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Cohomologia quântica tangencial e geometria enumerativa de curvas racionais

por

JOACHIM KOCK

Til mindet om min morfar

Til familie og venner

Kunsten at tælle er den ældste og mest grundlæggende disciplin i matematik. To af de mest basale resultater, som de fleste kender, kan betragtes som tællespørgsmål: I algebra, "en andengradsligning har altid netop to løsninger (forudsat da diskriminanten er positiv)". Og i geometri, "gennem to punkter går der netop én ret linie". Denne sidste sætning kan opfattes som grundsætningen i den disciplin, der kaldes enumerativ geometri, og som er det felt min Ph.D.-afhandling falder ind under. Nærmere bestemt drejer det sig om at tælle rationelle kurver som går igennem givne punkter og tangerer andre givne linier eller flader.

For hver grad kan man stille nye spørgsmål. Grad 1 svarer til rette linier, grad 2 svarer til keglesnit, og så fremdeles — jo højere grad, des mere krøllede bliver kurverne. Mange af den slags spørgsmål, for kurver af lav grad, blev besvaret allerede i det nittende århundrede, blandt andet af den danske matematiker Hieronymus Zeuthen [50], og det har været en af de store udfordringer for det tyvende århundredes matematik at fundamentere den klassiske æras geometriske indsigt og teknikker, så de lever op til moderne krav om stringens.

I løbet af de sidste ti år har den enumerative geometri gennemlevet store omvæltninger. Helt uventet har idéer og metoder fra moderne teoretisk fysik fundet anvendelser i disciplinen og bragt ambitionerne op på et højere niveau: Man er ikke længere tilfreds med at løse problemerne grad for grad — det har jo ingen ende — istedet søger man nu at finde formler der løser problemerne for vilkårlig grad.

To af hovedresultaterne i min afhandling, Sætning 2.3.1 og 2.4.1, er netop formler af den slags. Imidlertid involverer formlerne ikke selve løsningerne til tælleproblemerne; symbolerne i formlerne refererer til abstrakte objekter og begreber med navne der røber deres oprindelse i fysik. Retfærdiggørelsen for dette abstraktionsniveau er at resultaterne kan anvendes i mange forskellige situationer, og de indeholder så meget geometrisk information, at man ved hjælp af en computer nemt kan løse konkrete tælleproblemer i bunkevis. I tabellerne på side 78–87 kan man se masser af eksempler på konkrete facit — og det er jo ganske almindelige hele tal!

JOACHIM

Para família e amigos

A arte de contar é a disciplina mais antiga e fundamental da matemática. Dois dos resultados mais básicos, conhecidos pela maioria das pessoas, podem ser considerados como questões de contagem: Na álgebra, "uma equação de segundo grau sempre possui duas soluções (assumindo que o discriminante é positivo)". E na geometria, "através de dois pontos passa uma única reta". Este último resultado pode ser visto como o teorema fundamental da disciplina chamada geometria enumerativa que é a área à qual pertence minha tese de doutorado. Mais precisamente se trata de contar curvas racionais que passam por certos pontos e tangenciam determinadas retas ou superfícies.

Para cada grau se pode fazer novas perguntas. Grau 1 corresponde a retas, grau 2 diz respeito a cônicas (por exemplo círculos), e assim por diante — quanto mais alto o grau, mais as curvas ficam enroladas. Muitas perguntas desse tipo, para grau baixo, foram resolvidas já no século XIX, pelo matemático dinamarquês Hieronymus Zeuthen [50], entre outros, e tem sido um grande desafio para a matemática do século XX fundamentar as técnicas e o conhecimento geométricos da era clássica, para responderem à exigência de rigor moderna.

Durante os últimos dez anos, a geometria enumerativa tem vivenciado uma grande revolução. De forma inesperada, idéias e métodos da física teórica moderna têm encontrado aplicação na disciplina e trazido as ambições para um nível mais elevado: não se contenta mais de resolver os problemas grau por grau — isso nunca teria fim — a meta agora é encontrar fórmulas que resolvem os problemas para grau arbitrário.

Dois dos resultados principais da minha tese, Teoremas 2.3.1 e 2.4.1, são justamente fórmulas desse tipo. Porém, as fórmulas não envolvem diretamente as soluções dos problemas de contagem; os símbolos das fórmulas referem a objetos e conceitos abstratos cujos nomes revelam sua origem na física. A justificação para este nível de abstração é que os resultados podem ser aplicados em muitas situações diferentes, e contêm tanta informação geométrica que fica fácil, ajudado por um computador, resolver em massa problemas de contagem concretos. Nas tabelas nas páginas 78–87 são compiladas centenas de exemplos de soluções concretas — e afinal estas são números inteiros familiares!

Joachim

For my family and friends

The art of counting is the oldest and most fundamental discipline of mathematics. Two of the most basic results, known to most people, can be viewed as counting questions: In algebra, “a second-degree equation has got two solutions (provided the discriminant is positive)”. And in geometry, “through two points there is a unique straight line”. This last statement can be considered the fundamental result in the discipline called enumerative geometry, which is the field my Ph.D. thesis belongs to. More precisely the purpose is to count rational curves which pass through given points and are tangent to specified lines or surfaces.

For each degree one can ask new questions. Degree 1 corresponds to straight lines, degree 2 corresponds to conics (for example circles), and so forth — the higher the degree, the more curly the curves become. Many of these problems, for curves of low degree, were solved already in the nineteenth century by the Danish mathematician Hieronymus Zeuthen [50], among others, and it has been one of the great challenges for the mathematics of the twentieth century to give solid foundations to the geometric insight and techniques of the classical era, to make them meet modern standards of rigour.

During the last decade, enumerative geometry has gone through big revolutions. Unexpectedly, ideas and methods from modern theoretical physics have found applications in the discipline and raised the ambitions to a much higher level: it is no longer satisfactory to solve the problems degree for degree — there is no end to this — instead one endeavours for formulae that solve the problems for arbitrary degree.

Two of the main results of my thesis, Theorem 2.3.1 and 2.4.1, are just formulae of this sort. However, the expressions do not involve the numerical solutions to the counting problems themselves; the symbols in the formulae refer to abstract objects and concepts whose names reveal their origin in physics. The justification for this level of abstraction is that the results can be applied in many different situations, and they contain so much geometric information that it is easy, assisted by a computer, to solve heaps of concrete counting problems. In the tables on page 78–87 there are lots of examples of such concrete answers — and after all they are just plain familiar integers!

JOACHIM

Officiel sammenfatning

Tangens kvante-kohomologi og rationelle kurvers enumerative geometri

Gromov-Witten invarianterne er integraler over moduli-rum af stabile afbildninger. I enumerativ geometri har de en fortolkning som løsninger på tællespørgsmål vedrørende rationelle kurver underlagt incidensbetingelser, som for eksempel: "hvor mange rationelle kubiske plane kurver går igennem otte givne punkter?" De nylige teorier om kvante-kohomologi (som har deres oprindelse i teoretisk fysik) tillader rekursiv beregning af invarianterne, og udgør på den måde en overraskende løsning på enumerative problemer som i over hundrede år har modstået intens forskningsaktivitet.

En vigtig generalisering af Gromov-Witten invarianter er at indføje de såkaldte psi-klasser i integralerne, hvilket i teoretisk fysik svarer til at inddrage tyngdekraften i kvanteteorierne.

Nærværende afhandling præsenterer en forbindelse mellem tyngdekraftsteorierne og den enumerative geometri for rationelle kurver underlagt betingelser af infinitesimal karakter, som for eksempel det at tangere en given linie. Grundideen er at introducere en modificering af psi-klasserne hvis egenskaber gør den velegnet til enumerative formål.

Første del af afhandlingen behandler de grundlæggende egenskaber ved disse nye klasser og etablerer visse differentialligninger for de frembringende funktioner hørende til deres integraler. Dette er tilstrækkelig information til effektiv beregning af invarianterne og afslører nye dybe strukturer i moduli-rummene. En af differentialligningerne kan opfattes som en associativitetsrelation for en ny kvante-multiplikation, kaldet tangens-kvante-produktet, der udgør en overraskende generalisering af det sædvanlige kvante-produkt.

Anden del af afhandlingen viser hvorledes de modificerede psi klasser virkelig er velegnede til at beskrive enumerative problemstillinger angående rationelle kurver underlagt tangensbetingelser (såvel som betingelser på deres singulariteter). Denne del indbefatter nye anvendelser af redskaber fra klassisk enumerativ geometri, som for eksempel Porteous' formel og Bertini's sætning. Når først forbindelsen til de modificerede psi klasser er etableret, giver differentialligningerne fra første del løsningerne, ikke bare på tælleproblemer angående kurver i projektivt rum, men mere generelt i homogene varieteter. Også i de simpleste tilfælde, som for eksempel optælling af plane kurver, hvor flere problemer er blevet løst inden for de sidste par år, repræsenterer teknikkerne i nærværende afhandling et betydeligt fremskridt, idet de forskelligartede problemer herved finder en samlet løsning, der ovenikøbet er enklere og mere begrebsmæssig.

Resumo oficial

Cohomologia quântica tangencial e geometria enumerativa de curvas racionais

Invariantes de Gromov-Witten são integrais sobre espaços de módulos de mapas estáveis. Na geometria enumerativa eles têm uma interpretação como soluções a problemas de contagem de curvas racionais sujeitas a condições de incidência, como por exemplo: "quantas cúbicas racionais planas passam por oito pontos gerais?" A recente teoria de cohomologia quântica (que tem sua origem na física teórica) permite o cálculo recursivo dos invariantes, providenciando desta forma uma solução inesperada a problemas enumerativos que há mais de cem anos resistem a intensa investigação.

Uma generalização importante dos invariantes de Gromov-Witten é a inclusão nas integrais das chamadas classes de psi, o que na física teórica corresponde à introdução da gravitação na teoria quântica.

Esta tese apresenta uma ligação entre a teoria gravitacional e a geometria enumerativa de curvas racionais sujeitas a condições de natureza infinitesimal, como por exemplo de tangência. A idéia é introduzir uma modificação das classes de psi, cujas propriedades a fazem bem-feita para fins enumerativos.

A primeira parte da tese trata das propriedades dessas novas classes e estabelece equações diferenciais para as funções geratrizes correspondentes às integrais, informação esta que permite uma computação efetiva dos invariantes e revela uma nova estrutura profunda dos espaços de módulos. Uma das equações tem uma interpretação natural como uma relação de associatividade de um novo produto quântico, chamado produto quântico tangencial, que constitui uma generalização surpreendente do produto quântico usual.

A segunda parte da tese mostra como de fato as classes de psi modificadas são convenientes para descrever problemas de enumeração de curvas racionais sujeitas a condições de tangência (bem como condições impostas sobre as singularidades das curvas). Esta parte traz novas aplicações de ferramentas clássicas de geometria enumerativa, como a fórmula de Porteous e o teorema de Bertini. Uma vez estabelecida a ligação com as classes de psi modificadas, as equações diferenciais da primeira parte fornecem as soluções, não apenas para problemas de contagem de curvas em espaços projetivos, e sim mais geralmente em variedades homogêneas. Também para os casos mais simples, como o de contagem de curvas planas, onde vários problemas vêm sendo resolvidos nos últimos anos, as técnicas deste trabalho representam um avanço, fornecendo aos diversos problemas uma solução unificada, muito mais simples e conceitual.

Official summary

Tangency quantum cohomology and enumerative geometry of rational curves

Gromov-Witten invariants are integrals over moduli spaces of stable maps. In enumerative geometry they have an interpretation as solutions to problems of counting rational curves subject to incidence conditions, like for example: "how many rational plane cubics pass through eight given points?" The recent theories of quantum cohomology (which originated in theoretical physics) allow for the recursive computation of the invariants, and provide in this way a surprising solution to enumerative problems that for more than a century have resisted intense investigation.

An important generalisation of the Gromov-Witten invariants is to include the so-called psi classes in the integrals, which in theoretical physics corresponds to introducing gravity into the quantum theory.

This thesis presents a connection between this gravitational quantum cohomology and enumerative geometry of rational curves subject to conditions of infinitesimal nature, like for example tangency. The idea is to introduce a modification of the psi classes whose properties make it well-suited for enumerative purposes.

The first part of the thesis treats the properties of these new classes and establishes differential equations satisfied by the generating functions of their integrals. This information allows for effective computation of the invariants and reveals a new deep structure in the moduli spaces. One of the equations has a natural interpretation as an associativity relation for a new quantum multiplication, called the tangency quantum product, which constitutes a surprising generalisation of the usual quantum product.

The second part of the thesis shows how in fact the modified psi classes are convenient for describing problems of enumeration of rational curves subject to tangency conditions, (as well as conditions imposed on the singularities). This part features new applications of classical tools from enumerative geometry, like Porteous' formula and Bertini's theorem. Once the connection to the modified psi classes is established, the differential equations of the first part provide the solutions, not only to problems of counting curves in projective space, but more generally in homogeneous varieties. Also for the simplest cases, like that of counting plane curves, where various problems have been solved over the past few years, the techniques of the present work represent an advance, providing the various problems with a unified, much simpler and more conceptual solution.

Preface

I am proud to present this final version of my Ph.D. thesis *Tangency quantum cohomology and enumerative geometry of rational curves*, which crowns four years of study at the University in Recife, under the supervision and guidance of Israel Vainsencher. As is custom in mathematics, the main text (starting on page 1) is presented as a *product*, ignoring completely the *process* that led to it. In this preface I would like to record at least an outline of the course of studies that led to this thesis, and collect a few events which were of particular importance for my mathematical apprenticeship as well as for the present thesis. I use this chronological skeleton to organise acknowledgements and thanks to the various people who have had importance for my work.

I started my Ph.D. studies in March 1996, and the first 18 months were spent fulfilling the requirements imposed by the regulations of the UFPE, following courses in algebraic and riemannian geometry, with emphasis on intersection theory. I passed my qualifying exam in august 1997, with expositions on the Grothendieck-Riemann-Roch theorem and on the classification of symmetric spaces. I would like to thank ***everybody at the math department*** for constituting a very nice working environment, a second (or third) home, not only during my Ph.D. studies, but also during earlier periods when I was not even an official student at the UFPE.

