B}_{0,r}$ and $\mathbf{B}_{1,r}$.

Theorem E. (1) *If $H^*(X; \mathbb{F}_p) \cong \mathbf{B}_{1,r}$ then there exists $0 \leq k \leq \infty$ such that $\hat{X}_p \simeq Y_{k,r}$.*
(2) *If $H^*(X; \mathbb{F}_p) \cong \mathbf{B}_{0,r}$ then there exists $0 < k \leq \infty$ such that $\hat{X}_p \simeq Z_{k,r}$.*

Finally we study suspensions of all the spaces constructed. It turns out that even after an l-fold suspension all the "fake" spaces are not homotopy equivalent to the genuine ones; i.e. the spaces which were constructed out of S^3.

Theorem F. *If $k \neq \infty$, for all $r|(p-1)$ and for all $0 \leq l < \infty$, the l-fold suspensions $\Sigma^l Y_{k,r}$ and $\Sigma^l Y_{\infty,r}$ are not homotopy equivalent.*

The analogous statement is true for the spaces $Z_{k,r}$.

The method used to prove most of the theorems stated above is based on the study of the mapping spaces $\mathrm{map}(B\mathbb{Z}/p, X)$, where X is a space whose cohomology is assumed to be of the form $\mathbb{F}_p[x] \otimes E(y)$. Here the techniques developed by Lannes ([**17**]) play a fundamental role.

In order to show in a simplified way the main ideas in the proofs of theorems A to E above, we present now a rough description of the homotopy classification of spaces with the same cohomology as $S^3\langle 3\rangle$. This will also illustrate where the fake $S^3\langle 3\rangle$ come from. Imagine we have a p-complete space X with the same mod p cohomology as $S^3\langle 3\rangle$. Take $Y = \mathrm{map}(B\mathbb{Z}/p, X)_f$ to be an appropriate component of the space of maps from $B\mathbb{Z}/p$ to X and compute, using the T functor, the mod p cohomology of Y. It turns out that Y is homotopy equivalent to X but the gain from X to Y is that Y exhibits a greater symmetry than X for Y belongs to a principal fibration

$$B\mathbb{Z}/p \to Y \to Y(1).$$

The cohomology of $Y(1)$ has the form $\mathbb{F}_p[x_2] \otimes E(\beta(x_2))$ hence either $H^*(Y(1); \mathbb{F}_p) \cong \mathbf{B}_{0,1}$ or $H^*(Y(1); \mathbb{F}_p) \cong \mathbf{A}_1$ and it turns out that both cases are possible. Hence, we already have two possibilities for X: the true $S^3\langle 3\rangle$ obtained by taking $Y(1)$ to be Cooke's realization of $\mathbf{B}_{0,1}$ and a fake one obtained by taking $Y(1)$ equal to the space $X_0(1)$ of theorem B. If

$H^*(Y(1);\mathbb{F}_p) \cong \mathbf{B}_{0,1}$ then we can apply the same technique again and obtain a principal fibration

$$B\mathbb{Z}/p \to Y(1) \to Y(2)$$

with again two possibilities for $Y(2)$. At the end we obtain either an infinite sequence

$$X \to Y \to Y(1) \to Y(2) \to \ldots \to Y(j) \to \ldots$$

with all spaces having mod p cohomology of type $\mathbf{B}_{i,r}^{(k)}$ or a finite sequence

$$X \to Y \to Y(1) \to Y(2) \to \ldots \to Y(j),$$

where the last space has mod p cohomology of type $\mathbf{A}_1^{(k)}$, stopping the inductive process because $\mathrm{map}(B\mathbb{Z}/p, Y(j))_f$ will not be homotopy equivalent to $Y(j)$. The first case forces $X \simeq S^3\langle 3\rangle\hat{_p}$ and in the second one we obtain an infinite family of fake $S^3\langle 3\rangle$. Moreover, the uniqueness of realizations of the algebras $\mathbf{A}_r^{(k)}$ yields the homotopy uniqueness of each of the fake spaces.

The paper is organized as follows. Section 2 deals with the algebraic problem of classifying unstable algebras over the Steenrod algebra of the form $\mathbb{F}_p[x] \otimes E(y)$ with $\beta(x) = y$. There we introduce the algebras $\mathbf{B}_{i,r}$ and $\mathbf{A}_r$. In section 3 we compute the T functor applied to these algebras, a computation that will be crucial for the rest of the paper. In section 4 we prove the Cooke conjecture, i.e. theorem A (cf. theorem 4.3). Section 5 is devoted to the construction of spaces whose cohomology is of the form $\mathbf{A}_r^{(k)}$. Here we prove theorem B (cf. theorem 5.5). In section 6 we obtain the homotopy classification of the spaces of section 5, proving theorem C (cf. theorem 6.1). Section 7 deals with the construction of spaces realizing $\mathbf{B}_{i,r}$ and in particular we obtain the family of fake $S^3\langle 3\rangle$ and we prove theorem D (cf. propositions 7.1 and 7.7 and corollary 7.6). In section 8 we show that there are no more p-complete spaces realizing the algebras $\mathbf{B}_{i,r}$ beside those constructed in section 7, by proving theorem E (cf. theorem 8.2). In section 9 we study suspensions of all the constructed spaces and prove theorem F (c.f. corollary 9.13 and corollary 9.15) using the localization functor of [**9**]. A final section 10 contains some tables which may help the reader through the rather intricated notation we use to denote the spaces we are dealing with and their cohomology algebras.

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Warning. *Throughout this paper p denotes an odd prime.*

2. Some unstable algebras over the Steenrod algebra. Through this section we say that A is a PE-algebra if A is a commutative graded $\mathbb{F}_p$-algebra which is the tensor product of a polynomial algebra on a generator of degree $2n$ and an exterior algebra on a generator of degree m. We say that A has type $(2n, m)$. We will usually call x one polynomial generator and y one exterior generator.

Our first example of an unstable algebra over the Steenrod algebra is a PE-algebra A of type $(2,3)$. We define an unstable action of the mod p Steenrod algebra over A by the Cartan formula and the identities

$$P^1 x = x^p \quad ; \quad P^i x = 0,\, i > 1 \quad ; \quad \beta x = y,$$

$$P^1 y = x^{p-1} y \quad ; \quad P^i y = 0,\, i > 1 \quad ; \quad \beta y = 0.$$

These formulas certainly define an unstable action of $\bar{\mathcal{A}}$ over A, where $\bar{\mathcal{A}}$ is the free associative algebra generated by P^i, $i > 0$, and β. In order to see that this action factors through the Steenrod algebra $\mathcal{A}$ we have to check that the Adem relations hold in A. This could be done directly using the techniques in [**24**] but it follows also from the following alternative description of A as a module over the Steenrod algebra.

Let H be the mod p cohomology of $B\mathbb{Z}/p$ and let $P \subset H$ be the even dimensional subalgebra. P is a polynomial algebra on one generator v in degree 2. Let us denote by P^+ the submodule of P formed by the elements of positive degree. Consider the diagram in $\mathcal{U}$ (the category of unstable modules over the Steenrod algebra):

$$\Sigma H \xrightarrow{\pi} \frac{\Sigma H}{\Sigma P^+} \xleftarrow{\phi} P$$

where π is the natural projection and ϕ is the homomorphism given by

$$\phi(v^n) = n\sigma(uv^{n-1})$$

where u is a one dimensional generator in H such that $\beta u = v$ and σ denotes suspension. One can easily check that ϕ is an $\mathcal{A}$-homomorphism. Actually, ϕ is the composition

$$P \xrightarrow{\Delta} P \otimes P \xrightarrow{k \otimes 1} \Sigma^2 \mathbb{F}_p \otimes P \cong \Sigma^2 P \xrightarrow{j} \frac{\Sigma H}{\Sigma P^+},$$

where Δ is the diagonal, k is the projection and j is an inclusion sending $\sigma^2 v^n$ to $\sigma(uv^n)$. Then if $\bar{A}$ is the pull back of the above diagram, $\bar{A}$ is an unstable module over the Steenrod algebra and a straightforward computation shows that $\bar{A} \cong A$ as $\bar{\mathcal{A}}$-modules. This shows that the Adem relations hold true in A since they hold true in $\bar{A}$.

If λ is a unit in $\mathbb{F}_p$ then the map $x \mapsto \lambda x$ induces an algebra automorphism of A which commutes with the Steenrod algebra action. Hence, for any r dividing $p-1$ we have an action of the cyclic group of order r on A and the algebra of invariants of this action is also an unstable algebra over the Steenrod algebra. We call this algebra $\mathbf{A}_r$. It is a PE-algebra of type $(2r, 2r+1)$ and the action of the Steenrod algebra is determined by:

$$\begin{aligned} P^1 X &= rX^{s+1}, \\ \beta X &= Y, \\ P^1 Y &= rX^s Y, \end{aligned}$$

where $s = (p-1)/r$.

Let $i \geq 0$. Our second example of an unstable algebra over the Steenrod algebra is an algebra B which is a PE-algebra of type $(2p^i, 2p^i+1)$. We define an unstable action of the Steenrod algebra over B by the Cartan formula and the identities

$$\begin{aligned} P^{p^i}x = x^p \quad &, \quad P^k x = 0,\ k \neq 0, p^i, \\ P^k y = 0 \quad &\text{for any } k > 0, \\ \beta x = y \quad &, \quad \beta y = 0. \end{aligned}$$

As before, a direct calculation as in [**24**] would check that the Adem relations hold, but we will instead use an alternative description of B as an unstable module over the Steenrod algebra. Let H be as before and let $P(i)$ be the subalgebra of H generated by v^{p^i}. Let $J(2)$ be the reduced mod p cohomology of $S^1 \cup_p e^2$. Consider the diagram in $\mathcal{U}$

$$\Sigma^{2p^i-1} J(2) \otimes P(i) \xrightarrow{\pi} \Sigma^{2p^i} P(i) \xleftarrow{\phi} P(i)$$

where the map π is the natural projection and ϕ is given by the composition

$$P(i) \xrightarrow{\Delta} P(i) \otimes P(i) \longrightarrow \Sigma^{2p^i} \mathbb{F}_p \otimes P(i) \cong \Sigma^{2p^i} P(i),$$

or, equivalently, by the formula

$$\phi(v^{p^i}) = n\sigma^{2p^i}(v^{(n-1)p^i}).$$

Then if $\bar{B}$ is the pull back of this diagram, $\bar{B}$ is an unstable module over the Steenrod algebra such that $\bar{B} \cong B$ as $\bar{\mathcal{A}}$-modules. As before, this shows that B is an unstable algebra over the Steenrod algebra.

If λ is a unit in $\mathbb{F}_p$ then the map $x \mapsto \lambda x$ induces an algebra automorphism of B which commutes with the Steenrod algebra action. This produces, in the same way as before, algebras $\mathbf{B}_{i,r}$ for any $i \geq 0$ and any r dividing $p-1$, with generators X and Y in degrees $2p^i r$ and $2p^i r + 1$, respectively, such that

$$P^{p^i} X = rX^{s+1}, \quad P^{p^j} X = 0,\ j \neq i,$$

$$\beta X = Y,$$

$$P^{p^i} Y = (r-1)X^s Y, \quad P^{p^j} Y = 0,\ j \neq i.$$

Notice that $\mathbf{A}_r$ and $\mathbf{B}_{0,r}$ are isomorphic as graded algebras, both being the tensor product of a polynomial algebra on one generator x in degree 2 and an exterior algebra on one generator y in degree 3. Moreover, the relation $\beta x = y$ holds in both algebras. However, $\mathbf{A}_r$ and $\mathbf{B}_{0,r}$ are not isomorphic as algebras over the Steenrod algebra, as one can easily check.

By construction, we see that if t divides r then $\mathbf{A}_r$ is a subalgebra of $\mathbf{A}_t$ and $\mathbf{B}_{i,r}$ is a subalgebra of $\mathbf{B}_{i,t}$. There are no further inclusions between these unstable algebras over the Steenrod algebra.

The next theorem proves that there are no more examples of PE-algebras which are unstable algebras over the Steenrod algebra and such that $\beta x = y$.

Theorem 2.1. *Let A be a PE-algebra which is an unstable algebra over the Steenrod algebra and such that $\beta x = y$. Then A is isomorphic, as an algebra over the Steenrod algebra, to one of the algebras $\mathbf{A}_r$, $\mathbf{B}_{i,r}$ constructed above.*

Proof. Notice that the ideal generated by y in A is closed under the action of the Steenrod algebra. Hence $A/\langle y\rangle \cong \mathbb{F}_p[x]$ should be an unstable algebra over the Steenrod algebra. It is well known (cf. [**25**]) that this implies that if A is of type $(2n, 2n+1)$ then $n = p^i r$ for some $i \geq 0$ and some r dividing $p-1$. Put $s = (p-1)/r$. Then we can choose the generator x such that

$$P^{p^i} x = r x^{s+1}.$$

This well known fact admits a tedious elementary proof using the Adem relations and is also a trivial consequence of the Adams-Wilkerson embedding theorem ([**1**]). If $i > 0$ we can use the Adem relation

$$P^1 \beta P^{p^i - 1} = -\beta P^{p^i} + P^{p^i} \beta$$

to deduce

$$P^{p^i} y = (r-1) x^s y.$$

By dimensional reasons and unstability, $P^{p^j} x = 0 = P^{p^j} y$ for any $j \neq i$. Hence A is isomorphic to $\mathbf{B}_{i,r}$.

In the case $i = 0$ if we write $P^1 y = \lambda x^s y$ then the Adem relation

$$2P^1 \beta P^1 = \beta P^1 P^1 + P^1 P^1 \beta$$

gives the following degree 2 equation for λ:

$$\lambda^2 + (1-2r)\lambda + r(r-1) = 0$$

whose roots are $\lambda = r$, $r-1$. In the first case A is isomorphic to $\mathbf{A}_r$ and in the second one it is isomorphic to $\mathbf{B}_{i,r}$. $\square$

Some of these PE-algebras appear as the mod p cohomology of some spaces. Let $S^3\langle 3\rangle$ denote the 3-connective covering of S^3, i. e. the fibre of the degree one map $S^3 \to K(\mathbb{Z}, 3)$. Then one can easily deduce from the spectral sequence of the fibration

$$K(\mathbb{Z}, 2) \to S^3\langle 3\rangle \to S^3$$

and theorem 2.1 that

$$H^*(S^3\langle 3\rangle; \mathbb{F}_p) \cong \mathbf{B}_{1,1}$$

as algebras over the Steenrod algebra. Moreover, since $S^3\langle 3\rangle$ is a loop space, the p-completion of $S^3\langle 3\rangle$ carries an action of the cyclic group of order $p-1$ and by taking the homotopy quotient of $S^3\langle 3\rangle^{\wedge}_p$ by the restriction of this action to the cyclic group of order r, for any r dividing $p-1$, we obtain a space X_r such that $H^*(X_r; \mathbb{F}_p) \cong \mathbf{B}_{1,r}$. (See [**7**] for further details on this construction.)

If Y is the fibre of the map $S^3 \to K(\mathbb{Z},3)$ of degree p then

$$H^*(Y;\mathbb{F}_p) \cong \mathbf{B}_{0,1}$$

as algebras over the Steenrod algebra. Again, the p-completion of Y carries an action of the cyclic group of order $p-1$ and this produces p-complete spaces Y_r for any r dividing $p-1$ such that $H^*(Y_r;\mathbb{F}_p) \cong \mathbf{B}_{0,r}$ as algebras over the Steenrod algebra. (See also [**7**] for details.)

Finally, in section 5 we will prove that all PE-algebras $\mathbf{A}_r$ are realizable.

Even if we are only interested in the PE-algebras with $\beta(x) = y$ it will be necessary to consider PE-algebras where the Bockstein homomorphism acts trivially. They will indeed play an important role in the forthcoming sections. We denote by $\mathbf{A}'_1$ the PE-algebra of type $(2,3)$ with an unstable action of the Steenrod algebra given by $\beta = 0$ and $P^1(y) = x^{p-1}y$. Let $\mathbf{A}'_r$ for r dividing $p-1$ be the algebra of invariants of $\mathbf{A}'_1$ by the action of the cyclic group of order r which sends x to λx and y to λy for λ an r-th root of unity. $\mathbf{A}'_r$ is a PE-algebra of type $(2r, 2r+1)$ with an unstable action of the Steenrod algebra. Let $\mathbf{B}'_{i,1}$ be the same graded algebra $\mathbf{B}_{i,1}$ but with the Steenrod algebra action given by $\beta = 0$ and $P^t(y) = 0$ for $t \geq 0$. Let $\mathbf{B}'_{i,r}$ for r dividing $p-1$ be the algebra of invariants of $\mathbf{B}'_{i,1}$ under the action of the cyclic group of order r which sends x to λx and y to λy for λ an r-th root of unity.

We denote by $\mathbf{C}_1$ a PE-algebra of type $(2,1)$ with an unstable action of the Steenrod algebra given by $\beta(x) = xz$ where x denotes a 2 dimensional generator and z denotes a one dimensional generator. Notice that $\mathbf{A}_1$ is isomorphic to a subalgebra of $\mathbf{C}_1$. The cyclic group of order r for r dividing $p-1$ acts on $\mathbf{C}_1$ leaving z and x^r invariants. We denote by $\mathbf{C}_r$ the algebra of invariants, which is a PE-algebra of type $(2r,1)$ with an unstable action of the Steenrod algebra. We also need a Bockstein-free version of these algebras which we denote by $\mathbf{C}'_r$.

If any of the above algebras with trivial Bockstein appears as the mod p cohomology of some space X it makes sense to ask about the order of the higher Bockstein which connects the polynomial and the exterior part and we indicate this order as a superscript. In this way, the notation

$$H^*(X;\mathbb{F}_p) \cong \mathbf{A}_r^{(k)}$$

means that $H^*(X;\mathbb{F}_p) \cong \mathbf{A}'_r$ as algebras over the Steenrod algebra and

$$\beta_{(i)}(x) = \begin{cases} 0, & i \leq k \\ y, & i = k+1. \end{cases}$$

where $\beta_{(i)}$ denotes the i-th order Bockstein, i.e. the i-th differential in the mod p Bockstein spectral sequence of X. In the same way, we introduce the notations $H^*(X;\mathbb{F}_p) = \mathbf{B}_{i,r}^{(k)}$ and $H^*(X;\mathbb{F}_p) = \mathbf{C}_r^{(k)}$.

For further reference, we summarize the algebras that we have considered so far in table 10.1.

3. Computing Lannes T functor. Let T denote the Lannes functor defined as left adjoint to $H\otimes -$ in the category $\mathcal{U}$ of unstable modules over the Steenrod algebra (see [**17**] for a full description of its properties.) Here H denotes the mod p cohomology of $\mathbb{Z}/p$ as in the previous section. When R is an unstable algebra over the Steenrod algebra then so is $T(R)$ and T becomes a functor in the category $\mathcal{K}$ of unstable algebras over the Steenrod algebra.