The initial problem Israel proposed to me at that time was that of determining the number of conics five-fold tangent to a smooth plane curve. (This problem is still open, even though it now seems that it should be possible to solve it via methods developed recently by Andreas Gathmann [19].) My work on this problem was initially delayed six months due to my complete absorption in a one-semester research seminar led by ***Dag Einar*** Sommervoll, on toric geometry and mirror symmetry. While the investigation of these subjects had no direct relation neither with the problem of multi-tangent conics, nor with the actual subject of my thesis, I would like to acknowledge that I learned a lot of algebraic geometry, and first of all, that I enjoyed it a lot.

My first effective step into the realm of Gromov-Witten theory was my participation in the mini-course *Intersection theory over moduli spaces of curves* [20] given by ***Letterio*** Gatto in January 1998. This course featured applications of Porteous' formula to jet bundles over moduli spaces of curves, and its influence is evident in Chapter 3 and 5 of this thesis. One lecture was devoted to stable maps and Kontsevich's formula for rational curves — unfortunately I missed just that lecture, but Letterio had the kindness to explain it all to me on the blackboard the following afternoon, and I understood that the theory of stable maps was not an inaccessible magic from physics, but that it fits perfectly into the tradition of enumerative geometry. In addition to this, I was deeply influenced by the whole spirit of this mini-course, making mathematics simple through the emphasis on examples. For these things, and for many later discussions of many mathematical questions, when Letterio came back to Recife to join the faculty, I am most indebted.

During the first semester of 1998, I studied the notes of Fulton and Pandharipande [16], and notably the early preprint of Rahul Pandharipande entitled *Intersection of $\mathbb{Q}$-divisors on Kontsevich's moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and enumerative geometry* [42]. The idea emerged that the conic problem could be attacked in the setting of stable maps as follows: define and compute the class of the condition of pointed tangency (cf. 3.1.1) to a smooth curve, and intersect five such cycles in $\overline{M}_{0,5}(\mathbb{P}^2, 2)$. This intersection is easily seen to have excess, and the project was to handle this problem. (This has not been accomplished.)

I did not have much success with these attempts, until August 1998 when I had the opportunity to visit ***Lars*** Ernström in Stockholm. He introduced me to gravitational descendants and topological recursion, just the concept and tools I needed at that stage. I am very thankful to Lars for inviting me and for three very fruitful days at Kungliga Tekniska Högskolan. Thanks are also due to ***Andreas*** Gathmann for sending me a copy of his thesis [17], where some similar arguments are carried out.

Soon after that, I began discovering things on my own: modified psi classes, and their basic properties — Chapter 1 of this thesis. Israel was spending the semester in Belo Horizonte, and I went there for a period of three weeks, which I remember as very productive: not only did it rain all the time, but first and foremost ***Dan*** Avritzer of the UFMG gave me a wonderful reception, instigated seminars, and showed a genuine interest in my budding work.

While my main concern at that time was the question of multiple contacts, the emphasis of my investigations turned in another direction when I discovered a recursion that reduces top products of modified psi classes and evaluation classes to pure gravitational descendants. This recursion was the main result of an article I posted to the Duke e-print server in February 1999 ([28]), but it has now been superseded by the recursion of 1.6.3.

During the spring 1999, I experimented with other applications of my new machinery, like conditions on cusps and double points, and the quick success with these undertakings practically killed the multiple contacts project. All that is left is Section 5.3 which treats multiple contacts of *linear* stable maps — unfortunately things are much more complicated in higher degree.

At the same time I began working with Israel on the notes on Kontsevich's formula [30], which were going to support the mini-course given at the XXII Colóquio Brasileiro de Matemática in Rio de Janeiro in July. The writing of these notes was very time consuming, and it may not have been the shortest path to this thesis, but it was very fruitful to be forced to thorough revision of my knowledge of the basics of the subject.

In July, I had the privilege to talk at the conference on Algebraic Geometry and Commutative Algebra, held in Campinas; I am grateful to ***the organisers*** for letting me speak.

In September–December 1999, I spent three months at the California Institute of Technology in Pasadena, California, studying with ***Rahul*** Pandharipande. While it was clear that Caltech would be the ideal place for me to go, it was less obvious that it could come true, and I am very grateful to Rahul for supporting the idea and encouraging its realisation, and to ***Laura*** Cashion for help with the administrative obstacles — and for being helpful in general and with many small problems. But first of all I am indebted to Rahul for his contribution to my mathematical apprenticeship: My stay at Caltech has been crucial for this thesis, and the whole of Chapter 2 as well as Section 3.3 represent work carried out at Caltech aided by encouragement and advice from Rahul. He urged me to upgrade to more general spaces, and to write down my recursions in terms of generating functions, and in fact these abstractions led to great simplifications. (I am embarrassed to admit that generating functions were not part of my working toolbox at that time.) Also I have profited a lot from many conversations with Rahul on much fancier things, like Hodge integrals, Virasoro constraints and Toda equations, and though I would not pretend that I have much understanding of these subjects, at least I broadened my horizons and acquired a better comprehension of which particular little corner of Gromov-Witten theory is occupied by my own work.

At different times and occasions, I have also benefited from conversations with ***Anders*** Kock, ***Anders*** Thorup, ***André*** Meireles, ***Aron*** Simis, Carlos ***Bahiano***, ***Eduardo*** Esteves, ***Elizabeth*** Gasparim, ***Fernando*** Xavier de Souza, especially ***Francesco*** Russo, ***Lorenz*** Wotzlau, ***Marzia*** Polito, ***Oscar*** Neto, ***Peter*** Johnson, ***Sérgio*** Santa Cruz, ***Steven*** Kleiman, and ***Y.-P.*** Lee, to all of whom I would like to express my thankfulness.

Throughout the past two years, I have spent a considerable amount of time on maple programming; including all the source code as an appendix would nearly double the length of this thesis. The first step was to implement the algorithm described in Pandharipande [42], and soon the topological recursion for tautological psi classes was incorporated, in order to speed up the computations. Next step was to include support for modified psi classes, reducing them to gravitational descendants, allowing for much more effective computations of characteristic numbers. Then came the codimension-2 boundary classes Π that are central to the applications of Chapter 4, and the diagonal classes, and then finally the recursion relation 1.6.3 took over, leaving many of the previous implementations obsolete.

It has been very satisfactory to see the programmes become faster and faster, and with the discovery of the differential equations of Chapter 2 and Section 3.3, the programming task has been reduced to a bare minimum. Computations of numbers that with the early algorithms and implementations could take many hours, can now be accomplished in seconds. But first and foremost, the importance of all this programming has been its rôle as experimental mathematics, and for sure: the quest for more effective algorithms in many cases was responsible for the discovery of better theory. Let me take the opportunity here to acknowledge also the C programme "farsta" developed by **Andrew** Kresch [34], which in some cases has been a useful complement to my own programmes.

My Ph.D. programme has been sponsored by a scholarship from the ***Natural Science Research Council of Denmark***, to which I am grateful for having provided the economical stability necessary to concentrate on the mathematics, most of the time.

Finally, and most importantly, I would like to express my deepest gratitude to my advisor ***Israel*** Vainsencher for his constant support, encouragement and patience, for sharing with me his deep insight in enumerative geometry, for always having time to listen to my questions or even less structured doubts, for reading and commenting my jungle of preliminary manuscripts, and for precious advice whose applicability is not limited to mathematical research or thesis writing.

Although not directly involved in the mathematics, the special prize goes to my beloved wife ***Andrea*** for her constant support and patience. (It is not always easy being married to a mathematician.)

I have dedicated this thesis to the memory of my maternal grandfather who was a scientist, inventor, and social critic, and from whom I learned so much when I was a boy.

January–March 2000
Recife, Pernambuco, Brazil

JOACHIM KOCK

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General conventions

Throughout we work over $\mathbb{C}$, the field of complex numbers. A variety means a quasi-projective, reduced, and equi-dimensional scheme of finite type over $\mathbb{C}$. All morphisms are $\mathbb{C}$-morphisms. A *point* of a variety is always a closed point (i.e. a $\mathbb{C}$-point).

Introduction

Background and main ideas

Enumerative geometry today

0.0.1 Enumerative geometry and its numbers. Enumerative geometry is the discipline of counting geometric objects that satisfy given conditions. Being concerned with *counting* it belongs to combinatorics, and virtually every solution comes as a sum involving binomial coefficients or factorials. In its formulations it is a discipline of *synthetic geometry* since the conditions imposed on the geometric objects make no reference to coordinates or intensive quantities. It is however in the context of algebraic geometry it has flourished, and today it is considered a part of *intersection theory*.

The charm of the subject is compelling: the problems are easy to state and the answers take the simplest possible form in mathematics — after all they are plain natural numbers! — but the techniques required to arrive at the solutions are often considerably sophisticated. At the same time this accounts for the paradoxical reality of enumerative geometry: that mostly, the methods are more important than the solutions, and it can be difficult to judge which problems are important and which are not. The number may not be important in itself, it is rather the method devised to determine it which is important; and the number is the witness of the success of the method.

While Zeuthen [50] and Schubert [43] stress that their main achievements are their *techniques*, their degeneration methods and incidence correspondences, they also list hundreds of *numbers*. Today, while few people understand their arguments and constructions, the numbers are still there, as a monument. Since Hilbert's inclusion of "vindicating results and methods of Schubert calculus" as one of the problems of the twentieth century, it has been a popular challenge for enumerative geometers to reproduce and prove those numbers —

often it seems to have been more interesting to reproduce old numbers than to pose and attack new problems. The relevance of the numerical solutions is probably this: that they allow for a convenient cross-check of new methods against old; a check-sum against silly mistakes. So today, when we continue the tradition of tabulating numbers, it is also meant as a service, an easy reference, for future enumerative geometers.

0.0.2 Stable maps and modern enumerative geometry. The advent of stable maps and quantum cohomology has revolutionised enumerative geometry, on various levels. On the most immediate level, simply many old problems have been solved, as well as new problems that were out of reach for previous techniques.

One virtue of the moduli spaces of stable maps, considered as parameter spaces for rational curves, is that there is a uniform description for the spaces for all degrees, in contrast to previous parameter spaces which often required a different construction for each degree — compactifications and blow-ups corresponding to the degenerations particular to that degree. This fact has favoured a major shift in emphasis, which at a deeper level distinguishes modern enumerative geometry from the classical: the problems are no longer posed and attacked degree for degree; rather all degrees are considered at once, and the emphasis has passed from the numbers themselves to the relations among the numbers — the *structure* that governs them. The objects of study of modern enumerative geometry are the *generating functions* for the numbers, and the solutions take form of differential equations.

At an even deeper level, these generating functions are not just considered convenient ways of storing numbers of curves passing through given points in some space X; they have passed to be considered "invariants" of the space itself. So stating that some generating function for X has certain properties is not only a statement about these curves but more generally a statement of X itself, a statement on how this particular space allows rational curves to move in it. This is one of the many view-points the discipline has inherited from physics, where for almost a decade the (full) Gromov-Witten potential has been referred to as the "total free energy" of the space.

0.0.3 Quantum cohomology. The starting point for these revolutions, and the textbook example of the point of view sketched above, is *quantum cohomology.* Gromov-Witten invariants are integrals over moduli spaces of stable maps, and when the target space is a projective homogeneous variety X, they are solutions to enumerative problems of counting rational curves subject to incidence conditions. The generating function for these numbers is called the *Gromov-Witten potential.* Exploring the recursive structure of the moduli spaces, differential equations for the Gromov-Witten potential are found, which allow for recursive computations of the numbers.

But much more important than the possibility of calculating the numbers is the fact that these differential equations constitute a deep structure on the cohomology space of X. They express the associativity of a new product on the cohomology space — a deep structure which just happens to have been constructed by means of enumerative geometry; other constructions are possible.

Considerations of this sort serve as guidelines for the research in enumerative geometry, and help judging which enumerative problems, and especially which methods and solutions, are important.

0.0.4 Recent trends in the enumerative geometry of rational curves. Gromov-Witten theory is a common denomination for everything that deals with the geometry of the moduli spaces of stable maps. It has grown into a very active research area at the confluence of many mathematical disciplines — geometry, theoretical physics, combinatorics — the ramifications are vast and the ultimate nature and scope of what is being developed can barely be glimpsed. For the parts directly concerned with enumerative geometry, the efforts of the research seem to address two main groups of questions:

Gromov-Witten invariants of difficult spaces (e.g. the quintic three-fold).

Finer enumerative problems in simple spaces (e.g. tangency in $\mathbb{P}^2$).

0.0.5 Gromov-Witten invariants via torus actions and localisation formulae. One of the most intricate enumerative problems has been that of counting rational curves on the quintic three-fold in $\mathbb{P}^4$. From the view-point of stable maps, this is a "difficult space" since the associativity relations are trivial in this case. Instead, the main techniques for treating these questions are those of torus actions and localisation formulae, introduced by Kontsevich [31] and Ellingsrud-Strømme [11], and further developed by Givental [23] (see Pandharipande [41] for a readable introduction to these theories). This line of research is at the very heart of Gromov-Witten theory because of its importance for the mirror conjecture.

It is not however the subject of the present thesis.

0.0.6 Finer enumerative geometry of simple spaces. The second trend, on the other hand, is a direct continuation of the tradition of the enumerative geometry of Zeuthen, Schubert and their century, and its revival with the advent of modern intersection theory à la Fulton. It is concerned with curves in simple spaces like $\mathbb{P}^2$ and $\mathbb{P}^3$.

Once the question of incidence conditions has been completely settled by the associativity relations, the next natural questions to attack are those of tangency conditions and characteristic numbers. Further natural questions include conditions imposed on the singularities, flexes, etc. — all questions that were posed and studied extensively in the nineteenth century.

The present thesis treats central questions in this second trend, so it is opportune to outline the previous work on this subject. Roughly, two strategies are possible, within the framework of stable maps. *Either* one can compute Gromov-Witten invariants of auxiliary spaces and interpret them as solutions to the original problem. This requires handling much more difficult spaces, but associativity relations are often sufficient to determine the sought for invariants. *Or* one can stick to working in the space $\overline{M}_{0,n}(\mathbb{P}^2, d)$, but then Gromov-Witten invariants and associativity relations are no longer sufficient to determine the numbers, and a finer analysis of the intersection theory of the space is needed.