Given a $\mathcal{K}$-map $f\colon R\to H$, its adjoint restricts to a $\mathcal{K}$-map $T^0(R)\to\mathbb{F}_p$, where $T^0(R)$ is the subalgebra of $T(R)$ of all elements of degree zero. We define the connected component of $T(R)$ corresponding to f as:

$$T_f(R) = T(R)\otimes_{T^0(R)}\mathbb{F}_p.$$

Furthermore, T_f may be thought as a functor defined on the category of R-$\mathcal{U}$-modules and with values in the category of $T_f(R)$-$\mathcal{U}$-modules (cf. [**13**].) We can also consider $T_f(M)$ as an R-$\mathcal{U}$-module induced by the natural $\mathcal{K}$-map $\varepsilon\colon R\to T_f(R)$ and then $\varepsilon\colon M\to T_f(M)$ becomes a natural transformation of R-$\mathcal{U}$-modules.

The purpose of this section is to compute for various PE-algebras A constructed in the previous section, namely, $\mathbf{A}_r$ for $r|(p-1)$ and $\mathbf{B}_{i,r}$ for $i\geq 0$ and $r|(p-1)$, the particular component of $T(A)$ that corresponds to a map $f\colon A\to H$ that can be uniformly described as the composition

$$A\xrightarrow{h}\mathbb{F}_p[x]\xrightarrow{k}H$$

where $h\colon A\to\mathbb{F}_p[x]$ is the projection onto the polynomial part of A and k is the obvious inclusion of $\mathbb{F}_p[x]$ in the even part of H. Our result is as follows.

Theorem 3.1. (1) $T_f(\mathbf{A}_1)\cong\mathbf{C}_1$ *and the natural map* $\varepsilon\colon\mathbf{A}_1\to T_f(\mathbf{A}_1)$ *is the inclusion of algebras given by* $\varepsilon(x)=x$ *and* $\varepsilon(y)=xz$.

(2) *For all* $i\geq 0$ *the natural map* $\varepsilon\colon\mathbf{B}_{i,1}\to T_f(\mathbf{B}_{i,1})$ *is an isomorphism.*

(3) *For any* $r|(p-1)$ *and all* $i\geq 0$*, the inclusions* $\mathbf{A}_r\to\mathbf{A}_1$ *and* $\mathbf{B}_{i,r}\to\mathbf{B}_{i,1}$ *induce isomorphisms* $T_f(\mathbf{A}_r)\cong T_f(\mathbf{A}_1)\cong\mathbf{C}_1$ *and* $T_f(\mathbf{B}_{i,r})\cong T_f(\mathbf{B}_{i,1})\cong\mathbf{B}_{i,1}$.

We will be using the following lemma that can be easily obtained:

Lemma 3.2. *Let A and B be two unstable algebras over the Steenrod algebra and $f\colon A\to B$ a $\mathcal{K}$-map that induces an isomorphism* $\mathrm{Hom}_{\mathcal{K}}(B,H)\cong\mathrm{Hom}_{\mathcal{K}}(A,H)$*. Then, for a $\mathcal{K}$-map $g\colon B\to H$ and any B-$\mathcal{U}$-module M*

$$T_{g\circ f}(M)\cong T_g(M)$$

and the $T_{g\circ f}(A)$-$\mathcal{U}$-module structure of $T_{g\circ f}(M)$ is induced by $T_{g\circ f}(A)\to T_g(B)$. Therefore, $T_{g\circ f}(M)$ is an A-$\mathcal{U}$-module through $A\to T_{g\circ f}(A)\to T_g(B)$ or equivalently through $A\to B\to T_g(B)$. $\square$

Proof of 3.1.(1). Recall that $\mathbf{A}_1=\mathbb{F}_p[x]\otimes E[y]$, $\deg(x)=2$, $\beta(x)=y$, $P^1(y)=yx^{p-1}$ and f has been defined as the composition $k\circ h$ where h is the projection $\mathbf{A}_1\to\mathbb{F}_p[x]$ and k identifies $\mathbb{F}_p[x]$ with the even part of H.

$\mathbf{A}_1$ sits in an exact sequence of $\mathbf{A}_1$-$\mathcal{U}$-modules

$$0 \to y\mathbb{F}_p[x] \to \mathbf{A}_1 \to \mathbb{F}_p[x] \to 0 \tag{1}$$

and T_f is exact so our first job will be the calculation of $T_f(y\mathbb{F}_p[x])$ and $T_f(\mathbb{F}_p[x])$. Both $y\mathbb{F}_p[x]$ and $\mathbb{F}_p[x]$ can actually be considered as $\mathbb{F}_p[x]$-$\mathcal{U}$-modules, with the $\mathbf{A}_1$-$\mathcal{U}$-module structure induced by the projection $h\colon \mathbf{A}_1 \to \mathbb{F}_p[x]$. According to lemma 3.2 what we have to do is to compute $T_k(y\mathbb{F}_p[x])$ and $T_k(\mathbb{F}_p[x])$ as $\mathbb{F}_p[x]$-$\mathcal{U}$-modules. $T(\mathbb{F}_p[x])$ is well known (see [**3**]) and $\varepsilon\colon \mathbb{F}_p[x] \to T_k(\mathbb{F}_p[x])$ turns out to be an isomorphism. For $y\mathbb{F}_p[x]$ we obtain:

Lemma 3.3. *$T_k(y\mathbb{F}_p[x]) \cong z\mathbb{F}_p[x]$ with $\deg(z) = 1$; that is, a $\mathbb{F}_p[x]$-$\mathcal{U}$-module on one generator of degree one on which the Steenrod operations act trivially. Moreover, $\varepsilon\colon y\mathbb{F}_p[x] \to T_k(\mathbb{F}_p[x])$ is an $\mathbb{F}_p[x]$-$\mathcal{U}$-module map given by $\varepsilon(y) = zx$.*

Proof. $y\mathbb{F}_p[x]$ might be identified to $\Sigma x\mathbb{F}_p[x]$ as $\mathbb{F}_p[x]$-$\mathcal{U}$-module (Σ denotes the suspension). Since T_k commutes with suspensions we must calculate $T_k(x\mathbb{F}_p[x])$ and for this we use the following exact sequence of $\mathbb{F}_p[x]$-$\mathcal{U}$-modules:

$$0 \to x\mathbb{F}_p[x] \to \mathbb{F}_p[x] \to \mathbb{F}_p \to 0.$$

$T_k(\mathbb{F}_p)$ is clearly trivial and we obtain $T_k(x\mathbb{F}_p[x]) \cong T_k(\mathbb{F}_p[x]) \cong \mathbb{F}_p[x]$. It also follows that ε is the inclusion $x\mathbb{F}_p[x] \to \mathbb{F}_p[x]$.

Finally we apply Σ and write $\Sigma\mathbb{F}_p[x]$ as $z\mathbb{F}_p[x]$ in order to get to the conclusion of the lemma. $\square$

The above computation together with lemma 3.2 give us $T_f(y\mathbb{F}_p[x])$ and $T_f(\mathbb{F}_p[x])$ as $T_f(\mathbf{A}_1)$-$\mathcal{U}$-modules and also as $\mathbf{A}_1$-$\mathcal{U}$-modules. Then, the exact sequence (1) induces a diagram of $\mathbf{A}_1$-$\mathcal{U}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & y\mathbb{F}_p[x] \arrow[r] \arrow[d, "\varepsilon"'] & \mathbf{A}_1 \arrow[r] \arrow[d, "\varepsilon"'] & \mathbb{F}_p[x] \arrow[r] \arrow[d, "\varepsilon"', "\cong"] & 0 \\
0 \arrow[r] & z\mathbb{F}_p[x] \arrow[r] & T_f(\mathbf{A}_1) \arrow[r] & \mathbb{F}_p[x] \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is an exact sequence of $T_f(\mathbf{A}_1)$-$\mathcal{U}$-modules. This diagram implies that $\varepsilon\colon \mathbf{A}_1 \to T_f(\mathbf{A}_1)$ is a $\mathcal{K}$-monomorphism and $T_f(\mathbf{A}_1) \cong \mathbb{F}_p[x] \otimes E[z]$ with ε determined by $\varepsilon(x) = x$ and $\varepsilon(y) = xz$. $\square$

Proof of 3.1.(2). Now we deal with the cases $\mathbf{B}_{i,1} \cong \mathbb{F}_p[x_i] \otimes E[y_i]$, $\deg x_i = 2p^i$, $\beta(x_i) = y_i$ and $P^{p^i}(y_i) = 0$, so that $\mathbf{B}_{i,1}$ sits in an exact sequence of $\mathbf{B}_{i,1}$-$\mathcal{U}$-modules:

$$0 \to y\mathbb{F}_p[x_i] \to \mathbf{B}_{i,1} \to \mathbb{F}_p[x_i] \to 0 \tag{2}$$

with $y\mathbb{F}_p[x_i]$ isomorphic as $\mathbb{F}_p[x_i]$-$\mathcal{U}$-module to $\Sigma^{2p^i+1}\mathbb{F}_p[x_i]$. In these cases $f = k \circ h$ with h the projection $\mathbf{B}_{i,1} \to \mathbb{F}_p[x_i]$ and $k\colon \mathbb{F}_p[x_i] \to H$ defined by $k(x_i) = v^{p^i}$, v a two dimensional generator of H.

Just as in the proof of the first part it is enough to compute $T_k(y\mathbb{F}_p[x_i])$ and $T_k(\mathbb{F}_p[x_i])$ and, in this case, both $\varepsilon\colon y\mathbb{F}_p[x_i] \to T_k(y\mathbb{F}_p[x_i])$ and $\varepsilon\colon \mathbb{F}_p[x_i] \to T_k(\mathbb{F}_p[x_i])$ are isomorphisms, thus the sequence (2) gives rise to the diagram:

$$\begin{tikzcd}
0 \arrow[r] & y\mathbb{F}_p[x_i] \arrow[r] \arrow[d, "\varepsilon"', "\cong"] & \mathbf{B}_{i,1} \arrow[r] \arrow[d, "\varepsilon"'] & \mathbb{F}_p[x_i] \arrow[r] \arrow[d, "\varepsilon"', "\cong"] & 0 \\
0 \arrow[r] & y\mathbb{F}_p[x_i] \arrow[r] & T_f(\mathbf{B}_{i,1}) \arrow[r] & \mathbb{F}_p[x_i] \arrow[r] & 0
\end{tikzcd}$$

and this implies the desired result. □

Proof of 3.1.(3). We will work out only the case of $\mathbf{A}_r$ for the case of $\mathbf{B}_{i,r}$ is essentially the same. Recall that $\mathbf{A}_r$ is the subalgebra of invariants of $\mathbf{A}_1$ by the action of $\mathbb{Z}/r$ and T commutes with taking invariants. In fact, we obtain:

$$T_f(\mathbf{A}_r) \cong T_f\big(\mathbf{A}_1^{\mathbb{Z}/r}\big) \cong \Big(\prod_{f_\lambda} T_{f_\lambda}\mathbf{A}_1\Big)^{\mathbb{Z}/r} \tag{1}$$

where f_λ runs through maps $\mathbf{A}_1 \to H$ that restrict as f to $\mathbf{A}_r$; that is, f_λ is the composition $\mathbf{A}_1 \to \mathbb{F}_p[x] \xrightarrow{k_\lambda} H$, with $k_\lambda(x) = \lambda v$, v a two dimensional generator of H and $\lambda \in \mathbb{F}_p$, such that $f_\lambda|_{\mathbf{A}_r} = f$, and this equality holds if and only if $\lambda^r = 1$, i.e. $\lambda \in \mathbb{Z}/r \subset {\mathbb{F}_p}^*$. Now it is clear that an element of $\mathbb{Z}/r$ induces a permutation of the factors in $\prod_{\lambda\in\mathbb{Z}/r} T_{f_\lambda}(\mathbf{A}_1)$ and then

$$\Big(\prod_{\lambda\in\mathbb{Z}/r} T_{f_\lambda}(\mathbf{A}_1)\Big)^{\mathbb{Z}/r} \cong T_f(\mathbf{A}_1). \quad \square$$

In a forthcoming section we will need a few variants of theorem 3.1.

Proposition 3.4. *Let $c : \mathbf{B}_{i,r} \to H$, $c : \mathbf{A}_r \to H$ denote the homomorphisms which are zero in positive degrees. Then $T_c(\mathbf{B}_{i,r}) \cong \mathbf{B}_{i,r}$ and $T_c(\mathbf{A}_r) \cong \mathbf{A}_r$.*

Proof. The proof is completely analogous to the proof of 3.1 except for two differences: We have $T_c(\mathbb{F}_p) = \mathbb{F}_p$ and this implies $T_c(\mathbf{A}_1) = \mathbf{A}_1$ and $T_c(\mathbf{B}_{i,1}) = \mathbf{B}_{i,1}$ by the same argument as in 3.1(1) and (2). On the other hand, the product in (1) has only one factor in the case of the trivial homomorphism c. □

One can check that the proofs of 3.1 and 3.4 work also for the algebras $\mathbf{A}'_r$ and $\mathbf{B}'_{i,r}$ and we have:

Proposition 3.5. *Let c and f be as in 3.1 and 3.4 respectively. Then $T_c(\mathbf{B}'_{i,r}) \cong \mathbf{B}'_{i,r}$, $T_c(\mathbf{A}'_r) \cong \mathbf{A}'_r$ and $T_f(\mathbf{A}'_r) \cong \mathbf{C}'_1$.* □

4. Non-realizability of PE-algebras. In this section we prove the Cooke conjecture as stated in the introduction. The proof will follow from a study of the transgression in the Serre spectral sequence of some fibration. We start with a lemma describing an interesting relation in the action of the Steenrod algebra on the mod p cohomology of $B^2\mathbb{Z}/p$. We use the notation

$$P^{\Delta_j} = P^{p^j} P^{p^{j-1}} \cdots P^1.$$

Lemma 4.1. *The following identities hold in $H^*(B^2\mathbb{Z}/p;\mathbb{F}_p)$:*

(1) $P^tP^{\Delta_r}\beta\iota = 0$ *for* $0 < t < p^{r+1}$.

(2) $P^{p^j}\beta P^{\Delta_{j-1}}\beta\iota = \beta P^{\Delta_j}\beta\iota \neq 0$ *for* $j > 0$.

Proof. Recall that $H^*(B^2\mathbb{Z}/p;\mathbb{F}_p)$ is a free graded–commutative algebra on free generators $P^I\iota$ where I is an admissible sequence of excess ≤ 2:

$$H^*(B^2\mathbb{Z}/p;\mathbb{F}_p) = \mathbb{F}_p[\iota,\beta P^1\beta\iota,\dots,\beta P^{\Delta_j}\beta\iota,\dots] \otimes E(\beta\iota,P^1\beta\iota,\dots,P^{\Delta_j}\beta\iota,\dots).$$

Hence $\beta P^{\Delta_j}\beta\iota$ is an indecomposable in $H^*(B^2\mathbb{Z}/p;\mathbb{F}_p)$. We prove first (1). If $r=0$ we have $P^tP^1\beta\iota = \lambda P^{t+1}\beta\iota = 0$ by unstability. We proceed then by induction:

$$P^tP^{\Delta_r}\beta\iota = P^tP^{p^r}P^{\Delta_{r-1}}\beta\iota = \sum_{s=0}^{[t/p]} \lambda_s P^{p^r+t-s}P^sP^{\Delta_{r-1}}\beta\iota.$$

In the right hand expression the term for $s=0$ vanishes by unstability and all other terms are zero by the induction hypothesis. The lemma follows now from the Adem relation for $P^{p^j}\beta P^{p^{j-1}}$ in the following way

$$\begin{aligned} P^{p^j}\beta P^{p^{j-1}}P^{\Delta_{j-2}}\beta\iota &= \sum_{t=0}^{p^{j-1}} \lambda_t\beta P^{p^j+p^{j-1}-t}P^tP^{\Delta_{j-2}}\beta\iota \\ &+ \sum_{t=0}^{p^{j-1}-1} \mu_t P^{p^j+p^{j-1}-t}\beta P^tP^{\Delta_{j-2}}\beta\iota. \end{aligned}$$

(If $j=1$, delete $P^{\Delta_{j-2}}$ in this formula.) By (1) and the unstability condition, the right hand term reduces to $\lambda_{p^{j-1}}\beta P^{p^j}P^{\Delta_{j-1}}\beta\iota$ and the proof ends by checking $\lambda_{p^{j-1}} = 1$. □

Proposition 4.2. *Assume $H^*(X;\mathbb{F}_p) \cong \mathbf{B}_{i+1,1}$, $i \geq 0$, as algebras over the Steenrod algebra. Assume also that there is a fibration*

$$X \longrightarrow E \xrightarrow{g} B^2\mathbb{Z}/p$$

such that x transgresses to $P^{\Delta_i}\beta\iota$ plus decomposables, where $\iota \in H^2(B^2\mathbb{Z}/p;\mathbb{F}_p)$ is the fundamental class. Then $i = 0$.

Proof. Consider the spectral sequence of the fibration $X \to E \to B^2\mathbb{Z}/p$. Since $\tau(x) = P^{\Delta_i}\beta\iota + d$ this element has to be killed by g^*. Hence $g^*(\beta P^{\Delta_i}\beta\iota + \beta d) = 0$. If we assume $i > 0$ we can apply the lemma and obtain

$$0 = g^*(P^1P^{p^i}\beta P^{\Delta_{i-1}}\beta\iota + P^1\beta d) = g^*(P^{p^i+1}\beta P^{\Delta_{i-1}}\beta\iota + P^1\beta d) = g^*([\beta P^{\Delta_{i-1}}\beta\iota]^p + P^1\beta d).$$

Notice that $\beta P^{\Delta_{i-1}}\beta\iota$ is an even dimensional indecomposable in $H^*(B^2\mathbb{Z}/p;\mathbb{F}_p)$. Using 4.1 we have the equalities

$$P^1P^{\Delta_j}\beta\iota = 0, \quad j \geq 0,$$

$$P^1\beta P^{\Delta_j}\beta\iota = \begin{cases} [\beta P^{\Delta_{j-1}}\beta\iota]^p, & j > 0 \\ 0, & j = 0 \end{cases}$$

and an elementary argument shows that $P^1\beta d$ cannot contain the term $[\beta P^{\Delta_{i-1}}\beta\iota]^p$. Hence, there is some differential in the spectral sequence, coming from an element in total degree $2p^{i+1}+2p-1$ which kills $[\beta P^{\Delta_{i-1}}\beta\iota]^p +$ other terms. By inspecting the E_2-term of the spectral sequence in total degree $2p^{i+1}+2p-1$ we see that only $x\otimes\iota^{p-2}\beta\iota$ and $y\otimes\iota^{p-1}$ may eventually kill this element.

We know by hypothesis that the first non vanishing differential maps x to $P^{\Delta_i}\beta\iota+d$. Hence it does not map $x\otimes\iota^{p-2}\beta\iota$ to $[\beta P^{\Delta_{i-1}}\beta\iota]^p +$ other terms. Since $\beta x = y$, the next differential maps y to $\beta P^{\Delta_i}\beta\iota+\beta(d)$ and so it cannot kill $[\beta P^{\Delta_{i-1}}\beta\iota]^p +$ other terms. In any case, $[\beta P^{\Delta_{i-1}}\beta\iota]^p +$ other terms survives, a contradiction that can only be avoided if $i=0$. □

Notice that the "elementary argument" mentioned in the above proof fails if $p=2$. This fact gives rise to a manifold of fascinating phenomena which will be studied in [**4**].