0.0.7 Enumerative geometry via Gromov-Witten invariants of auxiliary spaces. The first example was provided by Lotthar Göttsche and Rahul Pandharipande [24], and independently, Andreas Gathmann [18], who studied Gromov-Witten invariants of $\mathbb{P}^2$ blown up in general points, and interpreted them as numbers of plane curves with specified multiplicities at the blown-up points. (In Gathmann's Ph.D. thesis [17] there are also some rudimentary examples on how this technique can be used to account for some tangency conditions.)

The second example was given by Lars Ernström and Gary Kennedy [12] who constructed a *moduli space of stable lifts* whose points parametrise maps to the incidence variety of points and lines in $\mathbb{P}^2$. The Gromov-Witten invariants of this space encode the tangency behaviour of rational plane curves, for example the condition of being tangent to a line, or the condition of having a cusp. In their second joint paper [13] they also showed how this data is encoded in a ring called the *contact cohomology ring*, which specialises to the usual quantum cohomology ring of $\mathbb{P}^2$ when certain formal parameters are set to zero. Most recently, this lifting approach has succeeded in lifting the maps one step further, to the Semple bundle, which encodes second order curvilinear data, cf. Colley, Ernström and Kennedy [7]. The Gromov-Witten invariants of this space can in some cases be interpreted as numbers of plane curves with prescribed triple contact to a given curve. It is unclear to what extent these lifting techniques apply to other spaces than $\mathbb{P}^2$.

0.0.8 Divisor theory directly in $\overline{M}_{0,n}(\mathbb{P}^r, d)$. In order to treat tangency conditions (or anything beyond incidence conditions), further knowledge of the intersection theory of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is required. The foundational paper on this issue is Pandharipande [42], which accomplishes three important steps. First a set of generators of the Picard group of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is described. The generators are the irreducible boundary divisors, the pull-backs of a hyperplane along the evaluation morphisms, and the divisor of maps incident to a given codimension-2 plane in $\mathbb{P}^r$. Second, an algorithm is given for computing any top intersection of these divisors. The engine is the recursive structure of the boundary: if there is a boundary divisor in the integral it can be split up in a product of two integrals over moduli spaces of lower dimension. When no more boundary divisor are present, associativity is invoked to determine the integrals. The third step is to express the divisor of "maps tangent to a hyperplane" in terms of the generators. This yielded the first algorithm for computing characteristic numbers of rational curves in $\mathbb{P}^r$. It should be noted that this recursion is a low-level recursion whose pieces and moves are arbitrary boundary divisors and their intersections, which cannot be interpreted as relations among characteristic numbers. On the other hand, this makes for applicability in a wide range of situations.

It is characteristic for this approach that the marks are of no importance; in fact they are only a burden since the number of boundary divisors grows very quickly as more marks are introduced. Further work in this spirit is represented by Pandharipande [40], and in the articles of Ravi Vakil [46] and [47], some of the techniques are extended to work also in genus 1, 2 and 3.

0.0.9 Psi classes: Gravitational quantum cohomology. A very powerful extension of Gromov-Witten invariants is that of gravitational descendants (or descendent Gromov-Witten invariants), which incorporate the tautological psi classes, in addition to the classes pulled back from the target space. This notion was devised by Edward Witten [49] in order to introduce gravity into topological sigma models. Recently there has been a lot of interest in these invariants, (in the much more general setting of arbitrary genus), especially stemming from the Virasoro conjecture, which states that certain differential operators annihilate the generating function of the descendants. See Getzler [22] for the precise formulation, a survey, and a proof of the conjecture in genus 0.

In genus 0, 1, and 2, the psi classes admit an expression in terms of boundary divisors; since the psi classes restrict to a boundary divisor in a straightforward way, this allows for recursions among the invariants, known as topological recursion relations, which determine the descendants from the Gromov-Witten invariants.

The gravitational descendants are not directly interpretable in enumerative geometry. I his thesis [17], Andreas Gathmann devoted a section to this issue and found that correction terms at the boundary must be taken into account if certain products of psi classes and evaluations classes are to be interpreted in terms of tangency conditions. Only if no more than one tangency condition is imposed does this correction term not contribute.

Main ideas of this thesis

The point of departure of this thesis is the article of Pandharipande [42]. That is, we stick to the moduli spaces themselves, so the strategy is this: impose conditions on the image curves to define a cycle in the moduli space; compute the class of this cycle in terms of known cohomology classes; intersect the cycles, and use recursions to compute the integral.

0.0.10 Pointed conditions. Now the first difference in the present work is the *consequent use of pointed conditions.* So instead of requiring e.g. tangency to a given line (codimension 1) we will require tangency to the line at the mark p_1 (defining a cycle of codimension 2), and so forth.

An obvious drawback of this is that the spaces become much bigger, and that the cycles get codimension 2 instead of 1. For example in the case of quartics, the un-pointed space $\overline{M}_{0,0}(\mathbb{P}^2, 4)$ has dimension 11 and Picard number 3, while the space $\overline{M}_{0,11}(\mathbb{P}^2, 4)$ (one mark for each condition we want to impose) has dimension 22 and Picard number 5065. (cf. [42]).

The advantage is that we get a handle to each condition so we can control where they are fulfilled, and in particular under what circumstances they come together. On one hand we can force two conditions together getting a compound condition. For example, given the condition of having a cusp at mark p_1, we can easily get the condition of having this marked cusp mapping to a specified point Q, simply by combining the two condition "p_1 is a cusp" and "p_1 maps to Q". On the other hand it is also important to be able to separate conditions that tend to come together and contribute where they shouldn't. For example, in a family where all maps are cuspidal, when imposing the condition of being tangent to a line H, the case where the cusp maps to H tends to count as a tangency, although it is not in general the limit of a honest tangency. (The two conditions come together and interfere.) But when both conditions are pointed, such that the cusp is always p_1 and the tangency is required to occur at p_2, then it is easy to detect the unwanted contribution: in that situation the marks must have come together, and then by stability we are talking about the boundary divisor of maps where p_1 and p_2 are together on a twig of degree zero. The class of this configuration plays a crucial rôle exactly to separate unwanted contributions.

It turns out many of the cycles defined by such pointed geometrical conditions are in fact intersections of divisors, so in theory the algorithm of Pandharipande [42] can compute the top products. However, the abundance of boundary divisors in the relevant spaces makes this unfeasible in practice, except in very low degree. For example, when intersecting 11 such pointed tangency cycles in $\overline{M}_{0,11}(\mathbb{P}^2, 4)$, each cycle will be expressed in terms of almost all of the boundary divisors, resulting in a product whose expansion has approximately 5000^{11} terms! The problem is that there are too many basic building blocks, and that they do not correspond to geometric properties of the image curves we impose conditions on.

0.0.11 Pull-back invariance of cycles coming from enumerative geometry. Impose the condition that something happen to the image curve at the mark p_1. This defines a cycle in each of the moduli spaces $\overline{M}_{0,n}(\mathbb{P}^r, d)$ whose set of marks contains p_1. Now it is a simple but crucial observation that all these cycles are compatible with pull-back along the forgetful morphisms that relate these spaces, since the behaviour of the image curves is independent of the presence of marks. Precisely, if $\alpha_1 \in \overline{M}_{0,\{p_1\}}(\mathbb{P}^r, d)$ and $\alpha_1' \in \overline{M}_{0,n}(\mathbb{P}^r, d)$ are two such cycles, and $\pi : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,\{p_1\}}(\mathbb{P}^r, d)$ is the morphism that forgets all marks but p_1 then

$$\alpha_1' = \pi^* \alpha_1.$$

0.0.12 New basic building blocks. While the evaluation classes share this pull-back property, the psi classes do not, and this explains why it is difficult to interpret them in terms of the geometry of the image curves. A new set of basic building blocks is introduced (to use together with the evaluation classes): the *modified psi classes* $\overline{\boldsymbol{\psi}}_i$, and the *diagonal classes* $\boldsymbol{\delta}_{ij}$. Their features are: first, they enjoy the pull-back property just described, and this makes them suitable for describing cycles arising in enumerative geometry. Indeed the Chapters 3–5 show how a wealth of enumerative conditions (tangency conditions, conditions on cusps, flexes, and osculating planes, double point and secant conditions, etc.) admit natural — and short! — expressions in terms of these new classes. And second, there are nice recursions among their top products, which (because of the close relation with the geometry of the image curves) are easily interpretable in terms of the geometry of the curves. This is established in Chapter 2.

Let me briefly explain why (and in which sense) this approach turns out to be at the same time very classical and very modern.

0.0.13 These new classes versus classical theory of curves. Concerning the close relation between the new classes and the geometry, there is the following conceptual way of thinking of the new classes, demonstrating their classical roots. While $\overline{M}_{0,1}(X,\beta) \to \overline{M}_{0,0}(X,\beta)$ plays the rôle of a universal curve $\mathfrak{X} \to B$, it is fruitful to think of $\overline{M}_{0,n}(X,\beta)$ as the analogue of the n-fold fibred product of $\mathfrak{X}$ over B. The modified psi class $\overline{\boldsymbol{\psi}}_i$ then corresponds to (the pull-back of) the relative dualising sheaf of the i'th universal curve. In the classical theory of curves, the partial diagonals of $\mathfrak{X} \times_B \cdots \times_B \mathfrak{X}$ play an important rôle, as correction terms in degeneracy loci (see Gatto [20] for a readable introduction to these techniques). Their counterpart in the setting of stable maps is exactly the notion of diagonal classes (whence their name). They enjoy properties expected from this comparison, like pushing down to the fundamental class, as well as

$$-\boldsymbol{\delta}_{ij}^2 = \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_i = \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_j,$$

which properties do *not* hold for the tautological psi classes.

0.0.14 The tangency quantum potential. On the other hand, the recursion for top products of the new classes is much more than an algorithm. The principle is that of topological recursion, but in contrast to the topological recursion for the tautological psi classes, this one involves the diagonal classes at intermediate steps, in an essential way which complicates the recursion. The full structure of these complications have not yet been understood, but already in the easier case of integrals where the exponents of the modified psi class are at most one, a deep structure on the cohomology module of X is revealed. I call these integrals *degree-1 enumerative descendants*, and their generating function the *tangency quantum potential*, since it is closely related to the generating functions for the characteristic numbers.

While the usual quantum cohomology, as well as the topological recursion for tautological psi classes relate to the Poincaré metric $g_{ij} := \int_X T_i \cup T_j$, the tangency quantum potential turns out to relate to a certain deformation of this metric, which is one of the main conceptual innovations of this work.

Amazingly, this metric encodes all the combinatorics of the $\boldsymbol{\delta}_{ij}$-corrections of each step in the topological recursion: in this metric, the topological recursion for the degree-1 enumerative descendants takes a simple form, comparable to the usual topological recursion for tautological psi classes. Furthermore: the WDVV-equations hold for this tangency quantum potential in the deformed metric, in exactly the same form as they hold for the usual Gromov-Witten invariants (and more generally for gravitational descendants). This defines a ring structure on the cohomology space of X, the tangency quantum product, generalising the contact product for $\mathbb{P}^2$ constructed in Ernström-Kennedy [13].

0.0.15 Remark. It should be pointed out that I am not the first to use the notion of modified psi classes: they were previously studied briefly by Rahul Pandharipande (unpublished), who used them to establish the differential equation for the generating function of the characteristic numbers of plane curves (e-mail to Lars Ernström, May 1997).

0.0.16 A word about generality. The treatment of this thesis is restricted to the case of rational curves in projective homogeneous varieties. This choice is instilled by enumerative geometry, since for more general target spaces or higher genus, the enumerative interpretation of Gromov-Witten invariants is still dubious.

However, many of the theoretical results of this thesis should allow direct generalisations to a wider range of situations. The notions of modified psi class and diagonal class make sense for any projective smooth target space, and using the virtual fundamental class instead of the topological one, the topological recursions 1.6.3 and 2.3.1 should still hold, as should the WDVV equations 2.4.1. So even in this generality, the tangency quantum potential should be an important object, though perhaps no longer directly related to tangency and enumerative geometry. As to the questions of higher genus, the most obvious generalisations would be to genus 1 and 2 where topological recursion is available. That should work also for modified psi classes. Rahul Pandharipande (personal communication) has suggested a way of "correcting" the invariants in order to get even the answers to the enumerative problems in these cases.

Overview and results

0.1 Enumerative divisor classes

Chapter 1 introduces the modified psi classes and the diagonal classes and proves their basic properties, culminating with the topological recursion relation 1.6.3.

The first couple of sections recall the basic notions of stable maps, following FP-NOTES, as well as the basic properties of psi classes, and establish notation. Section 1.3 establishes a couple results on boundary divisors needed for the following treatment of modified psi classes.

0.1.4-5 Modified psi classes and diagonal classes. On a moduli space $\overline{M}_{0,n}(X,\beta)$ with $\beta > 0$, let $\boldsymbol{\xi}_i$ denote the sum of all boundary divisors having p.36 mark p_i on a contracting twig. Now the *modified psi class* is defined as

$$\overline{\boldsymbol{\psi}}_i := \boldsymbol{\psi}_i - \boldsymbol{\xi}_i.$$

The motivation for this definition is that $\overline{\boldsymbol{\psi}}_i$ is invariant under pull-back along forgetful morphisms.

p.37 The ij'th diagonal class $\boldsymbol{\delta}_{ij}$ is by definition the sum of all boundary divisor having p_i and p_j together on a contracting twig. The name is justified by the following basic properties enjoyed by the diagonal classes: First, if π_i is the p.37 morphism that forgets p_i then $\pi_{i*}\boldsymbol{\delta}_{ij} = 1$, the fundamental class. And second,

$$\begin{aligned} \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{ik} &= \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{jk} \\ -\boldsymbol{\delta}_{ij}^2 = \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_i &= \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_j. \end{aligned}$$

With these properties it is easy to see that every top product involving evaluation classes, modified psi classes and diagonal classes can be reduced to

products involving only evaluation classes and modified psi classes. The diagonal classes appear as correction terms when restricting to a boundary divisor both of whose twigs are of positive degree: If x is the gluing mark of the twig carrying p_i then the restriction of $\overline{\boldsymbol{\psi}}_i$ is $\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ix}$, cf. 1.5.9 for the precise p.40
statement.