Theorem 4.3. *If $H^*(X;\mathbb{F}_p)\cong \mathbf{B}_{i,r}$ as algebras over the Steenrod algebra, then $i\le 1$.*

Proof. Since $H^1(X;\mathbb{F}_p)=0$ we can assume, without loss of generality, that X is p-complete. Let $f:\mathbf{B}_{i,r}\to H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ be the non trivial homomorphism considered in the last section. Then by [**17**; 3.1.1] there is a map $\phi: B\mathbb{Z}/p\to X$ inducing f in mod p cohomology. By theorem 3.1 we have

$$T_f\mathbf{B}_{i,r}\cong \mathbf{B}_{i,1},$$

where T denotes the T functor with respect to $V=\mathbb{Z}/p$. Then, [**17**; 3.2.1] shows that

$$H^*(\mathrm{map}(B\mathbb{Z}/p,X)_\phi;\mathbb{F}_p)\cong \mathbf{B}_{i,1}$$

where $\mathrm{map}(B\mathbb{Z}/p,X)_\phi$ is the space of all maps $B\mathbb{Z}/p\to X$ homotopic to ϕ. Observe now that $B\mathbb{Z}/p$ is a connected abelian simplicial group and the action of $B\mathbb{Z}/p$ on itself by right translations induces an action of $B\mathbb{Z}/p$ on the space $\mathrm{map}(B\mathbb{Z}/p,X)_\phi$. If Y is the homotopy quotient of $\mathrm{map}(B\mathbb{Z}/p,X)_\phi$ by this action, we have a fibration

$$\mathrm{map}(B\mathbb{Z}/p,X)_\phi\to Y\to B^2\mathbb{Z}/p. \tag{1}$$

If we denote by i the induced map $B\mathbb{Z}/p\to\mathrm{map}(B\mathbb{Z}/p,X)_\phi$ and by $e:\mathrm{map}(B\mathbb{Z}/p,X)_\phi\to X$ the evaluation map at the base point of $B\mathbb{Z}/p$ (which is the unit of $B\mathbb{Z}/p$ as a simplicial group) then one sees easily that $e\circ i=\phi$. In particular, $i^*(x)=v^{p^i}=P^{\Delta_{i-1}}\beta u$ where $x\in\mathbf{B}_{i,1}$ is the class in degree $2p^i$. Hence, the class x in the mod p cohomology of the fibre of (1) transgresses to $P^{\Delta_{i-1}}\beta\iota$ plus decomposables and proposition 4.2 shows that $i\le 1$. □

Remark 4.4. This method can be applied to give a new short proof of the well known fact that if $H^*(X;\mathbb{F}_p)\cong\mathbb{F}_p[x_{2p^i}]$ then $i=0$ for p odd and $i=0,1$ for $p=2$. In this case there exists also a fibration sequence

$$B\mathbb{Z}/p\to X\to\overline{X}\xrightarrow{g} B^2\mathbb{Z}/p.$$

Then $\tau(x)=P^{\Delta_i}\beta\iota_2+d$ where d is a decomposable and the result follows from $0=g^*(\beta P^{\Delta_i}\beta\iota_2+\beta d)$.

5. Spaces realizing $\mathbf{A}_r$. Let us denote by π the additive group of the p-adic integers. Let G be the automorphism group of π. G is isomorphic to the multiplicative group of the invertible elements in the ring structure of π and this is the direct product of the cyclic group of $(p-1)$th-roots of unity by the group $U_1 = 1 + p\pi$. The group U_1 is torsion free and there is a monomorphism

$$\phi : \pi \to G$$

given by $\phi(x)(y) = \exp(px)y$ where the product is taken in the ring structure of the p-adic integers. Moreover, after identifying G with the invertibles of π, ϕ maps onto U_1, the logarithm providing an inverse.

We obtain therefore a precise description of all possible actions of π on π, namely, all these actions are obtained by composing ϕ with multiplication by a p-adic number α. We will denote by $\phi\alpha$ the one defined by α and by $\pi^{\phi\alpha}$ the additive group of p-adic integers endowed with the action defined by $\phi\alpha$. Among them the ones of most interest for us correspond to $\alpha = p^k$ for $k \geq 0$ and we will abbreviate ϕp^k as ϕ_k.

Realizing $\mathbf{A}_r$. We will construct a space realizing $\mathbf{A}_r$ as well as other related spaces. We suggest to consider tables 10.2 and 10.3 in the appendix as a quick reference guide to the spaces introduced in this section. For this aim we consider π^{ϕ_k}, the p-adics endowed with the action defined by ϕ_k for $k \geq 0$. $B^2\pi^{\phi_k}$ inherits the action and we define spaces

$$E_k = E_{\phi_k} = B^2\pi^{\phi_k} \times_\pi EG$$

for all $k \geq 0$. (π acts on G through ϕ.) Let us compute the mod p cohomology of E_k. From the obvious fibration

$$B^2\pi^{\phi_k} \to E_k \to EG/\pi \simeq B\pi$$

we get a spectral sequence

$$H^*(\pi; H^*(B^2\pi^{\phi_k}; \mathbb{F}_p)) \quad \Rightarrow \quad H^*(E_k; \mathbb{F}_p).$$

Notice that $B^2\pi \simeq BS^1{}^\wedge_p$ and $B\pi \simeq S^1{}^\wedge_p$. Since π is q-divisible for any $q \neq p$, π can only act trivially on $H^*(B^2\pi^{\phi_k}; \mathbb{F}_p)$ which is either trivial or one-dimensional in each degree. Hence the spectral sequence yields immediately that for any $k \geq 0$

$$H^*(E_k; \mathbb{F}_p) \cong E(z) \otimes \mathbb{F}_p[x]$$

where z and x are classes in degrees 1 and 2, respectively. The Steenrod algebra should act trivially on z and the Steenrod powers act on $\mathbb{F}_p[x]$ as they do in $H^*(BS^1; \mathbb{F}_p)$. It only remains to determine the action of the Bockstein homomorphism on the class x. This will distinguish E_0 from E_k for $k \geq 1$. More in general, we will show that the action of the higher Bocksteins on x implies that all these spaces are different.

Proposition 5.1. *$H^*(E_k; \mathbb{F}_p) \cong \mathbf{C}_1^{(k)}$ for $k \geq 0$.*

Proof. To prove this, we will compute the cohomology of E_k with p-adic coefficients in low dimensions by means of the Serre spectral sequence. We need the following results on the homology of the p-adic integers.

Lemma 5.2. *(1) $H_1(\pi;\mathbb{Z}) = \pi$ and for $j \geq 2$, $H_j(\pi;\mathbb{Z})$ is a $\mathbb{Q}$-vector space.*

(2) $H^1(\pi;\pi) \cong \pi$ and for $j \geq 2$, $H^j(\pi;\pi) = 0$ (trivial coefficients).

(3) The cohomology of π with twisted p-adics coefficients is

$$\begin{aligned} H^0(\pi;\pi^{\phi\alpha}) &= 0, \\ H^1(\pi;\pi^{\phi\alpha}) &\cong \mathbb{Z}/p^{\nu(\alpha)+1}, \\ H^j(\pi;\pi^{\phi\alpha}) &= 0, \quad j \geq 2 \end{aligned}$$

where $\nu(\alpha)$ denotes the biggest power of p dividing α.

Proof. We obtain $H_1(\pi;\mathbb{Z}) \cong \pi$ by the Hurewicz theorem. Since $H^j(\pi;\mathbb{Z}/q) = 0$ for all $j \geq 2$ and all primes q the universal coefficient formula implies that $\mathrm{Hom}(H_j(\pi;\mathbb{Z}),\mathbb{Z}/q) \cong \mathrm{Ext}(H_j(\pi;\mathbb{Z}),\mathbb{Z}/q) \cong 0$ for all $j \geq 2$ and all primes q, hence statement (1) follows.

The statement (2) follows by the universal coefficient formula because $\mathrm{Hom}(\pi,\pi) \cong \pi$, $\mathrm{Hom}(A,\pi) = 0$ if A is p-divisible and $\mathrm{Ext}(A,\pi) = 0$ if A is torsion free.

To prove (3) note first that zero is the only invariant element of $\pi^{\phi\alpha}$ under the action of π so $H^0(\pi;\pi^{\phi\alpha}) = 0$. Next, we consider the well known description of the first cohomology group through derivations:

$$H^1(\pi;\pi^{\phi\alpha}) \cong \mathrm{Der}(\pi,\pi^{\phi\alpha})/\mathrm{Ider}(\pi,\pi^{\phi\alpha}).$$

A derivation $\pi \to \pi^{\phi\alpha}$ is determined by the image of $1 \in \pi$. In fact, for a given derivation $d\colon \pi \to \pi^{\phi\alpha}$, if $x \in \pi$ then $d(x) + \exp(p\alpha x)d(1) = d(x+1) = d(1+x) = d(1) + \exp(p\alpha)d(x)$ and this equation has a unique solution for $d(x)$ once $d(1)$ is fixed.

Moreover, the formula

$$d_a(x) = a\ \frac{\exp(p\alpha x) - 1}{p^{\nu(\alpha)+1}}$$

defines a derivation $\pi \to \pi^{\phi\alpha}$ for any $a \in \pi$. This derivation is inner precisely when $a \equiv 0\,(p^{\nu(\alpha)+1})$ and therefore $H^1(\pi;\pi^{\phi\alpha}) \cong \mathbb{Z}/p^{\nu(\alpha)+1}$.

It remains to compute $H^j(\pi,\pi^{\phi\alpha})$ for $j \geq 2$. We will see that $H^*(\pi;\pi^{\phi\alpha})$ is isomorphic to $H^*(\mathbb{Z};\pi^{\phi\alpha})$ with the action induced by restriction and then the result will follow because $\mathbb{Z}$ is free.

The isomorphism that we claim is induced by the inclusion $\mathbb{Z} \to \pi$ and it is proved in degrees 0 and 1 by direct computation. It would be also clear if the coefficients were $\mathbb{Z}/p^r$ for any $r > 1$. Then the Lyndon-Hochschild-Serre spectral sequence for $\mathbb{Z} \to \pi \to \pi/\mathbb{Z}$ shows first that $\tilde{H}^*(\pi/\mathbb{Z};\mathbb{Z}/p^r) = 0$ and since $\pi/\mathbb{Z}$ can only act trivially on $\mathbb{Z}/p^r$, also that $H^*(\pi;\pi^{\phi\alpha}) \cong H^*(\mathbb{Z};\pi^{\phi\alpha})$. □

As a consequence of this lemma, in the spectral sequence of the fibration $B^2\pi^{\phi_k} \to E_k \to B\pi$ with coefficients in $\hat{\mathbb{Z}}_p$ the only term that can contribute to $H^3(E_k;\hat{\mathbb{Z}}_p)$ is $H^1(\pi;H^2(B^2\pi;\hat{\mathbb{Z}}_p)) \cong \mathbb{Z}/p^{k+1}$. This finishes the proof of proposition 5.1. □

Notice now that for any r dividing $p-1$ there is an embedding of $\mathbb{Z}/r$ in G which gives an action of $\mathbb{Z}/r$ on $B^2\pi$ and EG. Since G is isomorphic to $\mathbb{Z}/p-1 \times \pi$ we obtain an induced free action of $\mathbb{Z}/r$ on E_k. Notice that since $\beta_{(k+1)}(x) = xz$ the action has to be trivial on $z \in H^1(E_k;\mathbb{F}_p)$. Let $E_k(r)$ be the quotient of E_k by this action. Since r is prime to p, it is clear that

$$H^*(E_k(r);\mathbb{F}_p) = H^*(E_k;\mathbb{F}_p)^{\mathbb{Z}/r} = E(z) \otimes \mathbb{F}_p[u]$$

where u corresponds to x^r in $H^*(E_k;\mathbb{F}_p)$ and $\beta_{(k+1)}(u) = ruz$. Hence,

Proposition 5.3. *$H^*(E_k(r);\mathbb{F}_p) \cong \mathbf{C}_r^{(k)}$ for $k \geq 0$, $r|(p-1)$.* $\square$

Finally, let us consider the composition

$$B\pi \xrightarrow{f} E_k \longrightarrow E_k(r)$$

where f is a section of the fibration $B^2\pi \to E_k \to B\pi$. If $E'_k(r)$ denotes the cofibre of this composition then we have $H^*(E'_k(r);\mathbb{F}_p) \cong \mathbb{F}_p[u] \otimes E(w)$ with $\deg u = 2r$, $\deg w = 2r+1$, $\beta_{(k+1)}(u) = w$ and $P^1(w) = ru^s w$, where $s = (p-1)/r$ as is usual in this paper. In particular

$$H^*(E'_k(r);\mathbb{F}_p) \cong \mathbf{A}_r^{(k)}.$$

Definition 5.4. *For $k \geq 0$, $r|p-1$, we define $X_k(r)$ as the p-completion of $E'_k(r)$.*

The next theorem establishes some properties of these spaces.

Theorem 5.5. *For any r dividing $p-1$ and $k \geq 0$,*

1. *$X_k(r)$ is a simply connected p-complete space whose homotopy groups are finite p-groups.*
2. *$H^*(X_k(r);\mathbb{F}_p) \cong \mathbf{A}_r^{(k)}$.*

Proof. Will be based in the following two propositions.

Proposition 5.6. *Let $R = \hat{\mathbb{Z}}_p$ or $\mathbb{Z}$ and X a space with cohomology of finite type over R. If $H^*(X;\mathbb{F}_p) = \mathbf{A}_r^{(k)}$ then in the R-cohomology Bockstein spectral sequence $\{B_l, d_l\}$ for X*

1. *the first non-trivial differential is d_{k+1} and $d_{k+1}(u) = w$,*
2. *u^{p^n} survives to B_{n+k+1} and $d_{n+k+1}([u^{p^n}]) = [u^{p^n-1}w]$ and*
3. *$B_\infty = 0$.*

Proof. This is a direct consequence of known results about the differential in the Bockstein spectral sequence (cf. [**16**; pag. 102]). $\square$

The next proposition might be of independent interest and we establish it for any prime number, either two or odd.

Proposition 5.7. *Let p be any prime and X a 1-connected, p-complete space, then:*

1. *The following conditions are equivalent:*
   - (i) *$H^j(X;\mathbb{F}_p)$ is finite for all j.*
   - (ii) *$\pi_j(X)$ is a finitely generated $\hat{\mathbb{Z}}_p$-module for all j.*
   - (iii) *$H^j(X;\hat{\mathbb{Z}}_p)$ is a finitely generated $\hat{\mathbb{Z}}_p$-module for all j.*
2. *The following conditions are equivalent:*
   - (i) *$\tilde{H}^j(X;\hat{\mathbb{Z}}_p)$ is a finite p-group for all j.*
   - (ii) *$\pi_j(X)$ is a finite p-group for all j.*

Proof. Let F be the fibre of the rationalization $X \to X_0$. We can first obtain some general facts about F. $H^*(F;\mathbb{F}_p) \cong H^*(X;\mathbb{F}_p)$ and $F \to X$ is the p-completion of F. Also, $H^*(F;\hat{\mathbb{Z}}_p) \cong H^*(X;\hat{\mathbb{Z}}_p)$ and $\pi_j(F)$ is a p-group for all j. Moreover, since F is the fibre of a map between 1-connected spaces, the fundamental group of F is abelian and acts trivially on the homology and homotopy of the universal cover of F (cf. [**15**]). Hence the mod $\mathcal{C}$ Hurewicz theorem can be applied to F and we obtain that $\tilde{H}_j(F;\mathbb{Z})$ is a p-group (i.e. a group all of whose elements are p-power torsion) for any j.

Now, the proof of part (1) of the proposition will consist in the following sequence of statements.

Claim 5.7.1: $H^j(X;\mathbb{F}_p)$ is finite for all j if and only if $\tilde{H}_j(F;\mathbb{Z})$ is a finitely cogenerated p-group for all j.

Let us write $\tilde{H}_j(F;\mathbb{Z})$ as an extension of a divisible p-group D_j by a pure subgroup P_j which is a direct sum of cyclic p-groups. Then one easily deduces that the mod p homology of F is of finite type over $\mathbb{F}_p$ if and only if both P_j and D_j contain finitely many summands for all j. Since a bounded pure subgroup is a direct summand this means that the (reduced) integral homology groups of F are a direct sum of finitely many cyclic p-groups and finitely many groups $\mathbb{Z}/p^\infty$; that is, they are finitely cogenerated p-groups.

Claim 5.7.2: $\tilde{H}_j(F;\mathbb{Z})$ is a finitely cogenerated p-group for all j if and only if $\pi_j(F)$ is so.

Since $\pi_1(F)$ is abelian $\pi_1(F) \cong H_1(F;\mathbb{Z})$ and then since the class of finitely cogenerated abelian p-groups is an acyclic ring of abelian groups this claim follows by the mod $\mathcal{C}$ Hurewicz theorem.

Claim 5.7.3: $\pi_j(F)$ is a finitely cogenerated p-group for all j if and only if $\pi_j(X)$ is a finitely generated $\hat{\mathbb{Z}}_p$-module for all j.

From the homotopy exact sequence for the fibration $F \to X \to X_0$ we obtain short exact sequences

$$0 \to \pi_{j+1}(X) \otimes \mathbb{Q}/\mathbb{Z} \to \pi_j(F) \to \operatorname{Tor}(\pi_j(X), \mathbb{Q}/\mathbb{Z}) \to 0$$

and then one of the implications. On the other hand, since X is the p-completion of the nilpotent space F we also have short split exact sequences ([**6**; VI.5.1])

$$0 \to \operatorname{Ext}(\mathbb{Z}/p^\infty, \pi_j(F)) \to \pi_j(X) \to \operatorname{Hom}(\mathbb{Z}/p^\infty, \pi_{j-1}(F)) \to 0,$$

hence the implication in the other direction is also true.

Claim 5.7.4: $H^j(X;\mathbb{F}_p)$ is finite for all j if and only if $H^j(X;\hat{\mathbb{Z}}_p)$ is a finitely generated $\hat{\mathbb{Z}}_p$-module for all j.

It suffices to show this equivalence for F, and this follows easily by the universal coefficients formula using claim 5.7.1.

This finishes the proof of part (1). Let us turn to the proof of part (2). With the same argument as in Claim 5.7.4 we obtain that $\tilde{H}^j(X;\hat{\mathbb{Z}}_p)$ is a finite p-group for all j if and only if $\tilde{H}_j(F;\mathbb{Z})$ is a finite p-group for all j. Again by the mod $\mathcal{C}$ Hurewicz theorem this is equivalent to $\pi_j(F)$ to be a finite p-group for all j and finally the same argument of Claim 5.7.3 shows that if the homotopy groups of either X or F are finite p-groups, then $F \to X$ is actually a homotopy equivalence. $\square$

We can now finish the proof of Theorem 5.5. The space $X_k(r)$ was defined as the p-completion of $E'_k(r)$. By construction we have $\pi_1(E_k(r)) \cong \mathbb{Z}/r \times \pi$ and then by the van Kampen theorem $\pi_1(E'_k(r)) \cong \mathbb{Z}/r$. Hence ([**6**; p. 206]) $E'_k(r)$ is $\mathbb{Z}/p$-good and $X_k(r)$ is p-complete, simply connected and has the same mod p cohomology as $E'_k(r)$. So we have proved part (2) of the theorem.