0.1.6 Enumerative descendants. We define an *enumerative descendant* to be a top product of modified psi classes and evaluation classes, and set p.40

$$\langle\, \overline{\tau}_{m_1}(\gamma_1) \cdots \overline{\tau}_{m_n}(\gamma_n)\,\rangle_\beta := \int \overline{\boldsymbol{\psi}}_1^{m_1} \cup \nu_1^*(\gamma_1) \cup \cdots \cup \overline{\boldsymbol{\psi}}_n^{m_n} \cup \nu_n^*(\gamma_n) \cap [\overline{M}_{0,n}(X,\beta)].$$

Since the modified psi classes admit an expression in terms of boundary divisors and we know how to restrict a modified psi class to the boundary, topological recursion works: take out one of the modified psi classes in the product, write it as a sum of boundary divisors and restrict the remaining factor of the product to each of these boundary divisors, getting a sum of integrals over moduli spaces of lower degrees and with fewer modified psi classes.

However, at each recursion step, a lot of diagonal classes arise, and they must be eliminated using the result of Section 1.5 before going on to next recursion step. This complicates the recursion, and its expression is rather lengthy and ugly, and it has not found its way into this Overview. See **Theorem 1.6.3** for the precise statement. Note however that algorithmi- p.42
cally it is very effective. The main ingredient in the proof is a splitting lemma for enumerative descendants which is established separately, as Lemma 1.6.2.

0.2 The tangency quantum potential

Chapter 2 constitutes the theoretical heart of the thesis.

0.2.1 Degree-1 enumerative descendants. A particularly well-behaved and useful sort of enumerative descendants are those where the exponent on each modified psi class is at most one. These are called *degree-1 enumerative descendants* and we use the following notation for them: p.45

$$\langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta := \langle\ \prod_{k=0}^{r} (\overline{\tau}_0(T_k))^{a_k} (\overline{\tau}_1(T_k))^{b_k}\ \rangle_\beta,$$

where $\mathbf{a} = (a_0, \ldots, a_r)$ and $\mathbf{b} = (b_0, \ldots, b_r)$ are vectors of non-negative integers, and $T_0, \ldots, T_r$ constitute a basis of the cohomology space of X.

We now introduce the main object of study. It is the generating function for the degree-1 enumerative descendants, which we call the *tangency quantum* p.46 *potential:*

$$\Gamma(\mathbf{x},\mathbf{y}) = \sum_{\beta>0} \sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!} \langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\; \overline{\boldsymbol{\tau}}_1(\mathbf{b}) \,\rangle_\beta$$

where $\mathbf{x} = (x_0, x_1, \ldots, x_r)$ and $\mathbf{y} = (y_0, y_1, \ldots, y_r)$ are formal parameters and the sum is over all effective $\beta \in A_1(X)$ and all pairs of vectors $\mathbf{a}, \mathbf{b}$ of non-negative integers. The variables $\mathbf{x}$ are the usual formal variables from quantum cohomology, so when $\mathbf{y}$ is set to zero, then Γ is the usual (quantum part of the) genus-zero Gromov-Witten potential.

0.2.2 Deformation of the Poincaré metric. Let $A = H^*(X) \otimes \mathbb{Q}$ be the cohomology space of X, with a homogeneous additive basis $T_0, \ldots, T_r$. While the usual quantum potential is based on the integral $\int_X : A \to \mathbb{Q}$ and the Poincaré metric $g_{ij} = \int_X T_i \cup T_j$, the tangency quantum potential relates more naturally to a deformation of these structures, a certain "metric" with values in $\mathbb{Q}[[\mathbf{y}]]$.

As a substitute for the integral, consider the linear functional $\langle\ \rangle_X : A \to \mathbb{Q}[[\mathbf{y}]]$ defined as

$$\langle\, \alpha \,\rangle_X := \sum_{\mathbf{b}} \frac{(-2\mathbf{y})^{\mathbf{b}}}{\mathbf{b}!} \int_X \mathbf{T}^{\mathbf{b}} \cup \alpha,$$

p.49 and define the new metric (γ_{ij}) by

$$\gamma_{ij} := \langle\, T_i \cup T_j \,\rangle_X .$$

Let (γ^{ij}) denote the inverse matrix to (γ_{ij}).

0.2.3 Topological recursion. Miraculously, the deformed metric encodes all the combinatorics of the diagonal classes that appear in each step of topological recursion. To state the differential equation we need the following notation: Let $\Gamma_{x_i} := \frac{\partial}{\partial x_i}\Gamma$ denote the partial derivative, and employ also the notation p.50 $\Gamma_{(x_i x_j)} := \sum_{k=0}^{r} g_{ij}^k\ \Gamma_{x_k}$. Now the first main result is **Theorem 2.3.1.** *The tangency potential satisfies the differential equations*

$$\Gamma_{y_k x_i x_j} = \Gamma_{x_k (x_i x_j)} - \Gamma_{(x_k x_i) x_j} - \Gamma_{(x_k x_j) x_i} + \sum_{e,f} \Gamma_{x_k x_e}\ \gamma^{ef}\ \Gamma_{x_f x_i x_j} .$$

These equations determine all degree-1 enumerative descendants from the Gromov-Witten invariants. Indeed, the fact that the left hand side involves derivation with respect to y_k, while the right hand side does not, means that invariants with one or more modified psi classes are expressed in terms of invariants with fewer such classes. The condition of the presence of the classes x_i and x_j is not serious for the application of the recursion: we can always use the divisor equation to introduce two new classes.

0.2.4-5 Tangency WDVV equations. The second main result of this type is **Theorem 2.4.1.** *For simplicity, set* $\Gamma_i = \Gamma_{x_i}$. *Then the following form of the WDVV equations hold for the tangency quantum potential:* p.52

$$\Gamma_{(ij)k\ell} + \Gamma_{ij(k\ell)} + \sum_{e,f} \Gamma_{ije}\, \gamma^{ef}\, \Gamma_{fk\ell} = \Gamma_{(jk)i\ell} + \Gamma_{jk(i\ell)} + \sum_{e,f} \Gamma_{jke}\, \gamma^{ef}\, \Gamma_{fi\ell}.$$

Another word for WDVV is associativity relation, and the Equation 2.4.1 can be interpreted as the associativity of a new product (on $A[[\mathbf{x},\mathbf{y}]]$) defined by the rule p.54

$$T_i * T_j := T_i \cup T_j + \sum_{e,f} \Gamma_{ije}\, \gamma^{ef}\, T_f.$$

0.2.6 Frobenius structure. The usual quantum potential can be split up in a positive-degree part (encoding the "honest" Gromov-Witten invariants) and a degree-zero part which is just the classical potential of X. Surprisingly, the same point of view can be applied to the tangency quantum product. The classical potential is the generating function for all integrals of triple products. Similarly we introduce its tangency analogue, the generating function for functionals $\langle\, T_i \cup T_j \cup T_k \,\rangle_X$ (cf. 2.2.4), which upon expanding reads p.55

$$\Phi^0(\mathbf{x},\mathbf{y}) = \sum_{|\mathbf{a}|=3} \sum_{\mathbf{b}} \frac{\mathbf{x^a}}{\mathbf{a}!} \frac{(-2\mathbf{y})^{\mathbf{b}}}{\mathbf{b}!} \int_X \mathbf{T}^{\mathbf{a}+\mathbf{b}}.$$

The key observation to this definition is that its relation to the cup product in the deformed metric is completely analogue to the relation between the classical potential and the cup product in the Poincaré metric. Precisely, its third derivatives are just the structure constants:

$$T_i \cup T_j = \sum_{e,f} \Phi^0_{ije}\, \gamma^{ef}\, T_f.$$

With this observation, we can introduce the bigger potential

$$\Phi(\mathbf{x},\mathbf{y}) = \Phi^0(\mathbf{x},\mathbf{y}) + \Gamma(\mathbf{x},\mathbf{y}),$$

in terms of whose third derivatives $\gamma_{ijk} + \Gamma_{x_i x_j x_k}$ the tangency product is

$$T_i * T_j \ = \ \sum_{e,f} \Phi_{ije}\, \gamma^{ef}\, T_f.$$

In terms of this potential, the WDVV equation 2.4.1 takes the usual simpler form

$$\sum_{e,f} \ \Phi_{ije}\ \gamma^{ef}\ \Phi_{fk\ell} = \sum_{e,f} \ \Phi_{jke}\ \gamma^{ef}\ \Phi_{fi\ell}.$$

Once the tangency product has been written in this pure potential form, (i.e. that its structure constants are derivatives of a single function), it is p.58 straight forward to establish **Corollary 2.6.6.** *The cohomology* $\mathbb{Q}[[\mathbf{y}]]$*-module* $H = A^*(X)[[\mathbf{y}]]$ *with bilinear non-degenerate pairing* $\gamma : H \otimes H \to \mathbb{Q}[[\mathbf{y}]]$, *equipped with the tangency quantum potential* $\Phi \in \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ *constitutes a formal Frobenius manifold over* $\mathbb{Q}[[\mathbf{y}]]$. More precisely, it is a deformation of the formal Frobenius manifold given by usual quantum cohomology, in such a way that the underlying space is just trivially deformed, while the metric and the multiplication vary non-trivially.

0.3 Characteristic numbers

Chapter 3–5 are directly concerned with enumerative questions. The results of Chapter 1 and 2 are applied to get solutions to various enumerative problems in terms of differential equations.

0.3.1 Tangency conditions. Let $H \subset X$ be a very ample hypersurface, given as the zero scheme of a section f of a line bundle L. The mark p_1 of a (one-pointed) stable map to X lands in H if and only if $\mu^* f$ vanishes at p_1. The locus of such maps is denoted $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_1(H)$. The map is tangent to H at p_1 if $\mu^* f$ vanishes to second order at p_1. Let $\Theta_1 = \Theta_1(H)$ denote the cycle of maps tangent to H at p_1. **Proposition 3.1.1.** *In a space* $\overline{M}_{0,n}(X, \beta)$ *whose general member is an immersion, the locus* $\Theta_i(H)$ *is the zero scheme of* p.60 *a regular section of a rank-two vector bundle* E*, and its class is*

$$\Theta_i := c_2(E) = \boldsymbol{\eta}_i(\boldsymbol{\eta}_i + \overline{\boldsymbol{\psi}}_i).$$

On a one-pointed space, the vector bundle E is the relative jet bundle $\sigma_1^* J^1_\pi \mu^* L$ which is exactly the global description of the second order vanishing of $\mu^* f$ at the mark. On a space with more marks, this vector bundle is not the correct one, since in this case the divisor $\boldsymbol{\xi}_1$ will count as tangency even though its general map is not the limit of a honest tangent. The correct vector bundle to use is then the pull-back of the one in one-pointed space, and this explains why the correct classes to use are the modified psi classes, not the tautological ones.

More generally, these arguments are shown to work in certain nice families of stable maps, thus allowing for iteratively imposing more tangency conditions (with respect to general hypersurfaces). *p.62*

0.3.2 Enumerative significance. The top intersection of such cycles (and cycles defined by incidence conditions) is a finite number of reduced points, whose number is exactly the number of stable maps satisfying the imposed conditions. This is the content of Section 3.2. The main ingredient in the *p.65* proof is the Bertini theorem of Kleiman [27].

For example, the number $N_d(a, b, c)$ of rational plane curves of degree d, passing through a points, tangent to b lines and tangent to c lines at given points (with $a + b + 2c = 3d - 1$), is the top intersection *p.66*

$$\int_{\overline{M}_{0,a+b+c}(\mathbb{P}^2,d)} \boldsymbol{\eta}_1^2 \cdots \boldsymbol{\eta}_a^2 \cdot \Theta_{a+1} \cdots \Theta_{a+b} \cdot \Theta_{a+b+1}\boldsymbol{\eta}_{a+b+1} \cdots \Theta_{a+b+c}\boldsymbol{\eta}_{a+b+c}.$$

0.3.3 Differential equations for characteristic numbers. The fact that the class of the tangency condition admits an easy expression in terms of evaluation classes and modified psi classes of exponent at most 1, allows for an interpretation of the differential equations of Chapter 2 in terms of characteristic numbers. This is carried out in this Section 3.3, in the natural generality of any projective homogeneous target space, where there is no canonical choice of which hypersurfaces one should impose tangency conditions with respect to.

In this Overview, let us just illustrate the idea through the example of $X = \mathbb{P}^2$. Let $G(u_1, u_2, v_1, v_2)$ be the generating function for the characteristic numbers $N_d(a_2, b_1, b_2)$, such that u_1 is a formal variable corresponding to the degree, u_2 corresponds to the incidence condition (of which there are a_2), v_1 correspond to the b_1 simple tangency condition, and v_2 corresponds to the condition of being tangent to a line at a given point (b_2 such conditions). Precisely,

$$G(u_1, u_2, v_1, v_2) = G(\mathbf{u}, \mathbf{v}) = \sum_{d>0} \sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!} \frac{\mathbf{v}^{\mathbf{b}}}{\mathbf{b}!} d^{a_1}\, N_d(a_2, b_1, b_2).$$

p.69 Then (and in general), **Lemma 3.3.3** states that *The generating function for the characteristic numbers is related to the tangency quantum potential by*

$$G(\mathbf{u}, \mathbf{v}) = \Gamma(\mathbf{x}, \mathbf{y}),$$

subject to a simple linear change of variables which is just the dual of that expressing the conditions in terms of the basis $\overline{\boldsymbol{\tau}}_0$ *and* $\overline{\boldsymbol{\tau}}_1$.

Now a straightforward application of the chain rule translates the differential equations 2.3.1 into equations for G (which are very similar to those p.71 for Γ). This is the content of Proposition 3.3.5. These equations constitute a transparent and effective algorithm for computing the characteristic numbers of rational curves in X (as explained further in Example 3.3.9.)

p.71 Three examples are carried out in detail: Examples 3.3.7–3.3.8 treat pro- p.74 jective space and in particular $\mathbb{P}^2$; Example 3.3.10 treats the example of $X = \mathbb{P}^1 \times \mathbb{P}^1$, with the following conditions: passing through a given point; being tangent to a given $(1,1)$-curve (recall that the transversality arguments require the fixed hypersurfaces to be very ample), and being tangent to such a curve at a specified point.

p.76 Finally, Example 3.3.11 does the same for $\mathrm{Gr}(2,4)$, the Grassmannian of lines in $\mathbb{P}^3$, including tangency to the hyperplane section, and all possible incidence conditions. The complexity increases slightly, due to the higher number of variables, but not much.

The Appendix to this chapter tabulates a lot of numbers produced by a maple implementation of the differential equations, for $\mathbb{P}^1 \times \mathbb{P}^1$ and for $\mathrm{Gr}(2,4)$.

0.4 Curves with cusp or prescribed triple contact

Chapter 4 introduces a slight generalisation of the degree-1 enumerative descendants which allows for treating problems concerning cuspidal curves, as well as curves with a single specified triple contact. The slightly enriched degree-1 enumerative descendants are those where in addition to the classes $\overline{\tau}_0$ and $\overline{\tau}_1$ we allow on single class which can have a square of a modified psi class or it can have a factor which is a certain codimension-2 boundary class that arises naturally in many applications.

0.4.1 Codimension-2 boundary classes. Let Π_i denote the sum of all codimension-2 boundary cycles whose middle twig has degree 0 and carries the mark p_i, while the two other twigs have positive degree. Similarly, let Π_{ij} denote the sum of all codimension-2 boundary cycles whose middle twig has degree 0 and carries the marks p_i and p_j, while the two other twig have positive degree. A couple of easy properties of these classes are listed. *p.89*

0.4.2 Slightly enriched degree-1 enumerative descendants. Let P^k denote the generating function for top products of classes of type $\overline{\tau}_0(T_i)$ and $\overline{\tau}_1(T_j)$ and a single factor (say at the first mark) of type $\Pi_1 \cup \nu_1^*(T_k)$. Precisely

$$P^k(\mathbf{x},\mathbf{y}) := \sum_{\beta>0} \sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}\frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!} \langle\, \Pi_1 \cup \nu_1^*(T_k)\ \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta.$$

For each fixed k consider the matrix $\gamma_k^{ij} = \phi_k^{ij}(2\mathbf{y})$. Now **Proposition 4.2.1** relates P^k to Γ: *p.92*

$$P^k = \tfrac{1}{2} \sum_{e,f} \Gamma_{x_e}\ \gamma_k^{ef}\ \Gamma_{x_f}.$$

Next consider the enumerative descendants which in addition to degree zero and one allow a single quadratic modified psi class. Let $Q^k(\mathbf{x},\mathbf{y})$ denote the corresponding generating function

$$Q^k(\mathbf{x},\mathbf{y}) := \sum_{\beta>0} \sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}\frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!} \langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b})\ \overline{\tau}_2(T_k)\,\rangle_\beta.$$

To relate this potential to the tangency quantum potential, one more notational convention is in place, $\Gamma_{(y_i x_j)} := \sum_{k=0}^r g_{ij}^k\ \Gamma_{y_k}$, extending the notation introduced in 2.1.2.1. Now **Proposition 4.2.4** gives the following relation with the tangency quantum potential: *p.94*

$$Q^k_{x_i x_j} = \Gamma_{(x_i x_j) y_k} - \Gamma_{(y_k x_i) x_j} - \Gamma_{(y_k x_j) x_i} + \sum_{e,f} \big(\Gamma_{y_k x_e} + \Gamma_{(x_k x_e)}\big)\ \gamma^{ef}\ \Gamma_{x_f x_i x_j}.$$

In Examples 4.2.2, 4.2.3, 4.2.7, and 4.2.8, all these equations are written out explicitly for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, for easy subsequent reference.

0.4.3 Characteristic numbers for cuspidal curves. This section is concerned with plane cuspidal curves. The locus of maps such that p_1 is a cusp is denoted K_1. The class of this cycle is given in **Proposition 4.3.2**: *p.97*

$$\mathrm{K}_1 = 3\boldsymbol{\eta}_1^2 + 3\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 - \Pi_1.$$

This is established realising the locus as the degeneracy locus of the vector bundle homomorphism $\sigma_1^* T_\pi \longrightarrow \sigma_1^* \nu_2^* T_{\mathbb{P}^2}$. This locus includes Π_1 with multiplicity 1, so we must correct for that.

p.98 It is straightforward to obtain the locus $\mathrm{Kl}_1 = 3\boldsymbol{\eta}_1^2\overline{\boldsymbol{\psi}}_1 + \boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1^2 - \boldsymbol{\eta}_1\Pi_1$ of maps with a cusp at p_1, mapping to a given line, and the locus $\mathrm{Kp}_1 = \boldsymbol{\eta}_1^2\overline{\boldsymbol{\psi}}_1^2 - \boldsymbol{\eta}_1^2\Pi_1$ of maps with p_1-marked cusp mapping to a given point.

In a family where all maps are cuspidal (and have the cusp marked by p_1), the description 3.1.1 of tangency to a given line H breaks down since the case where the cusp maps to H contributes to the tangency class, whereas it shouldn't, since such configurations are not in general limits of honest tangencies. However, in that case the marks must have come together, so the bad locus is $\boldsymbol{\delta}_{12} \cdot \boldsymbol{\eta}_1 \cdot \mathrm{K}_1$, and a 1-parameter family argument (given in the Appendix to Chapter 5) shows it appears with multiplicity 1. So in p_1-cuspidal *p.99* environment the class of tangency at p_2 is

$$\Theta_2' = \boldsymbol{\eta}_2(\boldsymbol{\eta}_2 + \overline{\boldsymbol{\psi}}_2 - \boldsymbol{\delta}_{12}).$$

A similar description holds for the locus of tangency to a given line at a given point.

This allows for the computation of characteristic numbers for cuspidal curves. Since in each case there is a single class $\overline{\boldsymbol{\psi}}_1^2$ or Π_1, the top products expressing these characteristic numbers are encoded by the potentials Q^k and P^k described in Section 4.2. The above differential equations then relate these to the tangency quantum potential, and finally to the potential G of usual characteristic numbers for nodal curves, yielding differential equation *p.99* for the cusp potentials. These are given in Lemma 4.3.6, 4.3.7, and 4.3.8.

In the final paragraph of this section, it is discussed to what extent the same arguments work for $\mathbb{P}^1 \times \mathbb{P}^1$. While it is easy to write down the equations, unfortunately the resulting numbers count also reducible curves, namely those two-twig maps whose marked twig is a multiple cover of one of the generators of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$. Only in the case where only incidence conditions are imposed *p.103* do these reducible curves not contribute, and we get the **Formula 4.3.12** for the number $C_{(m,n)}$ of cuspidal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree (m,n) through the appropriate number of points:

$$C_{(m,n)} = \frac{4(d-1)}{d} N_{(m,n)} + \sum \binom{2d-2}{2d'-1} \frac{(m'n'' + n'm'')(d'd'' - d)}{d} N_{(m',n')} N_{(m'',n'')}.$$

Here $d = m + n$, and similarly, $d' = m' + n'$ and $d'' = m'' + n''$. The sum is over degree splittings $m' + m'' = m$ and $n' + n'' = n$.

0.4.4 Characteristic numbers for curves with a specified triple contact. In this section, the locus of F_1 "maps with a triple contact to a given hypersurface H at p_1" is described as the vanishing of a section of a second jet bundle, by a construction analogue to that of 3.1.1. However, this locus must be corrected by a Pi-class, and the formula is (**Proposition 4.4.4**) p.105

$$\mathrm{F}_i = \boldsymbol{\eta}_i(\boldsymbol{\eta}_i + \overline{\boldsymbol{\psi}}_i)(\boldsymbol{\eta}_i + 2\overline{\boldsymbol{\psi}}_i) - [\mathrm{I}_i]_3 - \boldsymbol{\eta}_i \Pi_i.$$

Here $[\mathrm{I}_i]_3$ denotes a certain correction consisting of two-twig maps with a linear twig mapping into H, but it turns out never to give any contribution. The multiplicity 1 at Π_i is established in the Appendix to Chapter 5. Under mild initial assumptions this locus is sufficiently well-behaved for safely imposing additional tangency conditions, allowing for direct computation of characteristic numbers for curves having the triple contact. This is carried out explicitly for $\mathbb{P}^2$ in Example 4.4.6, and for $\mathbb{P}^1 \times \mathbb{P}^1$ in Example 4.4.8. p.106

As in the case of cuspidal curves, the top products corresponding to the characteristic numbers involve a single factor $\overline{\boldsymbol{\psi}}_1^2$ or Π_1, so the generating function for these flex characteristic numbers is expressed in terms of the potentials P^1 and Q^1 described in Section 4.2, which in turn are related to the tangency quantum potential, cf. 4.2.1 and 4.2.4. The final outcome of this is a differential equation for the flex characteristic number potential, which effectively determines the numbers from the usual characteristic numbers.

In the Appendix to this chapter, these flex characteristic numbers are listed in low degree, for $\mathbb{P}^2$ as well as $\mathbb{P}^1 \times \mathbb{P}^1$.

0.5 More enumerative geometry

This final chapter is not conclusive. Rather is it a catalogue of further applications of the techniques presented in the previous chapters, and many of them deserve a deeper treatment than time, space and competence have allowed me to give here. For simplicity we take $X = \mathbb{P}^r$ throughout.

0.5.1 Enumerative geometry via Porteous' formula. The jet bundle constructions used to compute the class of tangency and triple contact readily generalise to other degeneracy loci. Take a linear system $V^e \subset H^0(\mathbb{P}^r, \mathcal{O}(1))$ of affine dimension e; consider the $(f-1)$'st jet bundle $L^f := \sigma_1^* J_\pi^{f-1} \mu^* \mathcal{O}(1)$, and look at the natural vector bundle homomorphism $V^e \to L^f$. The degeneracy loci of such homomorphisms, varying e and f, correspond to various conditions imposed on tangents and osculating planes at the mark p_1. Porteous' formula gives the class of these loci in terms of the Chern classes of V^e and L^f, which in turn are polynomials in evaluation classes and (modified) psi classes, so the results are enumerative descendants.

This section is mostly made up of examples. For $X = \mathbb{P}^2$ the construction gives the class of the locus of maps whose marked tangent (5.1.1) or marked flex tangent (5.1.5) passes through a given point, as well as new constructions *p.117* of the loci of cusps, already computed in Chapter 4.

For $X = \mathbb{P}^3$, we get the class of the conditions of the marked tangent line to be incident to given linear subspaces (5.1.8–5.1.9), or the conditions of the *p.121* osculating plane containing a given line or point (5.1.11, 5.1.12).

0.5.2 Two-pointed conditions: secant conditions. This section is concerned with the situation of imposing conditions on two marks, but with respect to the same subvariety in $\mathbb{P}^r$. For $\mathbb{P}^2$, we get the locus Np_{12} of (doubly *p.125* marked) double point at a specified point. The class of this locus is

$$\mathrm{Np}_{12} = \boldsymbol{\eta}_1^2\boldsymbol{\eta}_2^2 - \boldsymbol{\delta}_{12}\overline{\boldsymbol{\psi}}_1\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_1^2\Pi_{12}$$

where Π_{12} is enumeratively irrelevant. The potential $GP(u_1, u_2, v_1, v_2, w)$ where w is a variable corresponding to the numbers of Np-classes in the product, is *p.129* easily shown to satisfy the simple equation

$$GP_w = GP_{u_2u_2} - GP_{v_2},$$

but the numbers encoded by it are only enumerative when *either* there is just one double point condition, *or* the number of tangency condition is at most *p.126* one and the number of double point conditions is at most $d-1$.

Another approach to secant-conditions is given in 5.2.13–5.2.20, getting the class of the condition of being secant to a codimension-2 plane, and in particular a new construction of the double point cycle. The idea here is to impose incidence conditions inductively, and correct for the wrong contributions along diagonal classes and Pi classes as they arise. Letting the marks come together we also get new constructions of the tangency conditions, and of the condition *p.131* of being tangent to a codimension-2 plane.

0.5.3 Linear stable maps. This last section treats linear stable maps to $\mathbb{P}^r$. It's a nice simple case where the jet bundle constructions work in full, without the need of worrying about correction terms. The locus of lines contained in a given hypersurface $Z \subset \mathbb{P}^r$ of degree z is of codimension $z+1$. The first series of examples are concerned with the case $z \leq 2r - 3$; then we impose a contact of order $z+1$, thus forcing the map entirely into Z. The cycle of such maps is analogue to the classical concept of the Fano scheme of lines in Z. It turns out that the recursion relation 1.6.3 boils down to a closed formula in p.134 this case, which in turn implies a closed formula for the degree of the Fano scheme, as well as other curious results about lines in hypersurfaces and some p.135 complete intersections. p.138

The second series of examples is concerned with hypersurfaces Z of degree $z > 2r - 3$, i.e., containing no lines. Now instead of imposing just one high contact, one can impose various contacts at various marks, extending the techniques used for the secant constructions in the previous section. We define the cycle $\mathbf{J}_{(z)}(m_1p_1 + \cdots + m_np_n)$ to be (the closure of) the locus of irreducible maps μ such that the divisor μ^*Z dominates $\sum m_ip_i$. As already seen in 5.2.13, the construction of this locus in terms of degeneracy loci involves correcting for diagonal classes. The final formula is (cf. **Lemma 5.3.13**) p.139

$$\mathbf{J}_{(z)}(m_1, \dots, m_n) = \prod_{i=1}^{n} \prod_{k=1}^{m_i} \big(\boldsymbol{\varphi}_i(k) - \sum_{j=1}^{i-1} m_j \boldsymbol{\delta}_{ij}\big).$$

where $\boldsymbol{\varphi}_i(m_i) := z\boldsymbol{\eta}_i + (m_i - 1)\overline{\boldsymbol{\psi}}_i$ are the Chern roots of the jet bundle of the construction.

A simple case of this locus is $\mathbf{J}_{(z)}(2p_1 + 2p_2) \subset \overline{M}_{0,2}(\mathbb{P}^2, 1)$ which is the (pointed) locus of bitangents to $Z \subset \mathbb{P}^2$. A lot of other formulae are given. Intersecting two such cycles (with respect to distinct hypersurfaces) we also give formulae for apparent nodes, tri-secants, and quadri-secants of a complete intersection space curve (reproducing classical formulae due to Schubert, Cayley and others). p.141

0.5.A Appendix: 1-parameter families. This Appendix gathers all the 1-parameter family arguments used in this chapter as well as in Chapter 4. These arguments establish the multiplicities of the various corrections terms encountered in the constructions. The principle is that the 1-parameter family draws a small arc in the moduli space which intersects the locus in question, and its multiplicity can then be read off in the 1-parameter family, which is given so explicitly that the multiplicity appears simply as an exponent of a variable. There are two series of arguments. One for determining multiplicities along loci supported at $\boldsymbol{\delta}_{ij}$ and one series for determining multiplicities along Π-loci.