By Proposition 5.7(1) $X_k(r)$ is of finite type over $\hat{\mathbb{Z}}_p$ hence the Bockstein spectral sequence applies and by Proposition 5.6 the cohomology groups $\tilde{H}^j(X_k(r);\hat{\mathbb{Z}}_p)$ are actually finite p-groups, hence by proposition 5.7(2) we obtain part (1) of the theorem. □

Remark 5.8. From Proposition 5.6 we can derive the integral cohomology of the spaces $X_k(r)$:

$$\tilde{H}^i(X_k(r);\hat{\mathbb{Z}}_p) \cong \tilde{H}^i(X_k(r);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p^{k+1+\nu(j)} & i = 2rj+1,\ j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Final remarks. Here is the reason for which we have been dealing with a certain collection among all possible actions of π on π.

Proposition 5.9. *Let π^ξ be the additive group of the p-adic integers together with a π action defined by a non trivial homomorphism $\xi\colon \pi \to \mathrm{Aut}(\pi) = G$ and define*

$$E_\xi = B^2\pi^\xi \times_\pi EG.$$

Then E_ξ is homotopy equivalent to a space $E_k = E_{\phi_k}$

Proof. From our discussion of the possible actions of π on π at the beginning of this section, ξ is of the form $\phi\alpha$ for a p-adic integer α. That is:

$$\xi(x)(y) = e^{p\alpha x}y.$$

Now, α might be written as $\alpha = p^{\nu(\alpha)}w$ where $w \in 1 + p\pi$. Since w is invertible it determines an automorphism

$$w\colon \pi \to \pi.$$

Now, the identity $B^2\pi \to B^2\pi$ is w-equivariant if we consider the action given by ξ on the source an by $\phi_{\nu(\alpha)}$ on the target:

$$\xi(x)(y) = e^{p\alpha x}y = e^{pwp^{\nu(\alpha)}x}y = \phi_k(wx)(y).$$

In this way we get a map $E_\xi \to E_{\nu(\alpha)}$ which is in fact a homotopy equivalence because w is invertible. □

Remark 5.10. Observe that until now all our constructions could be performed using $B\mathbb{Z}$ instead of $B\pi$ as base space of our fibrations, with actions of $\mathbb{Z}$ on π induced by restriction from the actions of π on π that we used. Also, we could use $B\mathbb{Z}/p^\infty$ instead of $B^2\pi$. However the above proposition would not be true in that case. We would need to complete our spaces before proving such a result.

6. Uniqueness of spaces realizing $\mathbf{A}_r$. In section 5 we have constructed the spaces $X_k(r)$ whose mod p cohomologies realize $\mathbf{A}_r^{(k)}$ for r dividing $p-1$ and $k \geq 0$ (theorem 5.5). In this section we show that up to p-adic completion these spaces are the only ones which realize the algebras with higher Bocksteins $\mathbf{A}_r^{(k)}$.

Theorem 6.1. *Let X be such that $H^*(X;\mathbb{F}_p) \cong \mathbf{A}_r^{(k)}$ for some $r|(p-1)$ and $k \geq 0$. Then $\hat{X}_p \simeq X_k(r)$.*

Proof. Let X be a space satisfying the hypothesis of the theorem. Since $H_1(X;\mathbb{F}_p) = 0$ we have that $\pi_1(X)$ is $\mathbb{Z}/p-$perfect. Hence ([**6**; p. 206]) $\hat{X}_p$ is a simply connected p-complete space with the same mod p cohomology as X itself. By 5.6 and 5.7 the homotopy groups of $\hat{X}_p$ are finite $p-$groups.

Let $f : B\mathbb{Z}/p \to \hat{X}_p$ be a map such that f^* is non-trivial in degree $2r$ and trivial in degree $2r+1$ and let Y denote the component of the mapping space $\mathrm{map}(B\mathbb{Z}/p, \hat{X}_p)$ containing the map f. There is an evaluation map $e : Y \to \hat{X}_p$. The next step in the proof of 6.1 will be to show that Y is homotopy equivalent to the space E_k of section 5. According to the computation of the T functor in section 3, $T_f(H^*(X;\mathbb{F}_p)) \cong E(z) \otimes \mathbb{F}_p[w]$ with $\deg(z) = 1$, $\deg(w) = 2$ and

$$\beta(w) = \begin{cases} zw, & k = 0 \\ 0, & k > 0. \end{cases}$$

Notice that $T_f(H^*(X;\mathbb{F}_p))$ is only an algebra over the Steenrod algebra and so higher Bocksteins do not make any sense in $T_f(H^*(X;\mathbb{F}_p))$ unless we show that it is the cohomology of some space.

The computed value of the functor T_f is interpreted by [**10**] as follows. Let $P_n\hat{X}_p$ denote the n-th stage of the Postnikov decomposition of $\hat{X}_p$. Then $\left\{ \mathrm{map}(B\mathbb{Z}/p, P_n\hat{X}_p)_{f_n} \right\}$ is a tower with

$$Y \simeq \varprojlim_n \mathrm{map}(B\mathbb{Z}/p, P_n\hat{X}_p)_{f_n}$$

and

$$\varinjlim_n H^*(\mathrm{map}(B\mathbb{Z}/p, P_n\hat{X}_p)_{f_n};\mathbb{F}_p) \cong \begin{cases} \mathbf{C}_1', & k > 0, \\ \mathbf{C}_1, & k = 0. \end{cases}$$

The natural homomorphism $H^*(X;\mathbb{F}_p) \to \varinjlim_n H^*(\mathrm{map}(B\mathbb{Z}/p, P_n\hat{X}_p)_{f_n};\mathbb{F}_p)$ sends x to w^r and y to $w^r z$.

Some information about the homotopy of the spaces $\mathrm{map}(B\mathbb{Z}/p, P_n\hat{X}_p)_{f_n}$ is provided by results of Thom ([**26**], revisited in [**21**]). The principal fibration $P_n\hat{X}_p \to P_{n-1}\hat{X}_p$ gives rise to a principal fibration

$$\mathrm{map}(B\mathbb{Z}/p, P_n\hat{X}_p)_{f_n} \to \mathrm{map}(B\mathbb{Z}/p, P_{n-1}\hat{X}_p)_{f_{n-1}}$$

with fibre a union of components of $\mathrm{map}(B\mathbb{Z}/p, K(\pi_n\hat{X}_p, n))$. But each of these components has the homotopy type of a product of Eilenberg-MacLane spaces

$$K(H^{n-j}(B\mathbb{Z}/p, \pi_n\hat{X}_p), j), \quad 1 \leq j \leq n.$$

Since the homotopy groups of $\hat{X}_p$ are finite p-groups and so are the homotopy groups of $\mathrm{map}(B\mathbb{Z}/p, P_{n-1}\hat{X}_p)_{f_{n-1}}$ by induction, those of $\mathrm{map}(B\mathbb{Z}/p, P_n\hat{X}_p)_{f_n}$ should also be finite p-groups.

Let us write $Y_n = \mathrm{map}(B\mathbb{Z}/p, P_n\hat{X}_p)_{f_n}$.

The class z in degree one in $\varinjlim_n H^*(Y_n;\mathbb{F}_p)$ is represented by a class $z_n \in H^*(Y_n;\mathbb{F}_p)$ for some n and also by its images in $H^*(Y_{n+i};\mathbb{F}_p)$. We fix such a sequence $\{z_n\}$. This sequence provides a map of towers:

$$\{Y_n\} \xrightarrow{\{\bar{z}_n\}} \{K(\mathbb{Z}/p^{\alpha(n)},1)\}.$$

We will prove that we can choose the sequence $\alpha(n)$ to be unbounded. In fact, any of these maps is a lifting of the classifying map $Y_n \longrightarrow K(\mathbb{Z}/p,1)$ of the class z_n. Suppose by induction that z_n is classified by $\bar{z}_n\colon Y_n \longrightarrow K(\mathbb{Z}/p^{\alpha(n)},1)$, that is $\bar{z}_n^*(\iota) = z_n$ if ι is the fundamental class of $H^*(K(\mathbb{Z}/p^{\alpha(n)},1);\mathbb{F}_p)$. Observe that we can as well assume that $\alpha(n)$ is the maximum possible such that this lifting exists. This is because all of the homotopy groups of Y_n are finite and then z_n should be dual to a torsion homology class. Now we look at the class z_{n+i}, $i \geq 1$. This is classified by $Y_{n+i} \longrightarrow Y_n \longrightarrow K(\mathbb{Z}/p^{\alpha(n)},1)$ and the obstructions for the existence of a lifting

$$Y_{n+i} \longrightarrow K(\mathbb{Z}/p^{\alpha(n+i)},1)$$

with $\alpha(n+i) > \alpha(n)$ are some higher Bocksteins. But no higher Bockstein can be non trivial on z_n for all big enough n because if this happens then w^r is in the image of some higher Bockstein, contradicting the fact that $x \in H^*(X;\mathbb{F}_p)$ has a non trivial Bockstein of order $k+1$. Hence, $\lim_{n\to\infty} \alpha(n) = \infty$.

Consider now the inverse system of fibrations

$$\begin{tikzcd}
\dots \arrow[r] & F_n \arrow[r] \arrow[d] & F_{n-1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & Y_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & B\mathbb{Z}/p^{\alpha(n)} \arrow[r] & B\mathbb{Z}/p^{\alpha(n-1)} \arrow[r] & \dots
\end{tikzcd}$$

Notice that since the maps $Y_n \longrightarrow B\mathbb{Z}/p^{\alpha(n)}$ are liftings of non-trivial maps $Y_n \longrightarrow B\mathbb{Z}/p$, they induce epimorphisms between fundamental groups and so the spaces F_n are connected.

All homotopy groups involved in the above inverse system of fibrations are finite p-groups, hence, in the limit, we get a fibration:

$$F \longrightarrow Y \longrightarrow B\pi$$

where $F = \varprojlim_n F_n$ and π denotes as usual the additive group of the p-adic integers. Note that in all these fibrations the base space is not simply connected. Nevertheless, at any stage the Eilenberg-Moore spectral sequence of [**12**] starts with

$$E_2^{*,*} \cong \mathrm{Tor}^{*,*}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\left(H^*(Y_n;\mathbb{F}_p),\mathbb{F}_p\right)$$

and converges strongly to the mod p cohomology of the fibre F_n because ([**12**]) the fundamental group of the base is a p-group and thus it acts nilpotently on the mod p cohomology of the fibre. In the limit we have a spectral sequence

$$E_2^{*,*} \cong \varinjlim_n \operatorname{Tor}^{*,*}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(H^*(Y_n;\mathbb{F}_p),\mathbb{F}_p\big)$$

converging to $\varinjlim_n H^*(F_n;\mathbb{F}_p)$.

Lemma 6.3. $\varinjlim_n \operatorname{Tor}^{*,*}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(H^*(Y_n;\mathbb{F}_p),\mathbb{F}_p\big) \cong \mathbb{F}_p[w]$, $\deg(w)=2$ *and so therefore* $\varinjlim_n H^*(F_n;\mathbb{F}_p) \cong \mathbb{F}_p[w]$.

Proof. Tor is covariant with respect to any of its three variables and $\varinjlim$ is an exact functor. Hence one can easily derive a commutation formula for Tor and $\varinjlim$ which shows that

$$\varinjlim_n \operatorname{Tor}^{*,*}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(H^*(Y_n;\mathbb{F}_p),\mathbb{F}_p\big) \cong \operatorname{Tor}^{*,*}_{E(z)}\big(E(z)\otimes\mathbb{F}_p[w],\mathbb{F}_p\big) \cong \mathbb{F}_p[w].$$

Alternatively, one can directly compute the $\varinjlim$ as follows.

Let us denote

$$K_n = \ker\big\{H^*(Y_n;\mathbb{F}_p) \to \varinjlim_n H^*(Y_n;\mathbb{F}_p) \cong E(z)\otimes\mathbb{F}_p[w]\big\}.$$

K_n is an ideal of $H^*(Y_n;\mathbb{F}_p)$ and therefore a sub-$H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)$-module. The induced $H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)$-module structure of $E(z)\otimes\mathbb{F}_p[w]$ factors:

$$\begin{tikzcd}
 & H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p) \arrow[r] \arrow[d] & E(z_n) \arrow[d] \\
K_n \arrow[r] & H^*(Y_n;\mathbb{F}_p) \arrow[r] & E(z)\otimes\mathbb{F}_p[w]
\end{tikzcd}$$

Observe that $H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p) \cong E(z_n)\otimes\mathbb{F}_p[a_n]$, $\deg(a_n)=2$. As a consequence we have:

$$\begin{aligned}
(1)\quad & \operatorname{Tor}^{*,*}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(E(z)\otimes\mathbb{F}_p[w],\mathbb{F}_p\big) \\
&\qquad \cong \operatorname{Tor}^{*,*}_{E(z_n)}\big(E(z_n)\otimes\mathbb{F}_p[w],\mathbb{F}_p\big)\otimes\operatorname{Tor}^{*,*}_{\mathbb{F}_p[a_n]}(\mathbb{F}_p,\mathbb{F}_p) \\
&\qquad\qquad \cong \mathbb{F}_p[w]\otimes\operatorname{Tor}_{\mathbb{F}_p[a_n]}(\mathbb{F}_p,\mathbb{F}_p)
\end{aligned}$$

There is an exact sequence:

$$\begin{aligned}
(2)\quad & \ldots \to \operatorname{Tor}^{r,s}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(K_n,\mathbb{F}_p\big) \to \operatorname{Tor}^{r,s}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(H^*(Y_n;\mathbb{F}_p),\mathbb{F}_p\big) \to \\
&\quad \to \operatorname{Tor}^{r,s}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(E(z)\otimes\mathbb{F}_p[w],\mathbb{F}_p\big) \to \operatorname{Tor}^{r-1,s}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(K_n,\mathbb{F}_p\big) \to \ldots
\end{aligned}$$

which is natural with respect to maps $Y_{n+i} \to Y_n$. Observe that no element of K_n survives to the limit $\varinjlim H^*(Y_n;\mathbb{F}_p)$. Since K_n is finite dimensional there is a large enough i such that $K_n \to K_{n+i}$ is zero, hence so is

$$\operatorname{Tor}^{*,*}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(K_n,\mathbb{F}_p\big) \to \operatorname{Tor}^{*,*}_{H^*(\mathbb{Z}/p^{\alpha(n+i)};\mathbb{F}_p)}\big(K_{n+i},\mathbb{F}_p\big).$$

Then (2) implies

$$\varinjlim_n \operatorname{Tor}^{r,s}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(H^*(Y_n;\mathbb{F}_p),\mathbb{F}_p\big) \cong \varinjlim_n \operatorname{Tor}^{*,*}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(E(z)\otimes\mathbb{F}_p[w],\mathbb{F}_p\big).$$

For similar reasons (1) implies:

$$\varinjlim_n \operatorname{Tor}^{*,*}_{H^*(B\mathbb{Z}/p^{\alpha(n)};\mathbb{F}_p)}\big(E(z)\otimes\mathbb{F}_p[w],\mathbb{F}_p\big) \cong \mathbb{F}_p[w]. \quad \square$$

Lemma 6.4. *Let $\{Z_n\}$ be a tower of fibrations of pointed connected p-complete spaces with mod p cohomology of finite type. If $\varinjlim_n H^*(Z_n;\mathbb{F}_p)$ is a polynomial algebra on one generator w in degree 2 then $\varprojlim_n Z_n \simeq B^2\pi$.*

Proof. An argument similar to one used above shows that there is a map of towers:

$$\{Z_n\} \xrightarrow{\{g_n\}} \{K(\mathbb{Z}/p^{\gamma(n)},2)\} \tag{1}$$

with $\gamma(n)$ an unbounded sequence. Here, each g_n detects a class in degree two of $H^*(Z_n;\mathbb{F}_p)$ that represents w in the limit, hence $\{g_n\}$ induces an isomorphism

$$\varinjlim_n H^*(Z_n;\mathbb{F}_p) \cong \varinjlim_n H^*(K(\mathbb{Z}/p^{\gamma(n)},2);\mathbb{F}_p) \cong \mathbb{F}_p[w]$$

and dually

$$\varprojlim_n H_*(Z_n;\mathbb{F}_p) \cong \varprojlim_n H_*(K(\mathbb{Z}/p^{\gamma(n)},2);\mathbb{F}_p)$$

because all relevant (co)homology groups are finite. By the same reason, this implies that the induced map of towers:

$$\{H_*(Z_n;\mathbb{F}_p)\} \to \{H_*(K(\mathbb{Z}/p^{\gamma(n)},2);\mathbb{F}_p)\}$$

is a pro-isomorphism.

Now, according to [**6**; III.6.6, pg. 88] the map of towers $\{R_nZ_n\} \to \{R_nK(\mathbb{Z}/p^{\gamma(n)},2)\}$ is a weak pro-homotopy equivalence, where $\{R_nX\}$ is the tower which defines Bousfield-Kan p-completion. Hence,

$$\varprojlim_n Z_n = \varprojlim_n R_\infty Z_n = \varprojlim_n R_nZ_n \simeq \varprojlim_n R_nK(\mathbb{Z}/p^{\gamma(n)},2) = B^2\pi. \quad \square$$

This lemma applies immediately to the tower $\{F_n\}$ because the homotopy groups of each F_n are finite p-groups and a space whose homotopy groups are finite p-groups is necessarily p-complete, nilpotent and of finite mod p type. We have, therefore, obtained a fibration

$$B^2\pi \to Y \to B\pi. \tag{2}$$

Since π can only act trivially on $H_*(B^2\pi;\mathbb{F}_p)$, Y is p-complete (cf. [**6**; mod-$\mathbb{Z}/p$ fibre lemma]). The Serre spectral sequence and the injectivity of $H^*(X;\mathbb{F}_p) \to \varinjlim H^*(Y_n;\mathbb{F}_p)$ show that the natural map

$$H^*(Y;\mathbb{F}_p) \to \varinjlim_n H^*(Y_n;\mathbb{F}_p) \cong T_f(H^*(X;\mathbb{F}_p))$$

is an isomorphism. By naturality of the Bockstein homomorphisms, we deduce that $H^*(Y;\mathbb{F}_p) = \mathbf{C}_1^{(k)}$. We want to deduce from here that the fibration (2) is fibre homotopy equivalent to the fibration

$$B^2\pi \to E_k \to B\pi$$

of section 5. Fibrations with base space $B\pi$ and fibre $B^2\pi$ are classified by the homotopy set:

$$\big[B\pi, B\,\mathrm{Aut}(B^2\pi)\big]$$

where $\mathrm{Aut}(B^2\pi)$ is the topological monoid of the self homotopy equivalences of $B^2\pi$. According to [**22**] there is a fibration

$$B^2\pi \to B\,\mathrm{Aut}(B^2\pi) \to B\,\mathrm{Aut}(\pi)$$

having a section $B\,\mathrm{Aut}(\pi) \to B\,\mathrm{Aut}(B^2\pi)$. Then

$$\big[B\pi, B\,\mathrm{Aut}(B^2\pi)\big] \cong \big[B\pi, B\,\mathrm{Aut}(\pi)\big] \cong \mathrm{Hom}(\pi, \mathrm{Aut}(\pi)).$$

Therefore, any fibration $B^2\pi \to Z \to B\pi$ is determined by an action of π on π. All these actions were considered in section 5. From such classification we obtain an equivalence of fibrations

$$\begin{array}{ccccc}
B^2\pi & \longrightarrow & Y & \longrightarrow & B\pi \\
\Big\| & & \simeq\Big\uparrow g & & \simeq\Big\uparrow w \\
B^2\pi & \longrightarrow & E_k & \longrightarrow & B\pi
\end{array}$$

There is an action of $\mathbb{Z}/r$ on E_k considered in section 5 and also an action of $\mathbb{Z}/r$ on Y defined in the following way. We have $Y = \mathrm{map}(B\mathbb{Z}/p, \hat{X}_p)_f$ and $\mathbb{Z}/r$ acts on $B\mathbb{Z}/p$. Since f^* commutes with this action, we get an action on Y such that the evaluation map $e : Y \to \hat{X}_p$ is equivariant. Naturality of T shows that on mod p cohomology this action leaves z fixed and sends w to λw where λ is an r-th root of unity. In this form we obtain a map $l : E_k \to \hat{X}_p$

Lemma 6.5. *l is homotopic to an equivariant map.*

Proof. First of all we notice that it is enough to prove that l is equivariant up to homotopy for Wojtkowiak proved in [**27**] that when a finite group of order prime to p acts freely on a space and the target space is p-complete, nilpotent and of finite type over $\hat{\mathbb{Z}}_p$ then a map equivariant up to homotopy is homotopic to an equivariant map.