Chapter 1

Enumerative divisor classes

1.1 Notation and preliminaries

The exposition of this thesis assumes a working knowledge of stable maps and quantum cohomology equivalent to the content of the notes of Fulton and Pandharipande [16], henceforth referred to as FP-NOTES. The reader unacquainted with the subject might skim through [30] before turning to FP-NOTES. While no attempt will be given to summarise the needed results from there, at least this section runs through the most important definitions, merely to establish notation and terminology. The theory of psi classes, which is central to this exposition, is not covered in FP-NOTES, so a slightly more careful review is given of the basic results on this issue.

Stable maps

The level of generality is also the same as that of FP-NOTES:

1.1.1 The target space and its cohomology. Throughout we work over $\mathbb{C}$, the field of complex numbers. Let X denote a projective homogeneous variety, i.e. of the form G/P, where G is a connected linear algebraic group and $P \subset G$ is a parabolic subgroup. For example, X could be a Grassmannian or just a projective space.

The cohomology spaces $H^k(X, \mathbb{Q})$ are zero for k odd, and the even spaces $H^{2k}(X, \mathbb{Q})$ are naturally identified with the Chow groups $A^k(X, \mathbb{Q})$. The total cohomology space $A^*(X) := \oplus A^k(X, \mathbb{Q})$ has the structure of a ring (the Chow ring) under the cup product, denoted $\cup$.

Let $T_0, T_1, \ldots, T_r$ be the elements of a homogeneous $\mathbb{Q}$-basis for $A^*(X)$, such that T_0 is the fundamental class and T_r is the class of a point. The constants $g_{ij} := \int_X T_i \cup T_j$ form an invertible matrix often referred to as the *Poincaré metric*. Let (g^{ij}) denote the inverse matrix to (g_{ij}); it can also be characterised by saying that the class of the diagonal $\Delta \subset X \times X$ is given as $\sum_{e,f} g^{ef} T_e \otimes T_f$ in $A^*(X \times X) \simeq A^*(X) \otimes A^*(X)$.

1.1.2 Stable maps. An n-pointed map to X is the data of a morphism $\mu : C \to X$, where C is a connected nodal curve with n distinct smooth points singled out. More precisely, this structure consists of a labelling set, say $S = \{p_1, \ldots, p_n\}$, and an injection from S to the smooth part of C. We refer to the elements $p_i \in S$ as the *marks*. From this view-point, it makes sense that two distinct n-pointed maps share the same marks.

The degree of an n-pointed map $\mu : C \to X$ is the homology class $\beta = \mu_*[C] \in A_1(X)$. An n-pointed map μ is *stable* if any rational component that is contracted by μ has at least three special points (that is, marks or singular points). This condition is equivalent to the group of automorphisms of μ being finite.

We avoid the use of the word “map” for other morphisms of algebraic varieties.

This thesis is concerned exclusively with stable maps in *genus 0*, and henceforth reference to the genus will be suppressed. The source curve of a stable map (of genus 0) is a tree of projective lines. We will consequently refer to its irreducible components as *twigs*, reserving the term *component* for the components of cycles in the moduli space.

1.1.3 The moduli space $\overline{M}_{0,n}(X, \beta)$. Two stable maps μ and μ' are isomorphic if there is an isomorphism ϕ of their source curves preserving all the structure. That is, $\mu' \circ \phi = \mu$, and $\phi(p_i) = p_i$ for each mark p_i. Let $\overline{M}_{0,n}(X, \beta)$ be the moduli space parametrising isomorphism classes of n-pointed stable maps of degree β. It is an irreducible normal variety (or a smooth Deligne-Mumford stack) of dimension

$$\dim \overline{M}_{0,n}(X, \beta) = \dim X - \int_\beta K_X + n - 3.$$

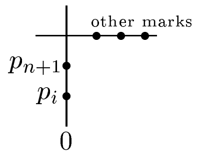
1.1.4 Structure morphisms. For each mark $p_i \in S$ there is an *evaluation morphism* $\nu_i : \overline{M}_{0,n}(X, \beta) \to X$ which sends the class of a map $[\mu : C \to X, p_1, \ldots, p_n]$ to $\mu(p_i) \in X$. Pull-backs of cohomology classes in X along these morphisms are called *evaluation classes*. Top products of evaluation classes (integrated against the fundamental class of $\overline{M}_{0,n}(X,\beta)$) are the (genus zero) *Gromov-Witten invariants* of X.

There is a *forgetful morphism* $\pi : \overline{M}_{0,n+1}(X,\beta) \to \overline{M}_{0,n}(X,\beta)$ which consists in forgetting the extra mark p_{n+1} (and stabilising, by contracting any twig that becomes unstable in the absence of p_{n+1}).

The diagram

$$\begin{tikzcd}
\overline{M}_{0,n+1}(X,\beta) \arrow[r, "\nu_{n+1}"] \arrow[d, shift left, "\pi"] & X \\
\overline{M}_{0,n}(X,\beta) \arrow[u, shift left, "\sigma_i"]
\end{tikzcd}$$

plays the rôle of a universal family (in the setting of stacks, it is a true universal family). Here σ_i is the section corresponding to the mark p_i. The image of σ_i is the irreducible *boundary divisor* $D_{i,n+1}$ characterised as the (closure of the) locus of maps whose source curve is made out of two twigs, one of which carries just the two marks p_i and p_{n+1} and is contracted by μ:



Throughout we will abuse of language like this and talk about a boundary divisor with a degree-0 twig, when we mean a boundary divisor whose general point represents a map whose source curve has two twigs, one of which is of degree 0.

1.1.5 The boundary. More generally, for each stable partition $S' \cup S'' = S$ and $\beta' + \beta'' = \beta$, there is an irreducible boundary divisor denoted $D(S',\beta' \mid S'',\beta'')$. A general point of $D(S',\beta' \mid S'',\beta'')$ represents a map μ from a two-twig curve $C = C' \cup C''$ (with the marks S' on C' and the marks S'' on C''), such that the restriction of μ to C' is a map of degree β' and the restriction of μ to C'' is of degree β''.

Let Δ be the boundary divisor $D(S', \beta' \mid S'', \beta'')$. Each twig corresponds to a moduli space of lower dimension, $\overline{M}' := \overline{M}_{0,S' \cup \{x'\}}(X, \beta')$ and $\overline{M}'' := \overline{M}_{0,S'' \cup \{x''\}}(X, \beta'')$; more precisely, Δ is the image of a finite morphism $\rho_\Delta : \overline{M}' \times_X \overline{M}'' \to \overline{M}_{0,n}(X, \beta)$ (which is an isomorphism in most cases cf. 1.3.1 below). The fibred product is over the two evaluation morphisms $\nu_{x'} : \overline{M}' \to X$ and $\nu_{x''} : \overline{M}'' \to X$, reflecting the fact that in order to glue, the two maps must agree at the mark. The fibred product is a subvariety in the cartesian product $\overline{M}' \times \overline{M}''$; let $\jmath_\Delta$ denote the inclusion. This set-up and notation is used throughout — summarised in the following diagram:

$$\begin{tikzcd} \overline{M}_{0,n}(X, \beta) & \\ \overline{M}' \times_X \overline{M}'' \arrow[u, "\rho_\Delta"] \arrow[r, hook, "\jmath_\Delta"'] & \overline{M}' \times \overline{M}''. \end{tikzcd}$$

Since the fibred product can be viewed as the pull-back of the diagonal in $X \times X$ along the evaluation morphisms at the gluing points, we get the following formula for the boundary divisor considered a cycle in $\overline{M}' \times \overline{M}''$

$$\jmath_*([\overline{M}' \times_X \overline{M}'']) = \sum_{e,f} \nu^*_{x'}(T_e)\, g^{ef}\, \nu^*_{x''}(T_f),$$

which is the expression arising in all recursions, expressing a top product involving one boundary divisor as a product of two top products corresponding to the twigs.

The part of the boundary made up of components both of whose twigs have positive degree is referred to as the *hard boundary*. It can also be characterised as the pull back of the boundary of $\overline{M}_{0,0}(X, \beta)$ along forgetting all marks. Correspondingly, the part of the boundary consisting of boundary divisors one of whose twigs is of degree zero is called the *soft boundary*.

Psi classes

For ulterior reference, let us collect some standard results on psi classes and gravitational descendants. An elementary introduction to these results can be found in my *Notes on psi classes* [29]. For generalisations, see Getzler [21], [22], or Pandharipande [41].

1.1.6 Cotangent lines and psi classes. Let ω_π be the relative dualising sheaf of π. The i'th *cotangent line* of $\overline{M}_{0,n}(X,\beta)$ is the line bundle $\sigma_i^*\omega_\pi$, whose fibre over a moduli point $[\mu : C \to X]$ is the cotangent line $T^*_{p_i}C$. The i'th *psi class* is its first Chern class:

$$\boldsymbol{\psi}_i := c_1(\sigma_i^*\omega_\pi).$$

The psi class $\boldsymbol{\psi}_i$ can also be thought of as "minus" the normal bundle of σ_i:

$$\boldsymbol{\psi}_i = -\sigma_i^* D_{i,n+1}. \tag{1.1.6.1}$$

This identity holds downstairs. Upstairs, the corresponding identity reads

$$D_{i,n+1}^2 = -D_{i,n+1} \cdot \pi^*\boldsymbol{\psi}_i. \tag{1.1.6.2}$$

1.1.7 Basic properties of psi classes. The psi class pulls back along the forgetful morphism like this:

$$\pi^*\boldsymbol{\psi}_i = \boldsymbol{\psi}_i - D_{i,n+1}. \tag{1.1.7.1}$$

In case $n \geq 3$ there is the following expression for the psi class:

$$\boldsymbol{\psi}_1 = (p_1 \mid p_2, p_3) \tag{1.1.7.2}$$

where $(p_1 \mid p_2, p_3)$ denotes the sum of all boundary divisors having the mark p_1 on one twig and p_2 and p_3 on the other twig. This formula is a consequence of the pull-back formula above, and the obvious fact that the psi classes are trivial on $\overline{M}_{0,3}$. (Of course, any two marks distinct from p_1 could replace p_2 and p_3 in this formula.)

Finally, a psi class $\boldsymbol{\psi}_i$ restricted to a boundary divisor Δ gives the psi class $\boldsymbol{\psi}_i$ of the twig carrying p_i. Precisely,

$$\rho_\Delta^*\boldsymbol{\psi}_i = \jmath_\Delta^*\boldsymbol{\psi}_i \tag{1.1.7.3}$$

where it is understood that the psi class on the right hand side is pulled back from $\overline{M}'$ (resp. $\overline{M}''$) if $p_i \in S'$ (resp. $p_i \in S''$).

1.1.8 Gravitational descendants. Top intersections of psi classes and classes pulled back from X are usually called *gravitational descendants*. The following notation has become standard:

$$\langle\, \tau_{u_1}(\gamma_1)\cdots\tau_{u_n}(\gamma_n)\,\rangle_\beta := \int \boldsymbol{\psi}_1^{u_1} \cup \nu_1^*(\gamma_1) \cup \cdots \cup \boldsymbol{\psi}_n^{u_n} \cup \nu_n^*(\gamma_n) \cap [\overline{M}_{0,n}(X,\beta)].$$

The pull-back formula 1.1.7.1 yields three important equations known as the string, dilaton and divisor equations. They apply when there is an extra class which is either the identity, a single psi class, or the pull-back of a single divisor class $T_k \in A^*(X)$. In each formula, the left hand side is on a space with one extra mark.
String equation:

$$\langle \prod_{i=1}^{n} \tau_{u_i}(\gamma_i) \cdot \tau_0(T_0) \rangle_\beta = \sum_{j=1}^{n} \langle \prod_{i=1}^{n} \tau_{u_i-\delta_{ij}}(\gamma_i) \rangle_\beta$$

Dilaton equation:

$$\langle \prod_{i=1}^{n} \tau_{u_i}(\gamma_i) \cdot \tau_1(T_0) \rangle_\beta = (-2+n) \cdot \langle \prod_{i=1}^{n} \tau_{u_i}(\gamma_i) \rangle_\beta$$

Divisor equation:

$$\langle \prod_{i=1}^{n} \tau_{u_i}(\gamma_i) \cdot \tau_0(T_k) \rangle_\beta = \textstyle\int_\beta T_k \cdot \langle \displaystyle\prod_{i=1}^{n} \tau_{u_i}(\gamma_i) \rangle_\beta + \textstyle\sum_{j=1}^{n} \langle \displaystyle\prod_{i=1}^{n} \tau_{u_i-\delta_{ij}}(\gamma_i \cup \delta_{ij} T_k) \rangle_\beta$$

In these formulae, δ_{ij} is the Kronecker delta, and any term with a negative exponent on a psi class is defined to be 0.

1.1.9 Topological recursion relations. Finally, the topological recursion relation applies when there are at least three marks. It is a consequence of the restriction formula 1.1.7.3 and the linear equivalence $\boldsymbol{\psi}_1 = (p_1 \mid p_2, p_3)$. The formula is (assuming $u_1 \geq 1$):

$$\langle \prod_{i=1}^{n} \tau_{u_i}(\gamma_i) \rangle_\beta = \sum \langle \tau_{u_1-1}(\gamma_1) \prod_{\substack{p_i \in S' \\ i \neq 1}} \tau_{u_i}(\gamma_i)\ \tau_0(T_e) \rangle_{\beta'}\ g^{ef}\ \langle \tau_0(T_f) \prod_{p_i \in S''} \tau_{u_i}(\gamma_i) \rangle_{\beta''}.$$

The sum is over all stable splittings $S' \cup S'' = S$ and $\beta' + \beta'' = \beta$, with $p_1 \in S'$ and $p_2, p_3 \in S''$, and over the splitting indices $e, f = 0, \ldots, r$, coming from the Künneth decomposition of the diagonal.

These equations (together with the divisor equation) completely determine the gravitational descendants from the Gromov-Witten invariants. Indeed, if there are three or more marks, use the topological recursion relation to reduce the number of psi classes. Otherwise, use first the divisor equation (backwards) to introduce more marks. (Note that this step may introduce rational coefficients.)

1.2 E-classes

When moduli spaces of stable maps are used in enumerative geometry, they play the rôle of parameter spaces. Imposing conditions on the image curves defines cycles in the moduli space, whose intersections are then interpreted as solutions to the counting problem. The essential structure of a stable map $[\mu : C \to X, p_1, \ldots, p_n]$ is then the *map* μ, and notably its *image curve* $\mu(C) \subset X$, while the marks can be viewed as a secondary structure or a mere bookkeeping device.

One response to this observation is to dispense of the marks altogether and work in the space $\overline{M}_{0,0}(X, \beta)$. The references [42], [40], [45], [46], [47] represent this approach. However, as explained in the introduction, there are good reasons for working consequently with *pointed conditions*. So for example, instead of considering the codimension-1 cycle $\boldsymbol{\tau} \in \overline{M}_{0,0}(X, \beta)$ of maps tangent to a hypersurface in X, consider the codimension-2 cycle in $\Theta_1 \in \overline{M}_{0,1}(X, \beta)$ of maps tangent to the hypersurface at the image of p_1. The two cycles are related by $\pi_* \Theta_1 = \boldsymbol{\tau}$.

Now we need one mark for each of the imposed conditions, and of course the number of conditions vary from problem to problem, so this approach involves various marked spaces and joggling with the forgetful morphisms between them. In this section some terminology and notation is set up, as a minimal framework for dealing with this situation.

1.2.1 Marks and hierarchies. For any inclusion of labelling sets $S \subset R$ there is a forgetful morphism between the corresponding moduli spaces

$$\overline{M}_{0,R}(X, \beta) \to \overline{M}_{0,S}(X, \beta).$$

These moduli spaces and morphisms constitute a category which clearly is equivalent to the category of finite sets and inclusions, with arrows reversed. The terminal object is of course $\overline{M}_{0,0}(X, \beta)$.

Given a set S one can consider the category of spaces corresponding to sets containing S as a subset, that is, the spaces lying over $\overline{M}_{0,S}(X, \beta)$. We call it the hierarchy over S.

It is often fruitful to think of the spaces in such a hierarchy not as separated objects but as different levels of one and the same thing.

1.2.2 Cohomology classes defined on hierarchies. Often the definition of a cohomology class or an algebraic cycle on a moduli space of stable maps is independent of the presence of more marks. That is, if the definition makes sense on one particular space then it also makes sense on all overlying spaces. In that case we say the class (or the cycle) is defined on the hierarchy.

As an example of such a definition consider "the cycle of maps that are not immersions", which makes sense on any space (the minimal space of definition is the un-marked space $\overline{M}_{0,0}(X,\beta)$). Another example, the psi class $\boldsymbol{\psi}_1$. It's definition 1.1.6 makes sense on any space whose labelling set includes p_1, i.e. the minimal space of definition is $\overline{M}_{0,\{p_1\}}(X,\beta)$. Note that this property is not related to the class itself but to its definition. For example, we have $\boldsymbol{\psi}_1 = (p_1 \mid p_2, p_3)$, but this second definition makes sense only in the hierarchy lying over $\overline{M}_{0,\{p_1,p_2,p_3\}}(X,\beta)$.

Thus we allow ourselves to use the same symbol $\boldsymbol{\psi}_1$ for the psi class on the space $\overline{M}_{0,\{p_1\}}(X,\beta)$ and for the first psi class on any space $\overline{M}_{0,n}(X,\beta)$ (with labelling set containing p_1). This practice can sometimes lead to confusion, and occasionally it is necessary to decorate the symbols further, in order to indicate on what space the class is meant to live. (The confusion would be total if this practice were used in conjunction with another abusive practice in intersection theory, namely that of omitting symbols for pull-backs along projections. Consider for example the formula 1.1.7.1 for pull-back of a psi class along a forgetful morphism.)

While many important classes are defined on hierarchies, one notable exception is that of the irreducible boundary divisors.

Definition. A cohomology class defined on a hierarchy of moduli spaces of stable maps is called an *E-class* (E for enumerative) if it is invariant under pull-back along the forgetful morphisms of the hierarchy.

An easy way to get plenty of such classes is to define the class initially on one particular space, and then let it be defined on any overlying space as the pull-back along the forgetful morphism.

If the minimal space of definition of an E-class is a k-pointed space then we will refer to it as a *k-pointed* E-class. (The symbol for such a class should involve an index referring to the k marks in question.) We will only need 1-pointed and 2-pointed classes.

1.2.3 Example. Evaluation classes are (1-pointed) E-classes. Indeed, the evaluation morphisms commute with the forgetful morphisms in the sense that the following diagram commutes:

$$\begin{tikzcd}
\overline{M}_{0,n+1}(X,\beta) \arrow[d, "\pi"'] \arrow[dr, "\nu_i"] & \\
& X \\
\overline{M}_{0,n}(X,\beta) \arrow[ur, "\nu_i"'] &
\end{tikzcd}$$

where π is the morphism that forgets p_{n+1}.

1.2.4 Enumerative cycles. More generally, any cycle defined in terms of the image curve is an E-class. For example: the condition of the image curve being tangent to a given hypersurface at the image of the mark p_1. This cycle is a central object in Chapter 3; in Chapter 5 a wealth of other examples are studied. An example of a 2-pointed E-class is provided by the condition that the two given marks map to the same point in X (cf. 5.2.1).

1.2.5 Example. The pull-back formula 1.1.7.1 shows that the tautological psi classes are not E-classes.

1.2.6 Extrinsic and intrinsic classes. While the classes defined in terms of the image curves may be called extrinsic classes, the following E-classes are intrinsic (following the terminology employed by Diaz and Harris [10] in their study of Severi varieties). They are introduced and studied in this chapter.

(i) The *modified psi classes* denoted $\overline{\boldsymbol{\psi}}_i$, which are defined in 1.4.1 below,

(ii) the *diagonal classes* $\boldsymbol{\delta}_{ij}$, defined in 1.5.1.

The main theme of Chapter 3–5 is showing how extrinsic classes admit simple expressions in terms of intrinsic classes.

1.3 Some results on the boundary

The semi-group of effective curve classes $\beta \in A_1(X)$ is partially ordered by declaring $\beta' < \beta''$ whenever there exists an effective class β''' such that $\beta' + \beta''' = \beta''$.

1.3.1 Lemma. *Let $\Delta = D(S', \beta' \mid S'', \beta'')$ be a boundary divisor such that S' is non-empty and $\beta' < \beta''$. Then ρ_Δ is an isomorphism onto Δ.*

This lemma completes Lemma 12 of FP-NOTES, where it is proved that if both S' and S'' are non-empty then ρ_Δ is an isomorphism onto its image. Together the two lemmas cover all the cases of isomorphism.

Note in particular that whenever a boundary divisor Δ has got a twig of degree zero, then ρ_Δ is an isomorphism onto Δ.

Proof. We need to show that given a map $\mu : C \to X$ belonging to the divisor Δ, there is only one node (referred to as the official node) that can be the limit of the node of generic elements in Δ; in other words, there is only one way to cut the curve.

START: Due to the lemma of FP-NOTES, we can suppose $B = \emptyset$. There is a minimal sub-tree $T' \subset C$ carrying all the marks. Let T' grow:

STEP 1: Whenever T' disconnects the rest, then there is only one of the remaining connected component that has total degree $\geq \beta''$; let T' grow, incorporating the others.

STEP 2: As long as the total degree of T' has not reached β', we must incorporate the next twig. If this act disconnects the rest, repeat step 1.

STOP: If the total degree of T' is equal to β' the growth stops here, and we have found the unique spot to cut the curve. Indeed, if the next twig C° has positive degree, it is clear that we cannot include it in T'. If C° has degree zero, then by *stability* it must intersect another two connected components, each of which must have positive total degree. Including C° would disconnect the rest, and consequently force one of the remaining components to be included in T'. Anyway, the degree of T' would become too high.

This shows that ρ_Δ is bijective onto its image. The proof of normality of Δ is identical to that indicated in FP-NOTES and resorts to the locally rigidified moduli space $\overline{M}_{0,n}(X, \beta, \bar{t})$: For the divisors lying over Δ to intersect, there must be a symmetry along some degeneration of the configuration represented by Δ. The above arguments show such symmetries do not occur, so Δ is normal. Since ρ_Δ is bijective between normal varieties (and we are in characteristic zero) it must be an isomorphism. □

1.3.2 Lemma. *If Δ is a boundary divisor such that ρ_Δ is an isomorphism onto its image, then*

$$\rho_\Delta^* \Delta = \jmath_\Delta^*(-\boldsymbol{\psi}_{x'} - \boldsymbol{\psi}_{x''}).$$

Here $\boldsymbol{\psi}_{x'}$ (resp. $\boldsymbol{\psi}_{x''}$) denotes the pull-back to $\overline{M}' \times \overline{M}''$ from $\overline{M}'$ (resp. $\overline{M}''$) of the psi class of the gluing point.

This generalises 1.1.6.1. Note in particular that by Lemma 1.3.1, the formula holds for all boundary divisors with a contracting twig.

For the sake of completeness: If Δ is not normal, its pull-back along ρ_Δ has one more term which is the push-forth of the singular locus of Δ. I will not prove that formula here, since it is not needed in the sequel.

Proof. For short, set $\overline{M} := \overline{M}_{0,n}(X,\beta)$ and $U := \overline{M}_{0,n+1}(X,\beta)$, and let $\pi : U \to \overline{M}$ be the forgetful morphism. Its restriction $\bar{\pi}$ to $D_\Delta := \pi^{-1}(\Delta)$ is a tautological family over Δ; note that D_Δ is a divisor in U and it has two components D' and D'', corresponding to whether the new mark is on the $'$-twig or the $''$-twig. Now the key point is that the family $\bar{\pi} : D_\Delta \to \Delta$ has a section $\bar{\sigma}_x : \Delta \to D_\Delta$, defined by putting the new mark on the official node and stabilising. This works because Δ has only one official node, according to (the proof of) the previous lemma.

$$\begin{tikzcd} D_\Delta \arrow[r, hook, "\bar{\rho}"] \arrow[d, shift left, "\bar{\pi}"] & U \arrow[d, "\pi"] \\ \Delta \arrow[r, hook, "\rho"'] \arrow[u, shift left, "\bar{\sigma}_x"] & \overline{M} \end{tikzcd}$$

Note that D' and D'' intersect transversely along $\bar{\sigma}_{x*}\Delta$.

From the cartesian diagram we get

$$\begin{aligned} \rho^*\rho_*\Delta &= \rho^*(\pi_*\bar{\rho}_*\sigma_{x*}\Delta) \\ &= \bar{\pi}_*\bar{\rho}^*(\bar{\rho}_*\sigma_{x*}\Delta) \\ &= \bar{\pi}_*\bar{\rho}^*(D' \cap D'') \\ &= \bar{\pi}'_*\bar{\rho}'^*(D' \cap D'') + \bar{\pi}'_*\bar{\rho}''^*(D' \cap D''), \end{aligned} \tag{1.3.2.1}$$

where the primed morphisms in the last line are explained by the diagram

$$\begin{tikzcd} & U & \\ D' \arrow[ur, hook, "\bar{\rho}'"] \arrow[r, hook] \arrow[dr, "\bar{\pi}'"'] & D_\Delta \arrow[u, hook, "\bar{\rho}"] \arrow[d, "\bar{\pi}"] & D'' \arrow[ul, hook', "\bar{\rho}''"'] \arrow[l, hook'] \arrow[dl, "\bar{\pi}''"] \\ & \Delta & \end{tikzcd}$$

Note that the section $\bar{\sigma}_x$ factors through D' giving a section $\bar{\sigma}_{x'} : \Delta \to D'$. The image of $\bar{\sigma}_{x'}$ (which we denote by the same symbol) is cut out by the divisor D'' restricted to D'.

So, concerning the first term in 1.3.2.1, we continue

$$\begin{aligned} \bar{\pi}'_*(\bar{\rho}'^*(D' \cap D'')) &= \bar{\pi}'_*(\bar{\rho}'^* D' \cap \bar{\sigma}_{x'}) \\ &= \bar{\sigma}^*_{x'}(\bar{\rho}'^* D') \\ &= \bar{\sigma}^*_{x''}(\bar{\rho}''^* D') \\ &= \bar{\sigma}^*_{x''}(\bar{\sigma}_{x''}). \end{aligned}$$

By assumption we are identifying Δ with the fibred product $\overline{M}' \times_X \overline{M}''$; under this identification, $\bar{\pi}' : D' \to \Delta$ is the pull-back along $\jmath_\Delta$ of the tautological family $\pi' : U' \to \overline{M}'$. Similarly for $''$. The intersection $D' \cap D''$ in U corresponds to the image of the sections $\sigma_{x'}$ and $\sigma_{x''}$, along which the pulled-back families $\jmath^*_\Delta U'$ and $\jmath^*_\Delta U''$ are glued together. (These identifications are only natural up to non-unique isomorphism, but that is sufficient for our purposes.)

We find that

$$\begin{aligned} \bar{\sigma}^*_{x''}(\bar{\sigma}_{x''}) &= \jmath^*_\Delta \sigma^*_{x''} \sigma_{x''} \\ &= -\jmath^*_\Delta \boldsymbol{\psi}_{x''}. \end{aligned}$$

Same reasoning for the second term in 1.3.2.1 yields the result. $\square$

Definition. On a space $\overline{M}_{0,n}(X,\beta)$ with $\beta > 0$, let $\boldsymbol{\xi}_i$ denote the sum of all boundary divisors whose general point represents a map contracting the twig carrying the mark p_i. That is, $\boldsymbol{\xi}_i = \sum_{S' \ni p_i} D(S', 0 \mid S'', \beta)$.