E_k is a two stage Postnikov system and there is an exact sequence of Didierjan ([**8**]) for the group of homotopy classes of self homotopy equivalences of E_k:

$$1 \to H^2(\pi; \pi^{\varphi_k}) \to \mathcal{E}(E_k) \to \mathrm{Aut}(\pi) \oplus \mathrm{Aut}(\pi).$$

Since $H^2(\pi; \pi^{\varphi_k}) = 0$, this shows that a homotopy self equivalence of E_k is determined up to homotopy by its action on $\pi_1(E_k)$ and $\pi_2(E_k)$. The lemma is proved if we show that $\mathbb{Z}/r$ acts on $\pi_1(Y)$ and $\pi_2(Y)$ as it does on $\pi_1(E_k)$ and $\pi_2(E_k)$.

The action of $\mathbb{Z}/r$ on $\pi_1(Y) = \pi$ is determined by the action on $H^1(Y; \mathbb{F}_p)$ which can only be trivial. Similarly, the action of $\mathbb{Z}/r$ on $\pi_2(Y) = \pi$ is determined by the action on $H^2(Y; \hat{\mathbb{Z}}_p) \cong \hat{\mathbb{Z}}_p$ and this action is determined by the action on $H^2(Y; \mathbb{F}_p)$. $\square$

Hence we obtain a map

$$h : E_k(r) = E_k \times_{\mathbb{Z}/r} E\mathbb{Z}/r \to \hat{X}_p.$$

Let now $k : B\pi \to E_k(r)$ be the map considered in section 5. If kh is trivial we obtain a map $E'_k(r) \to \hat{X}_p$ which induces isomorphism in mod p cohomology and the theorem is proved. But $\hat{X}_p$ is simply connected, its homotopy groups are finite p-groups and $H^i(\pi; P) = 0$ for $i > 1$ and any finitely generated $\hat{\mathbb{Z}}_p$-module P with trivial action. Hence, by obstruction theory, any map $B\pi \to \hat{X}_p$ is trivial. This ends the proof of theorem 6.1. $\square$

7. Spaces realizing $\mathbf{B}_{i,r}$. In section 2 we constructed for each algebra $\mathbf{B}_{i,r}$ a topological realization. It turns out that these are not the only ones. In this section we will construct several families of spaces, some of which will have cohomology isomorphic to $\mathbf{B}_{i,r}$. We suggest using tables 10.2 and 10.3 in the appendix as a quick reference guide to all these spaces.

Let $X_k = X_k(1)$ be the spaces introduced in section 5. Theorem 5.5 proves that

$$H^*(X_k; \mathbb{F}_p) \cong \mathbf{A}_1^{(k)}.$$

Hence, the two dimensional class x in $H^*(X_k; \mathbb{F}_p)$ can be represented by a map

$$X_k \to B^2\mathbb{Z}/p^{k+1}. \tag{1}$$

Let Y_k be the fibre of this map. Y_k is a p-complete space with finite homotopy groups. From the construction of X_k we see that there is an action of the cyclic group of order r on X_k, for any $r|p-1$. By [**27**] we can assume that the map (1) is equivariant with respect to this action and the natural action on $B^2\mathbb{Z}/p^{k+1}$. This yields an action of the cyclic group of order r on Y_k and we define $Y_{k,r}$ as the p-completion of the homotopy quotient of Y_k by this action:

$$Y_{k,r} = (E\mathbb{Z}/r \times_{\mathbb{Z}/r} Y_k)\hat{}_p.$$

Proposition 7.1. *$H^*(Y_{k,r};\mathbb{F}_p)\cong \mathbf{B}_{1,r}$ as algebras over the Steenrod algebra.*

Proof. We consider the sequence of fibrations

$$B\mathbb{Z}/p^{k+1}\to Y_{k,1}\to X_k\to B^2\mathbb{Z}/p^{k+1}.$$

In the spectral sequence of the first three terms, the classes u and v in $H^*(B\mathbb{Z}/p^{k+1};\mathbb{F}_p)$ are transgressive and are mapped onto the classes x and y of $H^*(X_k;\mathbb{F}_p)$. By degree reasons it follows that these are the only non vanishing differentials. Therefore, $H^*(Y_{k,1};\mathbb{F}_p)$ is a PE-algebra of type $(2p,2p+1)$.

We prove now $H^{2p+1}(Y_{k,1};\hat{\mathbb{Z}}_p)\cong\mathbb{Z}/p$.

We have seen that $Y_{k,1}$, being a p-complete space, is $(2p-1)$-connected. All homology groups of $Y_{k,1}$ are torsion groups, and therefore, $H^{2p}(Y_{k,1};\hat{\mathbb{Z}}_p)=0$. The long exact sequence of cohomology groups associated to the fibration $Y_{k,1}\to X_k\to B^2\mathbb{Z}/p^{k+1}$ contains

$$\begin{aligned}0\to H^{2p+1}(B^2\mathbb{Z}/p^{k+1};\hat{\mathbb{Z}}_p)&\to H^{2p+1}(X_k;\hat{\mathbb{Z}}_p)\\ &\to H^{2p+1}(Y_{k,1};\hat{\mathbb{Z}}_p)\to H^{2p+2}(B^2\mathbb{Z}/p^{k+1};\hat{\mathbb{Z}}_p)\to H^{2p+2}(X_k;\hat{\mathbb{Z}}_p)=0.\end{aligned}$$

The last group vanishes because of remark 5.8. The first two groups are isomorphic, because both measure which high order Bockstein acts nontrivially on ι_2, the generator of $H^*(B^2\mathbb{Z}/p^{k+1};\mathbb{Z}/p)$, or on x^p. In both cases this is $\beta_{(k+2)}$. Thus, we have to calculate $H^{2p+2}(B^2\mathbb{Z}/p^{k+1};\hat{\mathbb{Z}}_p)$. In dimension $2p+1$, the mod-p cohomology of $B^2\mathbb{Z}/p^{k+1}$ is generated by $\iota_2^{p-1}\beta_{(k+1)}(\iota_2)$ and $P^1\beta_{(k+1)}(\iota_2)$. All higher order Bocksteins vanish on $\iota_2^{p-1}\beta_{(k+1)}(\iota_2)$, which therefore comes from an integral class, and $\beta P^1\beta_{(k+1)}(\iota_2)\neq 0$. Thus $H^{2p+2}(B^2\mathbb{Z}/p^{k+1};\hat{\mathbb{Z}}_p)\cong\mathbb{Z}/p$ and $H^{2p+1}(Y_{k,1};\hat{\mathbb{Z}}_p)\cong\mathbb{Z}/p$ as claimed.

Hence, the two generators of $H^*(Y_{k,1};\mathbb{F}_p)$ are connected via the Bockstein. The only algebra over the Steenrod algebra of this type is $\mathbf{B}_{1,1}$ (theorem 2.1).

For $r|p-1$, the space $Y_{k,r}$ fits into the fibration $Y_{k,1}\to Y_{k,r}\to B\mathbb{Z}/r$. A spectral sequence argument establishes the isomorphisms

$$H^*(Y_{k,r};\mathbb{F}_p)\cong H^*(Y_{k,r};\mathbb{F}_p)\cong H^*(Y_{k,1};\mathbb{F}_p)^{\mathbb{Z}/r}\cong \mathbf{B}_{1,1}^{\mathbb{Z}/r}=\mathbf{B}_{1,r}.\quad\square$$

For any map $f:BA\to Y$, A an abelian group, the connected group BA acts on the mapping space $\mathrm{map}(BA,Y)_f$. The Borel construction

$$\mathrm{Bor}(Y,f):=EBA\times_{BA}\mathrm{map}(BA,Y)_f$$

sits in a sequence of fibrations

$$BA\to\mathrm{map}(BA,Y)_f\to\mathrm{Bor}(Y,f)\to B^2A.$$

Lemma 7.2. *Let $f : BA \to Y$ be a map, A a compact abelian group. Then, $e_Y : \mathrm{map}(BA, Y)_f \simeq Y$ if and only if there exists a principal fibration $BA \xrightarrow{f} Y \to \overline{Y}$ and $e_{\overline{Y}} : \mathrm{map}(BA, \overline{Y})_c \simeq \overline{Y}$ where c denotes the constant map. Moreover, $\overline{Y} \simeq \mathrm{Bor}(Y, f)$.*

Proof. First let us assume that there exists a principal fibration $BA \xrightarrow{f} Y \to \overline{Y}$. By 'Thom–theory' ([**26**], revisited in [**21**]) this principal fibration establishes a diagram of principal fibrations

$$(*)\qquad \begin{tikzcd} \mathrm{map}(BA, BA)_c \arrow[r] \arrow[d, "e_{BA}"'] & \mathrm{map}(BA, Y)_f \arrow[r] \arrow[d, "e_Y"'] & \mathrm{map}(BA, \overline{Y})_c \arrow[d, "e_{\overline{Y}}"'] \\ BA \arrow[r] & Y \arrow[r] & \overline{Y} \end{tikzcd}$$

The product $h \cdot g$ of two maps $h : BA \to BA$ and $g : BA \to Y$ is given by the action of BA on Y. In general the fiber in the top row consists of all maps $h : BA \to BA$ such that $h \cdot f \simeq f$, in particular it contains the component of the constant map. The fundamental group $\pi_1(\overline{Y})$ acts on BA via maps homotopic to the identity. The fundamental group $\pi_1(\mathrm{map}(BA, \overline{Y})_c)$ acts on the fiber via this action, which therefore also acts via maps homotopic to the identity. Because the total space of the fibration is connected, this action also must permute the components of the fiber, which is therefore connected and consists only of the component of the constant map. Moreover, because A is a compact abelian group the map e_{BA} is an equivalence.

If $e_{\overline{Y}}$ also is an equivalence, then e_Y is also an equivalence, which proves one half of the statement.

Now we assume that $\mathrm{map}(BA, Y)_f \simeq Y$. The space BA acts on $\mathrm{map}(BA, Y)_f$ with homotopy orbit $\overline{Y} := \mathrm{Bor}(Y, f)$. This establishes the desired principal fibration

$$BA \longrightarrow \mathrm{map}(BA, Y)_f \simeq Y \longrightarrow \overline{Y}\ .$$

Applying 'Thom-theory' again, yields the diagram $(*)$ of principal fibrations. This time the first two vertical maps are equivalences and so is the third one. Moreover, the equivalence of both rows in $(*)$ proves that $\overline{Y} \simeq \mathrm{Bor}(Y, f)$. $\square$

The following lemma may also be found in [**20**].

Lemma 7.3. *Let $K \to G \to H$ be an exact sequence of topological groups. If the evaluation map $\mathrm{map}(BK, Y)_c \to Y$ is an equivalence, then*

$$\mathrm{map}(BH, Y) \to \coprod_{g|_{BK} \simeq c} \mathrm{map}(BG, Y)_g$$

is an equivalence, where c indicates a constant map.

Proof. H acts on $\widetilde{BK} := EG/K \simeq BK$ freely, and on Y trivially. The canonical map $Y \to \mathrm{map}(BK, Y)_c$ is equivariant and an equivalence. Therefore

$$\mathrm{map}(BH, Y) \simeq Y^{hH} \simeq (\mathrm{map}(\widetilde{BK}, Y)_c)^{hH} \simeq \coprod_{g|_{BK} \simeq const} \mathrm{map}(BG, Y)_g.$$

Here Y^{hH} denotes the homotopy fixed point set. The last equivalence follows from [**14**]. □

Now we can state and prove the main result of this section. For $j \leq k+1$, let $Y_k(j)$ denote the homotopy fibre of the composition

$$X_k \to B^2\mathbb{Z}/p^{k+1} \xrightarrow{B^2q} B^2\mathbb{Z}/p^{k+1-j},$$

where $q : \mathbb{Z}/p^{k+1} \to \mathbb{Z}/p^{k+1-j}$ is the projection. These spaces fit into a sequence

$$Y_k := Y_k(0) \to Y_k(1) \to \cdots \to Y_k(k+1) = X_k.$$

The realization $B\mathbb{Z}/p \to X_k$ of the composition $H^*(X_k;\mathbb{F}_p) \to \mathbb{F}_p[x] \to H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ can be lifted to a map $B\mathbb{Z}/p^{k+2} \to Y_k$.

Table 10.4 in the appendix displays the above sequences. In table 10.5 one can read the cohomology algebras of the spaces involved in table 10.4.

Proposition 7.4. (1)

$$H^*(Y_k(j);\mathbb{F}_p) \cong \begin{cases} \mathbf{B}_{1,1}, & j = 0 \\ \mathbf{B}_{0,1}, & j = 1 \\ \mathbf{B}'_{0,1}, & 2 \leq j \leq k \\ \mathbf{A}'_1, & j = k+1. \end{cases}$$

(2) *The spaces fit into fibrations*

$$B\mathbb{Z}/p^l \xrightarrow{g_{j,l}} Y_k(j) \xrightarrow{f_{j,j+l}} Y_k(j+l) \xrightarrow{a_2} B^2\mathbb{Z}/p^l,$$

where $l \leq k-j+1$. The last map classifies the two dimensional class x and is an $H^2(\ ;\mathbb{F}_p)$-isomorphism. The first map is a realization of the composition

$$H^*(Y_k(j);\mathbb{F}_p) \to \mathbb{F}_p[x] \to H^*(B\mathbb{Z}/p^l;\mathbb{F}_p).$$

(3) *For $l \leq k-j+1$, the evaluation $e : \mathrm{map}(B\mathbb{Z}/p^l, Y_k(j))_{g_{j,l}} \to Y_k(j)$ is a homotopy equivalence. Moreover, $Y_k(j+l) \simeq \mathrm{Bor}(Y_k(j), g_{j,l})$.*
(4) *For $l = k-j+2$, there is a fibration*

$$B\mathbb{Z}/p^{l-1} \to \mathrm{map}(B\mathbb{Z}/p^l, Y_k(j))_{g_{j,l}} \to E_k \to B^2\mathbb{Z}/p^{l-1}.$$

(5) *There exists a map $B^2\pi \simeq BS^1{}^{\wedge}_p \to Y_k$, which is a realization of the composition $H^*(Y_k;\mathbb{F}_p) \to \mathbb{F}_p[x] \to H^*(BS^1{}^{\wedge}_p;\mathbb{F}_p)$*

Proof. For $j \geq 1$, a Serre spectral sequence argument for the fibrations

$$B\mathbb{Z}/p^{k+1-j} \to Y_k(j) \to X_k \to B^2\mathbb{Z}/p^{k+1-j}$$

shows that $H^*(Y_k(j);\mathbb{F}_p)$ is a PE-algebra of type (2,3). The action of the Steenrod algebra will be calculated later.

(2) follows from the commutativity of the diagram

$$\begin{tikzcd}
B\mathbb{Z}/p^l \arrow[r] \arrow[d, equal] & B\mathbb{Z}/p^{k+1-j} \arrow[r] \arrow[d] & B\mathbb{Z}/p^{k+1-j-l} \arrow[r] \arrow[d] & B^2\mathbb{Z}/p^l \arrow[d, equal] \\
B\mathbb{Z}/p^l \arrow[r] \arrow[d] & Y_k(j) \arrow[r] \arrow[d] & Y_k(j+l) \arrow[r] \arrow[d] & B^2\mathbb{Z}/p^l \arrow[d] \\
* \arrow[r] \arrow[d] & X_k \arrow[r, equal] \arrow[d] & X_k \arrow[r] \arrow[d] & * \arrow[d] \\
B^2\mathbb{Z}/p^l \arrow[r] & B^2\mathbb{Z}/p^{k+1-j} \arrow[r] & B^2\mathbb{Z}/p^{k+1-j-l} \arrow[r] & B^3\mathbb{Z}/p^l
\end{tikzcd}$$

The conditions on the maps can be easily obtained by looking at the differentials of the associated Serre spectral sequences.

The classifying map $Y_k(j) \to B^2\mathbb{Z}/p^j$ of the fibration $Y_k(0) \to Y_k(j)$ is an $H^*(\ ;\mathbb{F}_p)$-isomorphism in low dimensions. This proves that $\beta(x) = \beta(y) = 0$ for $j \geq 2$ and that $\beta(x) = y$ for $j = 1$, which determines one part of the Steenrod algebra action.

(3) follows from (2) and lemma 7.2. We only have to show that $\mathrm{map}(B\mathbb{Z}/p^l, Y_k(j))_c \simeq Y_k(j)$ for all l and j. For $l = 1$ this is a consequence of theorem 3.4 and [**17**]. Now, lemma 7.3 and an induction over l proves the statement.

To prove (4), we use again lemma 7.3. In this case, it is $l = k - j{+}2$ and $i = k - j{+}1$,

$$\mathrm{map}(B\mathbb{Z}/p^{k-j+2}, Y_k(k+1))_{f_{j,k+1}g_{j,k-j+2}} \simeq \mathrm{map}(B\mathbb{Z}/p, Y_k(k+1))_{g_{k+1,1}},$$

because $f_{j,k+1}g_{j,k+2}|_{B\mathbb{Z}/p^{k-j+1}} \simeq c$, $Y_k(k+1) \simeq X_k$ and $\mathrm{map}(B\mathbb{Z}/p, X_k)_{g_{k+1,1}} \simeq E_k$ (see the proof of 6.1). We can apply the results of section 6 and get a principal fibration

$$\begin{aligned} \mathrm{map}(B\mathbb{Z}/p^{k-j+2}, B\mathbb{Z}/p^{k-j+1})_q \to\ & \mathrm{map}(B\mathbb{Z}/p^{k-j+2}, Y_k(j))_{g_{k-j+2,j}} \\ & \to \mathrm{map}(B\mathbb{Z}/p, Y_k(k+1))_{g_{k+1,1}} \simeq E_k. \end{aligned}$$

The first mapping space is equivalent to $B\mathbb{Z}/p^{k-j+1}$. This establishes the fibration of (4).

The composition

$$B^2\pi \simeq BS^1{}^{\wedge}_p \xrightarrow{p^{k+1}} BS^1{}^{\wedge}_p \longrightarrow E_k \simeq \mathrm{map}(B\mathbb{Z}/p, X_k)_{g_{k+1,1}} \longrightarrow X_k$$

can be lifted to $BS^1{}^{\wedge}_p \to Y_k$. Obviously, this map induces the desired map in mod p cohomology of (5).

To complete the proof of (1), we finally have to calculate $P^1(y)$. For $j = k+1$, there is nothing to show. $P^1(y) \neq 0$, for $j \leq k$, contradicts the fact that $\mathrm{map}(B\mathbb{Z}/p, Y_k(j))_{g_{j,1}} \simeq Y_k(j)$, as theorem 3.1 shows. $\square$

To get a complete picture, we define $X_\infty := S^3\hat{{}_p}$ and $Y_\infty := S^3\langle 3\rangle\hat{{}_p}$. Then,

$$BS^1\hat{{}_p} \to Y_\infty \to X_\infty \to B^2S^1\hat{{}_p} \simeq (B^2\mathbb{Z}/p^\infty)\hat{{}_p}$$

are fibrations. There exists a long sequence of maps

$$Y_\infty \to Y_\infty(1) \to Y_\infty(2) \to \cdots,$$

where $Y_\infty(j)$ is the homotopy fibre of the map $S^3 \xrightarrow{p^j} B^2S^1\hat{{}_p}$ of degree p^j. Moreover, proposition 7.4 holds for Y_∞. The proof is analogous.

Corollary 7.5. *For every $k \leq \infty$ and every $0 \leq j \leq k$, the homotopy type of $Y_k(j)$ determines the homotopy type of every space in the sequence associated to Y_k.*

Proof. Proposition 7.4 (2) and (3). $\square$

Corollary 7.6. *The spaces $Y_{k,r}$ are of different homotopy type.*

Proof. For $r = 1$, this follows from proposition 7.4 (3) and (4). For $r > 1$, the map $Y_{k,1} = Y_k \to Y_{k,r}$ induces an equivalence $Y_k \simeq \mathrm{map}(B\mathbb{Z}/p, Y_k)_{g_{0,1}} \simeq \mathrm{map}(B\mathbb{Z}/p, Y_{k,r})_g$, where $g : B\mathbb{Z}/p \xrightarrow{g_{0,1}} Y_k \longrightarrow Y_{k,r}$. This follows from theorem 3.1 and [**17**]. $\square$

Next we construct a list of spaces realizing the algebras $\mathbf{B}_{0,r}$, for $r|p-1$. Let $s \in \mathbb{Z}/r \subset (\mathbb{Z}/p)^* \cong Aut(\mathbb{Z}/p)$ be a generator. The diagram

$$\begin{tikzcd}
\mathbb{Z}/p \times \mathbb{Z}/p \arrow[r] \arrow[d, "s\times s"'] & \mathbb{Z}/p \arrow[d, "s"'] \\
\mathbb{Z}/p \times \mathbb{Z}/p \arrow[r] & \mathbb{Z}/p
\end{tikzcd}$$

commutes, because s is given by a multiplication. The horizontal arrows are given by addition. Passing to classifying spaces and mapping spaces and taking adjoints yields a commutative diagram

$$\begin{tikzcd}
B\mathbb{Z}/p \times \mathrm{map}(B\mathbb{Z}/p, Y_{k,r})_g \arrow[r] \arrow[d, "s\times \mathrm{map}(s,id)"'] & \mathrm{map}(B\mathbb{Z}/p, Y_{k,r})_g \arrow[d, "\mathrm{map}(s,id)"'] \\
B\mathbb{Z}/p \times \mathrm{map}(B\mathbb{Z}/p, Y_{k,r})_g \arrow[r] & \mathrm{map}(B\mathbb{Z}/p, Y_{k,r})_g.
\end{tikzcd}$$

Here, g denotes the composition $B\mathbb{Z}/p \xrightarrow{g_{0,1}} Y_k \longrightarrow Y_{k,r}$. Because $H^*(Y_{k,r};\mathbb{F}_p)$ is a PE-algebra of type $(2pr, 2pr+1)$, the component of g is fixed by the $\mathbb{Z}/r$-action. Thus, we get a $\mathbb{Z}/r$-action on the quotient $\mathrm{Bor}(Y_{k,r}, g) \simeq \mathrm{Bor}(Y_k, g_{0,1}) \simeq Y_k(1)$, $k \geq 0$. The first equivalence follows from theorem 3.1 and [**17**] and the second equivalence is from lemma 7.2.

Now, we define $Z'_{k,r} := E\mathbb{Z}/r \times_{\mathbb{Z}/r} Y_k(1)$ and $Z_{k,r} := (Z'_{k,r})\hat{{}_p}$. We also put $Z_k = Z_{k,1}$.

Proposition 7.7. *For $0 < k \leq \infty$ and $r|p-1$, all the spaces $Z_{k,r}$ are pairwise not homotopy equivalent, and $H^*(Z_{k,r};\mathbb{F}_p) \cong \mathbf{B}_{0,r}$ as algebras over the Steenrod algebra.*

Proof. For $r=1$, $Z_{k,r} \simeq Y_k(1)$, and $H^*(Y_k(1);\mathbb{F}_p) \cong \mathbf{B}_{0,1}$ by proposition 7.4 (1). For $r>1$, the calculation of $H^*(Z_{k,r};\mathbb{F}_p)$, is analogous to the calculation of $H^*(Y_{k,r};\mathbb{F}_p)$ in the proof of proposition 7.1. Let $f : B\mathbb{Z}/p \to Z_{k,1} \to Z_{k,r}$ be the obvious composition. By theorem 3.1 and [**17**],

$$\mathrm{map}(B\mathbb{Z}/p, Z_{k,r})_f \simeq Z_{k,1} \simeq Y_k(1).$$

Now, the statement follows from corollary 7.5. □

8. Classification of spaces realizing $\mathbf{B}_{i,r}$. In this section we classify up to p-completion the possible homotopy types of spaces realizing $\mathbf{B}_{i,r}$. By theorem 4.3 we only have to consider the cases $i=0,1$. Let $Y_{k,r}$ and $Z_{k,r}$ be the spaces constructed in section 7 with

$$H^*(Y_{k,r};\mathbb{F}_p) \cong \mathbf{B}_{1,r}$$
$$H^*(Z_{k,r};\mathbb{F}_p) \cong \mathbf{B}_{0,r}$$

We will show that these spaces form a complete list of p-complete homotopy types realizing $\mathbf{B}_{1,r}$ and $\mathbf{B}_{0,r}$, respectively.

The next proposition is an immediate consequence of well know properties of the Bockstein spectral sequence (see section 5).

Proposition 8.1. *Let X be a space with p-adic cohomology of finite type over $\hat{\mathbb{Z}}_p$. If $H^*(X;\mathbb{F}_p) = \mathbf{B}_{i,r}$ then in the $\hat{\mathbb{Z}}_p$-cohomology Bockstein spectral sequence $\{B_l, d_l\}$ for X we have $B_\infty = 0$.* □

Theorem 8.2. (1) *If $H^*(X;\mathbb{F}_p) \cong \mathbf{B}_{1,r}$ then there exists $0 \leq k \leq \infty$ such that $\hat{X}_p \simeq Y_{k,r}$.*
(2) *If $H^*(X;\mathbb{F}_p) \cong \mathbf{B}_{0,r}$ then there exists $0 < k \leq \infty$ such that $\hat{X}_p \simeq Z_{k,r}$.*

Proof. (cf. tables 10.4 and 10.5.) Let Y be the p-completion of a space X realizing $\mathbf{B}_{1,r}$. Then Y is 1-connected, p-complete and realizes also $\mathbf{B}_{1,r}$. By 5.7 the p-adic cohomology of Y is of finite type over $\hat{\mathbb{Z}}_p$ and by 8.1 and 5.7 all homotopy groups of Y are finite p-groups.

Consider first the case of $\mathbf{B}_{1,1}$. We will construct a sequence of maps

$$Y := Y(0) \;\to\; Y(1) \;\to\; Y(2) \;\to\; \cdots \quad ,$$

such that $H^*(Y(j);\mathbb{F}_p) \cong \mathbb{F}_p[a_j] \otimes E(b_j)$ isomorphic to either $\mathbf{A}_1^{(j-1)}$ or $\mathbf{B}_{0,1}^{(j-1)}$, $j \geq 1$, and such that there exists a fibration sequence $B\mathbb{Z}/p^j \longrightarrow Y(0) \longrightarrow Y(j) \xrightarrow{a_j} B^2\mathbb{Z}/p^j$, where the last map is algebraically given as in Proposition 7.4 (2).

Let us assume that we already constructed $Y(j)$, $j \geq 1$, with $H^*(Y(j);\mathbb{F}_p) \cong \mathbb{F}_p[a_j] \otimes E(b_j)$ isomorphic to either $\mathbf{A}_1^{(j-1)}$ or $\mathbf{B}_{0,1}^{(j-1)}$. Let $g_{j,1} : B\mathbb{Z}/p \;\to Y(j)$ be the realization of the composition $H^*(Y(j);\mathbb{F}_p) \to \mathbb{F}_p[a_j] \to H^*(B\mathbb{Z}/p;\mathbb{F}_p)$. Then the computation of the T functor on the algebras $\mathbf{A}$ and $\mathbf{B}$ in 3.1 and 3.5 and the results of [**17**] imply that a necessary and sufficient condition for

$$\mathrm{map}(B\mathbb{Z}/p, Y(j))_{f_j} \simeq Y(j)$$

is $P^1(b_j) = 0$, i.e. $H^*(Y(j);\mathbb{F}_p) = \mathbf{B}_{0,1}^{(j-1)}$. Hence, for $P^1(b_j) = 0$, there exists a principal fibration (lemma 7.2)

$$B\mathbb{Z}/p \to Y(j) \to \mathrm{Bor}(Y(j), f_j) =: Y(j+1) \to B^2\mathbb{Z}/p,$$

which fits into the commutative diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & Y(0) \arrow[r, equal] \arrow[d] & Y(0) \arrow[r] \arrow[d] & * \arrow[d] \\
B\mathbb{Z}/p \arrow[r, "f_j"] \arrow[d, equal] & Y(j) \arrow[r] \arrow[d, "a_j"'] & Y(j+1) \arrow[r] \arrow[d, "a_{j+1}"'] & B^2\mathbb{Z}/p \arrow[d, equal] \\
B\mathbb{Z}/p \arrow[r, "a_j f_j"] & B^2\mathbb{Z}/p^j \arrow[r] & B^2\mathbb{Z}/p^{j+1} \arrow[r] & B^2\mathbb{Z}/p\ .
\end{tikzcd}$$

A short calculation shows that $\mathrm{Bor}(B^2\mathbb{Z}/p^j, a_j f_j) \simeq B^2\mathbb{Z}/p^{j+1}$. This establishes the map a_{j+1}. The differentials in the Serre spectral sequence of the fibration given by the right three terms in the middle row are given by the equations $d^2(a_j) = \beta(\iota)$ and $d^2(b_j) = 0$. The equations follow from a comparison with the spectral sequence of the fibration in the bottom row. Now a straightforward calculation shows that $H^*(Y(j+1);\mathbb{F}_p) \cong \mathbb{F}_p[a_{j+1}] \otimes E(b_{j+1})$ is a PE–algebra of type $(2,3)$ with the relation $\beta_{j+1}(a_{j+1}) = b_{j+1}$; i.e. $H^*(Y(j+1);\mathbb{F}_p)$ is isomorphic to either $\mathbf{A}_1^{(j)}$ or $\mathbf{B}_{0,1}^{(j)}$. The relation on the Bockstein follows from the fact $H_2(Y(j+1);\mathbb{Z}) \cong \pi_2(Y(j+1)) \cong \pi_2(B^2\mathbb{Z}/p^{j+1}) \cong \mathbb{Z}/p^{j+1}$. This finishes the induction step.

The construction of $Y(1)$ does not fit into this picture, but it is done in the obvious way by starting with a map $B\mathbb{Z}/p \to Y(0)$ which also is algebraically given as in proposition 7.4.

We can continue as long as $P^1(b_j) = 0$. Let us first assume that we can construct only a finite sequence of spaces and let $Y(k+1)$ denote the last space. Then $H^*(Y(k+1);\mathbb{F}_p) \cong \mathbb{F}_p[x] \otimes E(y)$ is a PE-algebra of type $(2,3)$ such that $\beta_{(k+1)}(x) = y$ and $P^1(y) \neq 0$. Since $P^2(y) = 0$ by unstability, the only possibility for $P^1(y)$ is $P^1(y) = x^{p-1}y$ and so $H^*(Y(k+1);\mathbb{F}_p) \cong \mathbf{A}_1^{(k)}$. This implies that $Y(k+1) \simeq X_k$ (theorem 6.1) and $Y = Y(0) \simeq Y_k = Y_{k,1}$ (corollary 7.5).

If the sequence is infinite, we define $Y(\infty) := \mathrm{hocolim}\, Y(j)$. In the Milnor sequence

$$1 \to \varprojlim{}^1 H^{*+1}(Y(j);\mathbb{F}_p) \to H^*(Y(\infty);\mathbb{F}_p) \to \varprojlim H^*(Y(j);\mathbb{F}_p) \to 1\ ,$$

the first term vanishes because all the groups are finite, and

$$\varprojlim H^*(Y(j);\mathbb{F}_p) \cong H^*(S^3;\mathbb{F}_p).$$

Hence $Y(\infty)\hat{{}_p} \simeq S^3\hat{{}_p}$. Let F be the homotopy fibre of $Y(0) \to Y(\infty)$. Since the direct limit of a directed system of fibrations is again a fibration, the fibration $F \to Y(0) \to Y(\infty)$ is the direct limit of the fibrations

$$B\mathbb{Z}/p^j \to Y(0) \to Y(j).$$

Hence, $F \simeq \varinjlim_j B\mathbb{Z}/p^j = B\mathbb{Z}_{p^\infty}$ and by taking the p-completion we obtain a fibration

$$BS^1{}^\wedge_p \to Y(0) \to S^3{}^\wedge_p$$

classified by a map $S^3{}^\wedge_p \to B\operatorname{Aut}(BS^1{}^\wedge_p)$ into the classifying space of the monoid of self homotopy equivalences of $BS^1{}^\wedge_p$ ([**23**]), which lifts to a map

$$S^3{}^\wedge_p \to BS\operatorname{Aut}(BS^1{}^\wedge_p) \simeq B^2S^1{}^\wedge_p \simeq K(\pi,3) \ .$$

where $S\operatorname{Aut}$ denotes the self homotopy equivalences which are homotopic to the identity. This map is classified by degree and the p adic units are the only possible ones that produce the right cohomology of $Y(0)$ and therefore $Y = Y(0) \simeq Y_\infty$.

Now let Y' be the p-completion of a space realizing $\mathbf{B}_{1,r}$. Let $f : B\mathbb{Z}/p \to Y'$ be a realization of the composition $\mathbf{B}_{1,r} \to \mathbf{B}_{1,1} \to H^*(B\mathbb{Z}/p;\mathbb{F}_p)$. By the computation of the T functor in theorem 3.1 and the results of [**17**], $\operatorname{map}(B\mathbb{Z}/p, Y')_f$ is a realization of $\mathbf{B}_{1,1}$. Thus, there exists an equivalence $h : \operatorname{map}(B\mathbb{Z}/p, Y')_f \simeq Y_k$ for some $0 \le k \le \infty$. The space $\operatorname{map}(B\mathbb{Z}/p, Y')_f$ inherits a $\mathbb{Z}/r$-action from the $\mathbb{Z}/r$-action on $B\mathbb{Z}/p$. The component of f is fixed under this action because $H^*(Y';\mathbb{F}_p) \cong \mathbf{B}_{1,r}$. h induces an equivariant map in $H^*(\ ;\mathbb{F}_p)$ because the canonical map $B\mathbb{Z}/p \to \operatorname{map}(B\mathbb{Z}/p, Y')_f$ is equivariant. By lemma 8.3 below, we can replace h by an equivariant equivalence. Taking homotopy orbits gives equivalences

$$Y' \simeq (E\mathbb{Z}/r \times_{\mathbb{Z}/r} \operatorname{map}(B\mathbb{Z}/p, Y')_f)^\wedge_p \simeq (E\mathbb{Z}/r \times_{\mathbb{Z}/r} Y_k)^\wedge_p \simeq Y_{k,r}.$$

This finishes the proof of part (1).

To prove (2), let Z be the p-completion of a space X realizing the algebra $\mathbf{B}_{0,1}$. The homotopy fibre of the classifying map $x : Z \to B^2\mathbb{Z}/p$ of the 2-dimensional class x is a realization of $\mathbf{B}_{1,1}$ and hence, equivalent to some Y_k, $k \ge 1$. By corollary 7.5 it follows that $Z \simeq Z_k$.

If the p-complete space Z' realizes the algebra $\mathbf{B}_{0,r}$, we can proceed as in the case of $\mathbf{B}_{1,r}$. We have an equivalence $\operatorname{map}(B\mathbb{Z}/p, Z')_f \simeq Z_k$ for some k and for a suitable map $f : B\mathbb{Z}/p \to Z'$. Now all the above arguments go through with minor changes. This shows that $Z' \simeq Z_{k,r}$ and finishes the proof. □

Lemma 8.3. (1) *Let Y be a space equipped with a $\mathbb{Z}/r$ action, and let $h : Y \to Y_k$ be an equivalence, such that $H^*(h;\mathbb{F}_p)$ is equivariant. Then, there exists an equivalence $h' : Y \to Y_k$, which is equivariant.*

(2) *Let Z be a space equipped with a $\mathbb{Z}/r$ action, and let $h : Z \to Z_k$ be an equivalence, such that $H^*(h;\mathbb{F}_p)$ is equivariant. Then, there exists an equivalence $h' : Z \to Z_k$, which is equivariant.*

Proof. We only prove (1), the proof of (2) is analogous. There exists a map $g : B\mathbb{Z}/p^k \to Y$ and an equivalence $f : Y \simeq \operatorname{hofib}(a : \operatorname{Bor}(Y,g) \to B^2\mathbb{Z}/p^k)$, where hofib denotes the homotopy fibre. Analogously to the construction of the spaces $Y_{k,r}$, the $\mathbb{Z}/r$ action on Y passes to $\operatorname{Bor}(Y,f)$ and the map a is equivariant up to homotopy. $\mathbb{Z}/r$ acts on $\mathbb{Z}/p^k$ via the inclusion $\mathbb{Z}/r \subset \mathbb{Z}/p{-}1 \subset \hat{\mathbb{Z}}_p{}^*$. By [**27**] (see lemma 6.5), we can replace a by an equivariant map. This induces a $\mathbb{Z}/r$-action on the homotopy fibre. The homotopy equivalence f is equivariant up to homotopy. This construction can be applied to Y_k as well.

The Borel construction yields an equivalence $\mathrm{Bor}(Y,g) \simeq \mathrm{Bor}(Y_k, g_{0,k})$ which is also equivariant in mod-p cohomology. Both spaces are equivalent to X_k. As in lemma 6.5 we can replace this equivalence by an equivariant equivalence. Taking homotopy fibers produces an equivalence $Y \to Y_k$, which is equivariant up to homotopy. Again, the Wojtkowiak argument establishes an equivalence $Y \to Y_k$ which is equivariant. $\square$

9. Homotopy properties of the constructed spaces.

In [**9**], for any map $f : A \to B$ between spaces, Dror Farjoun constructed a localisation functor

$$L_f : \mathcal{S}paces \;\to\; \mathcal{S}paces \,.$$

Here, $\mathcal{S}paces$ means the category of topological spaces, of CW–complexes, or the simplicial category. In this section we will, among other things, compute the value of this functor when applied to some of the spaces constructed in the previous sections, in the particular case in which f is the map $B\mathbb{Z}/p \to *$. The functor L_f is coaugmented, homotopically idempotent, and takes values among the f–local spaces. A space Y is called f–local, if the map

$$f^*\colon \mathrm{map}(B,Y) \;\to\; \mathrm{map}(A,Y)$$

is a homotopy equivalence. Moreover, the coaugmentation $l : X \to L_f X$ into the localisation $L_f X$ is homotopically universal, i.e. for any map $X \to Z$ into a f–local space Z, there exists a map $L_f X \to Z$, unique up to homotopy, such that

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "l"'] & X \arrow[d] \\ L_f X \arrow[r] & Z \end{tikzcd}$$

commutes up to homotopy. Actually, l induces a homotopy equivalence

$$l^*\colon \mathrm{map}(L_f X, Z) \xrightarrow{\simeq} \mathrm{map}(X,Z)$$

for any f-local space Z. Such functors satisfy several properties by general nonsense arguments; e.g we have

Lemma 9.1. *Let $g : X \to Y$ be a map between spaces, then the following statements are equivalent:*

(1) *g induces a homotopy equivalence $L_f X \simeq L_f Y$.*
(2) *For any f-local space Z, $g^*\colon [Y,Z] \to [X,Z]$ is a bijection.*
(3) *For any f-local space Z, $g^*\colon \mathrm{map}(Y,Z) \to \mathrm{map}(X,Z)$ is a homotopy equivalence.* $\square$

Lemma 9.2. (1) *For any small category $\mathcal{C}$ and for any functor $F : \mathcal{C} \to \mathcal{S}paces$,*

$$L_f(\mathrm{hocolim}_{\mathcal{C}} F) \simeq L_f(\mathrm{hocolim}_{\mathcal{C}}(L_f \circ F)) \ .$$

(2) *The homotopy inverse limit over any small category of f-local spaces is f-local.*

Proof. For an f-local space Z, the map $\mathrm{map}(\mathrm{hocolim}_{\mathcal{C}} L_f \circ F, Z) \to \mathrm{map}(\mathrm{hocolim}_{\mathcal{C}} F, Z)$ can be factored as

$$\mathrm{map}(\mathrm{hocolim}_{\mathcal{C}} L_f \circ F, Z) \simeq \mathrm{holim}_{\mathcal{C}} \mathrm{map}(L_f \circ F, Z) \simeq$$
$$\simeq \mathrm{holim}_{\mathcal{C}} \mathrm{map}(F, Z) \simeq \mathrm{map}(\mathrm{hocolim}_{\mathcal{C}} F, Z).$$

Then, (1) is a consequence of 9.1. The proof of statement (2) is similar. $\square$

We will also use some facts about L_f that we quote from [**11**]:

Lemma 9.3. *Assume that $F \to E \to B$ is a fibration.*

(1) *If $L_f F \simeq *$ then $L_f E \simeq L_f B$.*
(2) *For $f \colon W \to *$, if B is f-local, then L_f preserves the fibration.* $\square$

For a space W, we denote the localisation with respect to the map $W \to *$ by L_W. Then, a space X is W-local if and only if $\mathrm{map}(W, X) \simeq X$ or equivalently, for X connected, if and only if $\mathrm{map}_*(W, X) \simeq *$. In this section we are interested in the localisation with respect to $B\mathbb{Z}/p$.

Some elementary calculations of $L_{B\mathbb{Z}/p}$ are provided by the next two results.

Lemma 9.4. *Let π denote a discrete group,*

(1) *$K(\pi, 1)$ is $B\mathbb{Z}/p$-local if and only if π is p-torsion free.*
(2) *$K(\pi, n)$ is $B\mathbb{Z}/p$-local for all $n \geq 1$ if and only if π is a uniquely p-divisible abelian.*
(3) *If π is a p-group then $L_{B\mathbb{Z}/p}(K(\pi, n)) \simeq *$ if $n > 1$ or π is finite.*

Proof. In general a direct computation of homotopy groups shows that the connected component containing the constant map of $\mathrm{map}(B\mathbb{Z}/p, K(\pi, 1))$ is homotopy equivalent to $K(\pi, 1)$. Now the set of components of $\mathrm{map}(B\mathbb{Z}/p, K(\pi, 1))$ is $\mathrm{Rep}(\mathbb{Z}/p, \pi)$, hence there is a unique component if and only if π is p-torsion free. This proves (1).

From a computation of homotopy groups it follows that $K(\pi, n)$, $n \geq 2$ is $B\mathbb{Z}/p$-local if and only if $H^r(\mathbb{Z}/p; \pi) = 0$ for $1 \leq r \leq n$, that is, if and only if π is uniquely p divisible. This is (2).

Finally we prove (3). Clearly $L_{B\mathbb{Z}/p}(B\mathbb{Z}/p) \simeq *$, then we use induction on the order of π and n in order to get the result for any finite p-group. A general p-group is direct limit of its finite subgroups, hence the result follows by 9.2(1). $\square$

Remark 9.5. The following explicit calculations will be useful later.

(1) For any $n \geq 2$, $L_{B\mathbb{Z}/p} K(\mathbb{Z}, n) \simeq K(\mathbb{Z}[\frac{1}{p}], n)$, and
(2) For any $n \geq 2$, $L_{B\mathbb{Z}/p} K(\hat{\mathbb{Z}}_p, n) \simeq K(\hat{\mathbb{Q}}_p, n)$.

This is computed using the exact sequences $0 \to \mathbb{Z} \to \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}/p^\infty \to 0$ and $0 \to \hat{\mathbb{Z}}_p \to \hat{\mathbb{Q}}_p \to \mathbb{Z}/p^\infty \to 0$ and then applying 9.3 and 9.4.

Examples of $B\mathbb{Z}/p$-local spaces are provided by the Sullivan conjecture:

Lemma 9.6.

(1) *Any finite CW-complex is $B\mathbb{Z}/p$-local. (This is the Sullivan conjecture: [**19**].)*
(2) *Let X be a connected nilpotent space with $H^*(X;\mathbb{F}_p)$ of finite type, then X is $B\mathbb{Z}/p$-local if and only if $H^*(X;\mathbb{F}_p)$ is locally finite as module over the Steenrod algebra. ([**18**])* $\square$

The next lemma lists several properties of $B\mathbb{Z}/p$-local spaces.

Lemma 9.7.

(1) *If Z is $B\mathbb{Z}/p$-local, then Z is $B\mathbb{Z}/p^k$-local for every $1 \leq k \leq \infty$. If, in addition, Z is p-complete, then Z is also BS^1-local.*
(2) *Let Z be a connected space and $\widetilde{Z} \to Z$ the universal covering. If Z is $B\mathbb{Z}/p$–local, then $\widetilde{Z}$ is also $B\mathbb{Z}/p$–local. Reciprocally, if $\pi_1(Z)$ is p-torsion free and $\widetilde{Z}$ is $B\mathbb{Z}/p$-local, then Z is also $B\mathbb{Z}/p$-local.*

The proof is based on the following lemma of Zabrodsky [**28**] (see also [**19**]).

Lemma 9.8. *Let G be a toplogical group and $G \to E \to B$ a principal fibration. If, for a space X, $\mathrm{map}(G,X)_{const} \simeq X$, then*

$$\mathrm{map}(B,X) \simeq \mathrm{map}(E,X)_{f|_G \simeq const} \ .$$

The mapping space $\mathrm{map}(E,X)_{f|_G \simeq const}$ consists of the components of all maps $f : E \to X$, whose restriction $f|_G$ is homotopic to the constant map.

Proof of lemma 9.7. (1). The principal fibration $B\mathbb{Z}/p \to B\mathbb{Z}/p^{k+1} \to B\mathbb{Z}/p^k$, the Zabrodsky lemma and an induction prove (1) for $k < \infty$. Moreover, for a $B\mathbb{Z}/p$–local space Z, the canonical map $\mathrm{map}(B\mathbb{Z}/p^{k+1}, Z) \to \mathrm{map}(B\mathbb{Z}/p^k, Z)$ is a homotopy equivalence. Therefore,

$$\mathrm{map}(B\mathbb{Z}/p^\infty, Z) \simeq \lim_{\overleftarrow{k}} \mathrm{map}(B\mathbb{Z}/p^k, Z) \simeq \lim_{\overleftarrow{k}} Z \simeq Z \ ,$$

and Z is $B\mathbb{Z}/p^\infty$–local. If Z is also p–complete $\mathrm{map}(BS^1, Z) \simeq \mathrm{map}(B\mathbb{Z}/p^\infty, Z) \simeq Z$, which shows that Z is BS^1–local and finishes the proof of (1).

(2). Assume that Z is connected and $B\mathbb{Z}/p$-local. Let $Z \to K := K(\pi_1(Z), 1)$ be the classifying map of the universal covering $\widetilde{Z} \to Z$. Applying the functor $\mathrm{map}(B\mathbb{Z}/p, \ \ )$ establishes a commutative diagram of fibrations

$$\begin{array}{ccccc}
\mathrm{map}(B\mathbb{Z}/p, \widetilde{Z}) & \longrightarrow & \mathrm{map}(B\mathbb{Z}/p, Z) & \longrightarrow & \mathrm{map}(B\mathbb{Z}/p, K)_{const} \\
{\scriptstyle e_{\widetilde{Z}}}\Big\downarrow & & {\scriptstyle e_Z}\Big\downarrow & & {\scriptstyle e_K}\Big\downarrow \\
\widetilde{Z} & \longrightarrow & Z & \longrightarrow & K \ .
\end{array}$$

The vertical arrows are given by the evaluation. In the upper middle term we do not have to consider particular components, because Z is $B\mathbb{Z}/p$ local; i.e. there is only the component of the constant map and e_Z is a homotopy equivalence. Since e_K is also a homotopy equivalence, this is also true for $e_{\widetilde{Z}}$. That is to say that $\widetilde{Z}$ is $B\mathbb{Z}/p$-local.

Finally, if $\pi_1(Z)$ is p-torsion free, $B\pi_1(Z)$ is $B\mathbb{Z}/p$-local by 9.4(1) and then, according to 9.3(2) Z is $B\mathbb{Z}/p$-local if and only if $\widetilde{Z}$ is $B\mathbb{Z}/p$-local. This finishes the proof of the second statement. □

Now we are prepared to start with the calculation of the localisations of the spaces we constructed in the previous sections.

The next result is actually a particular case of a more general result of Neisendorfer.

Lemma 9.9. *S^3, $L_{B\mathbb{Z}/p}S^3\langle 3\rangle$ and $L_{B\mathbb{Z}/p}(S^3\langle 3\rangle\hat{{}_p})$ are homotopy equivalent after completion.*

Proof. By lemma 9.6, S^3 and $S^3\hat{{}_p}$ are $B\mathbb{Z}/p$-local. So, by lemma 9.3 $L_{B\mathbb{Z}/p}$ preserves the fibration $BS^1 \to S^3\langle 3\rangle \to S^3$ as well as its p-completion, hence, by lemma 9.5 we obtain fibrations:

$$K(\mathbb{Z}[\frac{1}{p}], 2) \to L_{B\mathbb{Z}/p}S^3\langle 3\rangle \to S^3$$

and

$$K(\hat{\mathbb{Q}}_p, 2) \to L_{B\mathbb{Z}/p}(S^3\langle 3\rangle\hat{{}_p}) \to S^3\hat{{}_p}$$

and the p-completion of those gives the result. □

Recall that in section 5 the space E_k was defined as the total space of certain fibration

$$BS^1\hat{{}_p} \to E_k \to S^1\hat{{}_p}$$

that has a section $s\colon S^1\hat{{}_p} \to E_k$. Then E'_k is the homotopy cofibre of this section. Finally, X_k was defined as the p completion of E'_k. For the localization of that spaces we obtain:

Lemma 9.10. (1) $(L_{B\mathbb{Z}/p}E_k)\hat{{}_p} \simeq S^1\hat{{}_p}$.

(2) $(L_{B\mathbb{Z}/p}E'_k)\hat{{}_p} \simeq *$.

(3) $L_{B\mathbb{Z}/p}X_k \simeq *$.

Proof. Since $S^1\hat{{}_p}$ is $B\mathbb{Z}/p$-local, $L_{B\mathbb{Z}/p}$ preserves the above fibration and we obtain a fibration

$$K(\hat{\mathbb{Q}}_p, 2) \to L_{B\mathbb{Z}/p}E_k \to S^1\hat{{}_p}\,.$$

The p-completion of this fibration proves (1).

This fibration has also a section $s\colon S^1\hat{{}_p} \to L_{B\mathbb{Z}/p}E_k$. Let C be the homotopy cofibre of this section. Then C is simply connected and mod p acyclic, hence $B\mathbb{Z}/p$-local by lemma 9.6. Now, since a homotopy cofibre is a special sort of homotopy colimit, by 9.2 we obtain

$$L_{B\mathbb{Z}/p}E'_k \simeq L_{B\mathbb{Z}/p}C \simeq C$$

and therefore $(L_{B\mathbb{Z}/p}E'_k)\hat{{}_p} \simeq *$. This is statement (2).

According to the next lemma, (3) follows from (2) because X_k is 1-connected and $H^i(X_k; \hat{\mathbb{Z}}_p)$ is finite for all $i > 0$ (see 5.5 and 5.8.) □

Lemma 9.11. *Let X be a space for which $\hat{X}_p$ is 1-connected. If $H^i(\hat{X}_p;\hat{\mathbb{Z}}_p)$ is finite for all $i>0$ and $(L_{B\mathbb{Z}/p}X)^\frown_p \simeq *$, then $L_{B\mathbb{Z}/p}(\hat{X}_p)\simeq *$.*

Proof. We want to show that for any $B\mathbb{Z}/p$-local space Z, any map $\hat{X}_p \to Z$ factors through a point. that is $[\hat{X}_p, Z] = *$ for any connected $B\mathbb{Z}/p$-local space Z.

If Z is $B\mathbb{Z}/p$-local, the universal covering $\widetilde{Z}$, is also $B\mathbb{Z}/p$-local and since $\hat{X}_p$ is 1-connected

$$[\hat{X}_p, Z] \cong [\hat{X}_p, \widetilde{Z}]/\pi_1(Z)$$

hence it is enough to show that $[\hat{X}_p, Z] = *$ for all Z which is 1-connected and $B\mathbb{Z}/p$-local.

If Z is 1-connected and $B\mathbb{Z}/p$-local, so is $\hat{Z}_p$. Since $H^i(\hat{X}_p;\hat{\mathbb{Z}}_p)$ if finite for all $i>0$ and $\hat{X}_p$ is 1-connected, 5.7 implies that the homotopy groups of $\hat{X}_p$ are finite p-groups. Then, the arithmetic fracture lemma shows that $[\hat{X}_p, \hat{Z}_p] \cong [\hat{X}_p, Z]$ and

$$[\hat{X}_p, Z] \cong [\hat{X}_p, \hat{Z}_p] \cong [L_{B\mathbb{Z}/p}X, \hat{Z}_p] \cong [(L_{B\mathbb{Z}/p}X)^\frown_p, \hat{Z}_p] \cong [*, \hat{Z}_p] = * \qquad \square$$

The spaces $Y_k(j)$ for $k \geq 1$ and $0 \leq j \leq k+1$, were constructed out of X_k in section 7 in such a way that they fit in sequences of fibrations with fibre $B\mathbb{Z}/p$:

$$Y_k(0) \to Y_k(1) \to \ldots \to Y_k(k+1) = X_k.$$

Here $Y_k = Y_k(0)$ is the kth fake $S^3\langle 3\rangle$ and $Y_\infty = S^3\langle 3\rangle^\frown_p$ also fits in one such sequence $Y_\infty \to Y_\infty(1) \to Y_\infty(2) \to \ldots$ with $Y_\infty(\infty) = \mathrm{hocolim}_j\, Y_\infty(j)$ and $Y_\infty(\infty)^\frown_p \simeq S^3{}^\frown_p$.

Theorem 9.12. *For all $0 \leq j \leq k+1 \leq \infty$,*

$$L_{B\mathbb{Z}/p}Y_k(j) \simeq L_{B\mathbb{Z}/p}Y_k(0) = L_{B\mathbb{Z}/p}Y_k \simeq \begin{cases} Y_\infty(\infty) & \text{for } k=\infty \\ * & \text{for } k<\infty \end{cases}$$

Remark. Compare with 9.9 for $k=\infty$.

Proof. The principal fibrations $B\mathbb{Z}/p \to Y_k(j) \to Y_k(j+1)$ and lemma 9.2 establish equivalences $L_{B\mathbb{Z}/p}Y_k(j) \simeq L_{B\mathbb{Z}/p}Y_k(j+1)$.

Therefore, if $k<\infty$ we have $L_{B\mathbb{Z}/p}Y_k(j) \simeq L_{B\mathbb{Z}/p}X_k \simeq *$ by 9.10.

For $k=\infty$ we have first that $Y_\infty(\infty)$ is 1-connected and its mod p cohomology is finite hence it is $B\mathbb{Z}/p$-local by 9.6(2). Then the result follows from 9.1 because for any $B\mathbb{Z}/p$-local space Z the map $Y_\infty(j) \to \mathrm{hocolim}_j\, Y_\infty(j) = Y_\infty(\infty)$ induces

$$\begin{aligned} &\mathrm{map}(Y_\infty(\infty), Z) \xrightarrow{\simeq} \mathrm{holim}_j\, \mathrm{map}(Y_\infty(j), Z) \xrightarrow{\simeq} \\ &\qquad \xrightarrow{\simeq} \mathrm{holim}_j\, \mathrm{map}(L_{B\mathbb{Z}/p}Y_\infty(j), Z) \xrightarrow{\simeq} \mathrm{map}(L_{B\mathbb{Z}/p}Y_\infty(j), Z)\ . \qquad \square \end{aligned}$$

Corollary 9.13. *If $k \neq \infty$, the l-fold suspensions of $Y_\infty(j)$ and of $Y_k(j)$ are not homotopy equivalent for all l.*

Proof. By theorem 9.12, $\Sigma^l L_{B\mathbb{Z}/p} Y_\infty(j) \simeq \Sigma^l Y_\infty(\infty)$ and this is 1-connected with finite mod p cohomology hence $B\mathbb{Z}/p$-local. The suspensions are homotopy colimits. By lemma 9.2 we have

$$L_{B\mathbb{Z}/p}\Sigma^l Y_k(j) \simeq L_{B\mathbb{Z}/p}\Sigma^l L_{B\mathbb{Z}/p} Y_k(j) \simeq L_{B\mathbb{Z}/p}\Sigma^l Y_\infty(\infty) \simeq \Sigma^l Y_\infty(\infty) \ .$$

But for $k < \infty$, the same argument proves $(L_{B\mathbb{Z}/p}\Sigma^l Y_k(j)) \simeq *$. $\square$

Similiar arguments as in the proof of theorem 9.5 show that $L_{B\mathbb{Z}/p}(Y_\infty(j) \times_{\mathbb{Z}/r} E\mathbb{Z}/r) \simeq Y_\infty(\infty) \times_{\mathbb{Z}/r} E\mathbb{Z}/r$, where $\mathbb{Z}/r$ acts canonically on $Y_\infty(\infty)$. This space is mod–p acyclic and using lemma 9.11 we deduce that $L_{B\mathbb{Z}/p} Y_{\infty,r} \simeq *$. For $k < \infty$, one also can prove that $L_{B\mathbb{Z}/p} Y_{k,r} \simeq *$. Thus, the above application of the localisation functor does not see any difference between the suspensions of the different realisations of $\mathbf{B}_{i,r}$. But there is a way to distinguish between these spaces.

Let $g : B\mathbb{Z}/p \to Y_{k,r}$ be the map of section 7, and let $h : Y_k \times B\mathbb{Z}/p \to Y_{k,r}$ be the adjoint of the equivalence $Y_k \simeq \mathrm{map}(B\mathbb{Z}/p, Y_{k,r})_g$. $\mathbb{Z}/(p-1)$ acts on Y_k. For every $a \in \mathbb{Z}/p^*$ we have a map

$$\Sigma^l Y_k \times B\mathbb{Z}/p \xrightarrow{a \times id} \Sigma^l Y_k \times B\mathbb{Z}/p \longrightarrow \Sigma^l (Y_k \times B\mathbb{Z}/p) \xrightarrow{\Sigma^l h} \Sigma^l Y_{k,r} \ .$$

which has as adjoint a map

$$f_a : \Sigma^l Y_k \longrightarrow \mathrm{map}(B\mathbb{Z}/p, \Sigma^l Y_{k,r}) \ .$$

If a and b differ by a r-th power, the two associated maps f_a and f_b are homotopic, because $Y_{k,r}$ is the homotopy orbit of the $\mathbb{Z}/r$–action on Y_k. There is also an obvious map $\Sigma^l Y_{k,r} \to \mathrm{map}(B\mathbb{Z}/p, \Sigma^l Y_{k,r})$, which is the standard section of the evaluation.

Let $s = (p-1)/r$. Then, $\mathbb{Z}/s \subset \mathbb{Z}/p^*$ consists of the congruence classes modulo r-th powers. All these maps together fit into a map

$$f : \Sigma^l Y_{k,r} \vee \bigvee_{a \in \mathbb{Z}/s} \Sigma^l Y_k \ \to \ \mathrm{map}(B\mathbb{Z}/p, \Sigma^l Y_{k,r}) \ .$$

Theorem 9.14. *The map*

$$f : \Sigma^l Y_{k,r} \vee \bigvee_{a \in \mathbb{Z}/s} \Sigma^l Y_k \ \to \ \mathrm{map}(B\mathbb{Z}/p, \Sigma^l Y_{k,r})$$

is a homotopy equivalence.

Proof. We have to calculate the mapping space using the T–functor The T–functor is exact and commutes with suspensions. Hence,

$$\begin{aligned} T(H^*(\Sigma^l Y_{k,r}; \mathbb{F}_p)) &\cong T(\widetilde{H}^*(\Sigma^l Y_{k,r}; \mathbb{F}_p) \oplus T(\mathbb{Z}/p) \\ &\cong \Sigma^l T(H^*(\Sigma^l Y_{k,r}; \mathbb{F}_p)) \oplus \mathbb{Z}/p \\ &\cong (\bigoplus_{a \in \mathbb{Z}/s} \Sigma^l H^*(Y_k; \mathbb{F}_p)) \oplus \Sigma^l H^*(Y_{k,r}; \mathbb{F}_p) \oplus \mathbb{Z}/p \\ &\cong H^*((\bigvee_{a \in \mathbb{Z}/s} \Sigma^l Y_k) \vee \Sigma^l Y_{k,r}; \mathbb{F}_p) \ . \end{aligned}$$

All the isomorphism are obvious, but the second last. This one follows from theorem 3.1, the identity $T_{const}H^*(Y_{k,r};\mathbb{Z}/p) \cong H^*(Y_{k,r};\mathbb{Z}/p)$, the fact that every map $H^*(Y_{k,r};\mathbb{F}_p) \to H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ factors over $H^*(Y_{k,};\mathbb{Z}/p)$ and that every two factorizations differ by an r-th power. By construction, this series of isomorphisms is just the map induced by f, which shows that f is a mod–p equivalence [**17**]. The integral homology of $(\bigvee_{a\in\mathbb{Z}/s}\Sigma^l Y_k)\vee\Sigma^l Y_{k,r}$ consists of finite p-torsion in each dimension. Moreover, the space is 3-conected. The mod $\mathcal{C}$ Hurewicz theorem for the class of finite p-groups shows that $\pi_*((\bigvee_{a\in\mathbb{Z}/s}\Sigma^l Y_k)\vee\Sigma^l Y_{k,r})$ consists of finite p-torsion in each dimension. Hence, by [**6**] $(\bigvee_{a\in\mathbb{Z}/s}\Sigma^l Y_k)\vee\Sigma^l Y_{k,r}$ is p-complete. Because $Y_{k,r}$ is p–complete, f is an equivalence ([**17**]). □

Corollary 9.15. *If $k \neq \infty$, the l–fold suspensions $\Sigma^l Y_{\infty,r}$ and $\Sigma^l Y_{k,r}$ are not homotopy equivalent for all l.*

Proof. Applying localisation to the mapping space gives

$$\begin{aligned} L_{B\mathbb{Z}/p}(\mathrm{map}(B\mathbb{Z}/p,\Sigma^l Y_{k,r})) &\simeq L_{B\mathbb{Z}/p}(\Sigma^l Y_{k,r}\vee\bigvee_{a\in\mathbb{Z}/s}\Sigma^l Y_k) \\ &\simeq L_{B\mathbb{Z}/p}(L_{B\mathbb{Z}/p}(\Sigma^l Y_{k,r})\vee\bigvee_{a\in\mathbb{Z}/s}L_{B\mathbb{Z}/p}(\Sigma^l Y_k)) \\ &\simeq L_{B\mathbb{Z}/p}(\Sigma^l L_{B\mathbb{Z}/p}(Y_{k,r})\vee\bigvee_{a\in\mathbb{Z}/s}\Sigma^l L_{B\mathbb{Z}/p}(Y_k))\ . \end{aligned}$$

The last two equivalences follow from lemma 9.1 and lemma 9.2. By proposition 9.13 and the following remarks, for $k=\infty$ and $k<\infty$, these spaces cannot be homotopy equivalent. □

Remark 9.16. Using the same methods and ideas, one can also distinguish between the l–fold suspensions of $Z_{k,r}$, $k<\infty$, and $Z_{\infty,r}$.

Finally, we discuss the question of which of these spaces are H–spaces. The spaces $Y_\infty = S^3\langle 3\rangle\hat{}_p$ and $Z_\infty = Y_\infty(1)$ are loop spaces, in particuliar H-spaces. This follows easily from the construction. But these are the only ones among the spaces $Y_{k,r}$ and $Z_{k,r}$ which are H-spaces, as the following proposition shows.

Proposition 9.17. *For $k<\infty$ or $r>1$, the spaces $Y_{k,r}$ and $Z_{k,r}$ cannot carry an H-space structure.*

Proof. If the algebras $\mathbf{B}_{0,r}$ and $\mathbf{B}_{1,r}$ have the structure of a Hopf algebra, they are primitively generated. For $r>1$, the Steenrod power P^{p^i}, $i=0,1$, maps the primitive $2p^i r$-dimensional class on a nonprimitive class. Hence, $\mathbf{B}_{0,r}$ and $\mathbf{B}_{1,r}$ are Hopf algebras only for $r=1$.

Now, we assume that $Y_k(j)$, $0 \leq j \leq k$, is an H-space. We consider the diagram

$$\begin{tikzcd}
B\mathbb{Z}/p \times B\mathbb{Z}/p \arrow[r, "\mu"] \arrow[d, "g_{j,1}\times g_{j,1}"'] & B\mathbb{Z}/p \arrow[d, "g_{j,1}"'] \\
Y_k(j) \times Y_k(j) \arrow[r, "\mu"] \arrow[d] & Y_k(j) \arrow[d, "f_{j,j+1}"'] \\
Y_k(j+1) \times Y_k(j+1) & Y_k(j+1)
\end{tikzcd} \tag{3}$$

where μ denotes the multiplication. The upper square commutes in mod-p cohomology, because in $H^*(Y_k(j);\mathbb{F}_p)$ the 2-dimensional class for $j \geq 1$ and the $2p$-dimensional class for $j = 0$ are primitive. Thus, the upper square commutes up to homotopy. The obvious composition $f_{j,j+1}\mu(g_{j,1} \times g_{j,1}) : B\mathbb{Z}/p \times B\mathbb{Z}/p \to Y_k(j+1)$ is homotpic to the constant map and, by theorem 3.1 and taking the adjoint,

$$\begin{aligned}
\mathrm{map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, Y_k(j+1))_c &\simeq \mathrm{map}(B\mathbb{Z}/p, \mathrm{map}(B\mathbb{Z}/p, Y_k(j+1))_c)_c \\
&\simeq \mathrm{map}(B\mathbb{Z}/p, Y_k(j+1))_c \\
&\simeq Y_k(j+1).
\end{aligned}$$

Both vertical columns in (3) are principal fibrations. We can apply lemma 9.3, which establishes a map

$$\mu : Y_k(j+1) \times Y_k(j+1) \to Y_k(j+1)$$

making the lower square commutative up to homotopy. As easily shown, μ is an H-space structure on $Y_k(j+1)$.

If $Y_k = Y_k(0)$ is an H-space, the above induction procedure shows that $Y_k(k+1) = X_k$ carries also an H-space structure. But this is a contradiction, because the Steenrod power P^1 maps the 'primitive' 3-dimensional class of $H^*(X_k;\mathbb{F}_p)$ onto a nonprimitive class. $\square$

10. Appendix. Through this paper we have introduced several families of algebras over the Steenrod algebra $\mathbf{A}_r$, $\mathbf{B}_{i,r}$, $\mathbf{C}_r$, etc. as well as several families of spaces $E_k(r)$, $X_k(r)$, $Y_{k,r}$, etc. We think that the reader would find helpful to have the definitions of these algebras and spaces and the relationships between them displayed in a set of tables. In this appendix we include the following tables: Table 10.1 contains the definitions of the algebras over the Steenrod algebra introduced in section 2. Table 10.2 contains the definitions and the mod p cohomology of the spaces introduced in sections 5 and 7. In table 10.3 we list some fibrations between these spaces. Tables 10.4 and 10.5 display the spaces used in the proof of theorem 8.2 and their cohomology algebras.

$\mathbf{A}_r = \mathbf{A}_r^{(0)} = \mathbb{F}_p[x] \otimes E(y)$	$\deg(x) = 2r$ $\deg(y) = 2r+1$	$\beta(x) = y$ $P^1(y) = rx^s y$
$\mathbf{B}_{i,r} = \mathbf{B}_{i,r}^{(0)} = \mathbb{F}_p[x] \otimes E(y)$	$\deg(x) = 2p^i r$ $\deg(y) = 2p^i r + 1$	$\beta(x) = y$ $P^{p^i}(y) = (r-1)x^s y$
$\mathbf{C}_r = \mathbf{C}_r^{(0)} = \mathbb{F}_p[x] \otimes E(z)$	$\deg(x) = 2r$ $\deg(z) = 1$	$\beta(x) = xz$
$\mathbf{A}'_r$	same as $\mathbf{A}_r$ but with $\beta(x) = \beta(y) = 0$	
$\mathbf{B}'_{i,r}$	same as $\mathbf{B}_{i,r}$ but with $\beta(x) = \beta(y) = 0$	
$\mathbf{C}'_r$	same as $\mathbf{C}_r$ but with $\beta(x) = \beta(z) = 0$	
$\mathbf{A}_r^{(k)}$	equal to $\mathbf{A}'_r$ as algebras over the Steenrod algebra but with $\beta_{(j)}(x) = 0$ for $j \leq k$ and $\beta_{(k+1)}(x) = y$	
$\mathbf{B}_{i,r}^{(k)}$	equal to $\mathbf{B}'_{i,r}$ as algebras over the Steenrod algebra but with $\beta_{(j)}(x) = 0$ for $j \leq k$ and $\beta_{(k+1)}(x) = y$	
$\mathbf{C}_r^{(k)}$	equal to $\mathbf{C}'_r$ as algebras over the Steenrod algebra but with $\beta_{(j)}(x) = 0$ for $j \leq k$ and $\beta_{(k+1)}(x) = xz$	

Table 10.1

E	$E_k = EG \times_\pi B^2\pi^{\phi_k}$, $k \geq 0$ $E_k(r) = E_k/\mathbb{Z}/r$, $k \geq 0$, $r \mid p-1$ $E'_k(r) = \mathrm{Cofibre}\,(B\pi \to E_k(r))$, $k \geq 0$, $r \mid p-1$	$H^*(E_k;\mathbb{F}_p) \cong \mathbf{C}_1^{(k)}$, (5.1) $H^*(E_k(r);\mathbb{F}_p) \cong \mathbf{C}_r^{(k)}$, (5.3) $H^*(E'_k(r);\mathbb{F}_p) \cong \mathbf{A}_r^{(k)}$
X	$X_k(r) = (E'_k(r))^{\wedge}_p$, $k \geq 0$, $r \mid p-1$ $X_k = X_k(1)$, $k \geq 0$ $X_\infty \cong (S^3)^{\wedge}_p$	$H^*(X_k(r);\mathbb{F}_p) \cong \mathbf{A}_r^{(k)}$, (5.5)
Y	$Y_k = \mathrm{Fibre}\,(X_k \to B^2\mathbb{Z}/p^{k+1})$, $k \geq 0$ $Y_\infty = (S^3\langle 3\rangle)^{\wedge}_p$ $Y_{k,r} = (E\mathbb{Z}/r \times_{\mathbb{Z}/r} Y_k))^{\wedge}_p$, $0 \leq k \leq \infty$, $r \mid p-1$ $Y_k(j) = \mathrm{Fibre}\,(X_k \to B^2\mathbb{Z}/p^{k+1-j})$ $k \geq 0,\ 0 \leq j \leq k+1$ $Y_\infty(j) = \mathrm{Fibre}\,(S^3 \xrightarrow{p^j} (B^2S^1)^{\wedge}_p)$, $j \geq 0$ $Y_k(0) = Y_{k,1} = Y_k$ $Y_k(k+1) = X_k$	$H^*(Y_{k,r};\mathbb{F}_p) \cong \mathbf{B}_{1,r}$, (7.1) $H^*(Y_k(j);\mathbb{F}_p) \cong \begin{cases} \mathbf{B}_{1,1}, & j=0 \\ \mathbf{B}_{0,1}, & j=1 \\ \mathbf{B}'_{0,1}, & 2 \leq j \leq k \\ \mathbf{A}'_1, & j=k+1 \end{cases}$ (7.4)
Z	$Z_{k,r} = (E\mathbb{Z}/r \times_{\mathbb{Z}/r} Y_k(1))^{\wedge}_p$, $0 < k \leq \infty$, $r \mid p-1$ $Z_k = Z_{k,1} = Y_k(1)$, $0 < k \leq \infty$	$H^*(Z_{k,r};\mathbb{F}_p) \cong \mathbf{B}_{0,r}$, (7.7)

Table 10.2

$$\begin{gathered}
B^2\pi \to E_k \to B\pi \\
E_k \to E_k(r) \to B\mathbb{Z}/r \\
Y_k \to X_k \to B^2\mathbb{Z}/p^{k+1} \\
Y_k \to Z_k \to B^2\mathbb{Z}/p \\
Z_k \to X_k \to B^2\mathbb{Z}/p^k \\
Y_k(j) \to Y_k(j+l) \to B^2\mathbb{Z}/p^l, \qquad l \leq k-j+1 \\
Y_k(j) \to X_k \to B^2\mathbb{Z}/p^{k+1-j}, \qquad j \geq 1 \\
Y_\infty \to X_\infty \to (B^2S^1)^{\wedge}_p
\end{gathered}$$

Table 10.3

$$
\begin{array}{llllll}
S^3\langle 3\rangle^{\wedge}_p = Y_\infty \to Y_\infty(1) = Z_\infty \to Y_\infty(2) & \to Y_\infty(3) & \to \cdots & & & \cdots \to S^{3\wedge}_p \\
\quad\vdots & & & & & \\
Y_k \to Y_k(1) = Z_k \to Y_k(2) & \to Y_k(3) & \to \cdots & \to Y_k(k+1) = X_k & & \\
\quad\vdots & & & & & \\
Y_3 \to Y_3(1) = Z_3 \to Y_3(2) & \to Y_3(3) & \to Y_3(4) = X_3 & & & \\
Y_2 \to Y_2(1) = Z_2 \to Y_2(2) & \to Y_2(3) = X_2 & & & & \\
Y_1 \to Y_1(1) = Z_1 \to Y_1(2) = X_1 & & & & & \\
Y_0 \to Y_0(1) = X_0 & & & & &
\end{array}
$$

Table 10.4

$$
\begin{array}{lllllll}
\mathbf{B}_{1,1} & \mathbf{B}_{0,1} & \mathbf{B}^{(1)}_{0,1} & \mathbf{B}^{(2)}_{0,1} & \cdots & \cdots & \cdots \\
\vdots & & & & & & \\
\mathbf{B}_{1,1} & \mathbf{B}_{0,1} & \mathbf{B}^{(1)}_{0,1} & \mathbf{B}^{(2)}_{0,1} & \cdots & \mathbf{A}^{(k)}_1 & \\
\vdots & & & & & & \\
\mathbf{B}_{1,1} & \mathbf{B}_{0,1} & \mathbf{B}^{(1)}_{0,1} & \mathbf{B}^{(2)}_{0,1} & \mathbf{A}^{(3)}_1 & & \\
\mathbf{B}_{1,1} & \mathbf{B}_{0,1} & \mathbf{B}^{(1)}_{0,1} & \mathbf{A}^{(2)}_1 & & & \\
\mathbf{B}_{1,1} & \mathbf{B}_{0,1} & \mathbf{A}^{(1)}_1 & & & & \\
\mathbf{B}_{1,1} & \mathbf{A}_1 & & & & &
\end{array}
$$

Table 10.5

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