1.3.3 Lemma. *Let $\Delta = D(S', 0 \mid S'', \beta)$ be a boundary divisor with a contracting twig.*
(a) *If $p_i \in S'$ then*

$$\rho^*_\Delta \boldsymbol{\xi}_i = \jmath^*_\Delta \big((p_i \mid x') - \boldsymbol{\psi}_{x'} - \boldsymbol{\psi}_{x''} + \boldsymbol{\xi}_{x''} \big).$$

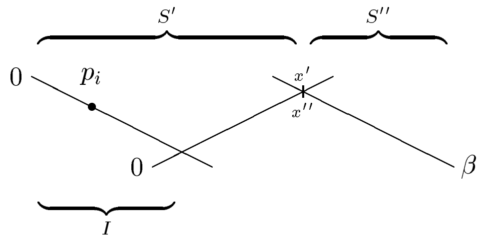
(b) *If $p_i \in S''$ then*

$$\rho^*_\Delta \boldsymbol{\xi}_i = \jmath^*_\Delta \boldsymbol{\xi}_i.$$

In (a), the symbol $(p_i \mid x')$ denotes the sum of all boundary divisors of $\overline{M}'$ with mark p_i on one twig and x' on the other. The divisor $\boldsymbol{\xi}_{x''}$ lives in $\overline{M}''$. In (b), the divisor $\boldsymbol{\xi}_i$ is understood to live in $\overline{M}''$, since $p_i \in S''$.

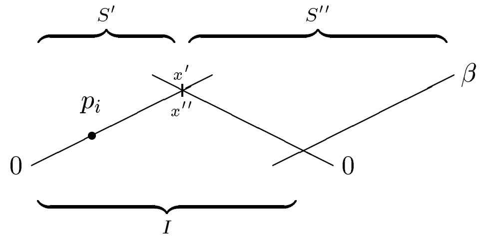
Proof. Re. (a): Note first of all that Δ itself is a component in $\boldsymbol{\xi}_i$. Therefore the product includes the self-intersection class $\Delta^2 = \jmath^*(-\boldsymbol{\psi}_{x'} - \boldsymbol{\psi}_{x''})$ (by 1.3.2). The other components of $\boldsymbol{\xi}_i$ are boundary divisors distinct from Δ, and their intersection with Δ is a reduced effective cycle. For such a divisor, let $I \subset S$ denote the degree-0 part, that is, the part containing p_i. If neither $I \subsetneq S'$ nor $I \supsetneq S'$, then clearly the intersection is empty.

Each component of $\boldsymbol{\xi}_i$ such that $I \subsetneq S'$, intersects Δ by breaking the $'$-twig like this:



such that the mark p_i is broken apart from the attachment mark x'. There is a 1–1 correspondence between the possible $I \subsetneq S'$ and these possible breaks of the $'$-twig. These codimension-2 boundary cycles correspond again to divisors in $\overline{M}'$ such that p_i is on one twig and x' is on the other twig. This accounts for the contribution $(p_i \mid x')$.

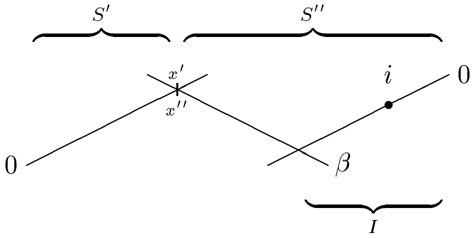
Next, the cases $I \supsetneq S'$ corresponds to breaking the $''$-twig in this way:



such that the marks of $I \cap S''$ are put on the new middle twig, of degree 0. In the space $\overline{M}''$ this corresponds to all ways of breaking off a twig of degree 0 carrying the mark x''. So these cases give the contribution $\boldsymbol{\xi}_{x''}$.

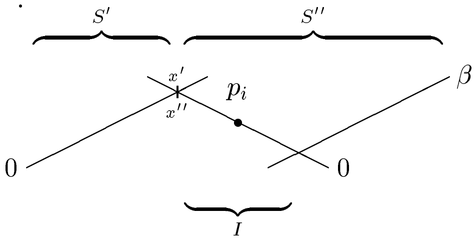
Now for item (b): This time, Δ has p_i on the degree-β twig, so all components in $\boldsymbol{\xi}_i$ are distinct from Δ. The intersection of such a divisor with Δ is empty unless $I \subset S''$ or $I \supset S' \cup \{p_i\}$.

The cases $I \subset S''$ corresponds to breaking the $''$-twig like this



— that is: break off a degree-0 twig containing p_i but not x''.

The cases $I \supset S' \cup \{p_i\}$ corresponds to breaking off a degree-0 twig containing both p_i and x''.



All together we get all the ways of breaking off p_i with degree 0, that is, the contribution is $\boldsymbol{\xi}_i$ (from the $''$-space). □

1.4 Modified psi classes

1.4.1 Definition. On a space $\overline{M}_{0,n}(X,\beta)$ with $\beta > 0$, the *modified psi classes* are by definition

$$\overline{\boldsymbol{\psi}}_i := \boldsymbol{\psi}_i - \boldsymbol{\xi}_i.$$

The motivation for this definition is the following observation:

1.4.2 Remark. Let $\pi : \overline{M}_{0,n+1}(X,\beta) \to \overline{M}_{0,n}(X,\beta)$ be the morphism that forgets the last mark. Then

$$\pi^*\overline{\boldsymbol{\psi}}_i = \overline{\boldsymbol{\psi}}_i.$$

in other words, the modified psi classes are E-classes.

1.4.3 Remark. On a space with just one mark, we have $\overline{\boldsymbol{\psi}}_1 = \boldsymbol{\psi}_1$, so an equivalent definition is $\overline{\boldsymbol{\psi}}_i := \hat{\pi}_i^*\boldsymbol{\psi}_i$, where $\hat{\pi}_i$ denotes the morphism that forgets all marks except p_i.

1.4.4 Lemma. *Let $\Delta = (S', 0 \mid S'', \beta)$ be a boundary divisor with a contracting twig. Then*

$$\rho_\Delta^*\overline{\boldsymbol{\psi}}_i = \begin{cases} \jmath_\Delta^*\overline{\boldsymbol{\psi}}_{x''} & \text{in case } p_i \in S' \\ \jmath_\Delta^*\overline{\boldsymbol{\psi}}_i & \text{in case } p_i \in S''. \end{cases}$$

In 1.5.9 below, the corresponding formula for the case where Δ is a hard boundary divisor is given. It depends on the concept of diagonal classes, introduced in the next section.

Proof. By definition, $\overline{\boldsymbol{\psi}}_i = \boldsymbol{\psi}_i - \boldsymbol{\xi}_i$. Suppose $p_i \in S'$. Applying the restriction formula 1.1.7.3 and the previous lemma gives $\sigma_\Delta^* \overline{\boldsymbol{\psi}}_i = \jmath_\Delta^* \big(\boldsymbol{\psi}_i - (p_i \mid x') + \boldsymbol{\psi}_{x'} + \boldsymbol{\psi}_{x''} - \boldsymbol{\xi}_{x''}\big)$. Now note that by stability, the $'$-twig must have yet another mark, say p_j, so by 1.1.7.2 we can write $\boldsymbol{\psi}_i = (p_i \mid p_j, x')$, and $\boldsymbol{\psi}_{x'} = (p_i, p_j \mid x')$. It follows that $\boldsymbol{\psi}_i + \boldsymbol{\psi}_{x'} = (p_i \mid x')$. This proves the first case. The second case is immediate from Lemma 1.3.3. □

1.4.5 Remark. The second case shows the geometric nature of the modified classes: If two marks come together, a new degree zero twig sprouts out and takes the marks, but the corresponding modified psi classes stick to the original twig instead of following the marks to the new one, which is illusory from the point of view of the image curve.

1.5 Diagonal classes

1.5.1 Diagonal classes. Let $\overline{D}_{ij}$ be the divisor in $\overline{M}_{0,n}(X, \beta)$ defined as the sum of all boundary divisors having p_i and p_j together on a contracting twig. It can also be described as the pull-back from $\overline{M}_{0,\{p_i,p_j\}}(X, \beta)$ of the divisor D_{ij}. By definition, the ij'th *diagonal class* $\boldsymbol{\delta}_{ij}$ is the class of $\overline{D}_{ij}$. From the pull-back description it is clear that diagonal classes are E-classes.

1.5.2 Lemma. *We have* $\pi_{i*}\boldsymbol{\delta}_{ij} = 1$, *the fundamental class.*

Proof. Indeed, $\boldsymbol{\delta}_{ij}$ is a sum of boundary divisors. One of them is $D_{ij} = \sigma_{j*}(1)$, and being a section, its push-down is the fundamental class. As to the others, they all have at least four special points on the contracting twig, so they are not destabilised when forgetting p_i. Therefore they drop dimension and have zero push-down. □

1.5.3 Lemma. *The diagonal classes enjoy the following properties.*

$$\begin{aligned} \boldsymbol{\delta}_{ij}\alpha_i &= \boldsymbol{\delta}_{ij}\alpha_j \\ \boldsymbol{\delta}_{ij}\alpha_{ik} &= \boldsymbol{\delta}_{ij}\alpha_{jk} \\ -\boldsymbol{\delta}_{ij}^2 = \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_i &= \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_j. \end{aligned}$$

Here α_i denote any 1-pointed E-class, and α_{ij} denotes any 2-pointed E-class.

These properties should justify the name “diagonal class”.

Proof. Consider $\overline{D}_{ij}$ as a subscheme of $\overline{M}_{0,n}(X,\beta)$, with inclusion $\iota : \overline{D}_{ij} \hookrightarrow \overline{M}_{0,n}(X,\beta)$, and consider the diagram

$$\begin{tikzcd}
& \boldsymbol{\delta}_{ij} \arrow[dl, "\hat{\tau}_{ik}"'] \arrow[dr, "\hat{\tau}_{jk}"] & \\
M_{\{p_i,p_k\}} \arrow[d] & \simeq & M_{\{p_j,p_k\}} \arrow[d] \\
M_{\{p_i\}} & \simeq & M_{\{p_j\}}.
\end{tikzcd}$$

Here $\overline{M}_S$ is short for $\overline{M}_{0,S}(X,\beta)$, the horizontal isomorphisms are change of label $p_i \leftrightarrow p_j$, and the vertical arrows are forgetful morphisms. Obviously the square is commutative.

In the triangle, $\hat{\tau}_{ik}$ is the restriction of $\hat{\pi}_{ik}$ to $\overline{D}_{ij}$ (and $\hat{\tau}_{jk}$ is the restriction of $\hat{\pi}_{jk}$). The triangle is also commutative. Indeed, when all marks except p_i and p_k are forgotten, an element in $\overline{D}_{ij}$ retains at most three special points on the contracting twig, and the position of three marks on a contracting twig is immaterial. So whether we keep p_i or p_j we end up with the same result.

Now the lemma follows from the commutativity of the triangle:

$$\begin{array}{c}
\boldsymbol{\delta}_{ij}\alpha_{ik} = \iota^*\alpha_{ik} = \iota^*\hat{\pi}_{ik}^*\alpha_{ik} = \hat{\tau}_{ik}^*\alpha_{ik} \\
\hfill \shortparallel \quad \\
\boldsymbol{\delta}_{ij}\alpha_{jk} = \iota^*\alpha_{jk} = \iota^*\hat{\pi}_{jk}^*\alpha_{jk} = \hat{\tau}_{jk}^*\alpha_{jk}
\end{array}$$

The case of α_i is just composing with the lower square.

As to the self-intersection, it follows from pull-back of the relation 1.1.6.1 $D_{12}^2 = -D_{12} \cdot \pi_2^* \boldsymbol{\psi}_1 = -D_{12} \cdot \overline{\boldsymbol{\psi}}_1$, where the last equality holds in a space with just two marks. $\square$

1.5.4 Remark. The argument holds also when $\overline{D}_{ij}$ is replaced by any of its irreducible components. For example, it holds for the irreducible boundary divisor D_{ij}.

1.5.5 Corollary. *For $k \geq 1$ we have $(\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ij})^k = \overline{\boldsymbol{\psi}}_i^k + \overline{\boldsymbol{\psi}}_i^{k-1}\boldsymbol{\delta}_{ij}$.*

Proof. The formula obviously holds for $k = 1$. Now by induction,

$$\begin{aligned}(\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ij})^k &= (\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ij})(\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ij})^{k-1} \\ &= (\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ij})(\overline{\boldsymbol{\psi}}_i^{k-1} + \overline{\boldsymbol{\psi}}_i^{k-2}\boldsymbol{\delta}_{ij}) \\ &= \overline{\boldsymbol{\psi}}_i^k + 2\overline{\boldsymbol{\psi}}_i^{k-1}\boldsymbol{\delta}_{ij} + \overline{\boldsymbol{\psi}}_i^{k-2}\boldsymbol{\delta}_{ij}^2 \\ &= \overline{\boldsymbol{\psi}}_i^k + \overline{\boldsymbol{\psi}}_i^{k-1}\boldsymbol{\delta}_{ij}.\end{aligned}$$

□

1.5.6 Corollary. *Any top intersection involving diagonal classes, modified psi classes and evaluation classes can be reduced to a product involving only modified psi classes and evaluation classes.*

Proof. If there is a diagonal class, say $\boldsymbol{\delta}_{ij}$, use the self-intersection formula to reduce to the case where there is only one factor of $\boldsymbol{\delta}_{ij}$. Next use the formulae 1.5.3 above to substitute all other indices i by j. Now all other classes in the product are pull-backs from the space without mark p_i, so pushing down forgetting p_i we get rid of $\boldsymbol{\delta}_{ij}$ via the projection formula and Lemma 1.5.2. □

The following lemma follows easily from the arguments of 1.3.3:

1.5.7 Lemma. *Let $\Delta = D(S', 0 \mid S'', \beta)$ be a boundary divisor with a contracting twig. Then*

$$\rho_\Delta^* \boldsymbol{\delta}_{ij} = \begin{cases} -\jmath_\Delta^* \overline{\boldsymbol{\psi}}_{x''} & \text{when } p_i, p_j \in S' \\ \jmath_\Delta^* \boldsymbol{\delta}_{x''j} & \text{when } p_i \in S' \text{ and } p_j \in S'' \\ \jmath_\Delta^* \boldsymbol{\delta}_{x''i} & \text{when } p_j \in S' \text{ and } p_i \in S'' \\ \jmath_\Delta^* \boldsymbol{\delta}_{ij} & \text{when } p_i, p_j \in S''. \end{cases}$$

□

1.5.8 Lemma. *Let $\Delta = D(S', \beta' \mid S'', \beta'')$ be a boundary divisor with $\beta' > 0$, $\beta'' > 0$, and $p_i \in S'$. Then*

$$\rho_\Delta^* \boldsymbol{\xi}_i = \boldsymbol{\xi}_i - \boldsymbol{\delta}_{ix'}.$$

Proof. Let $I \subset S$ denote the degree-0 part; that is, the part containing p_i. Unless $I \subsetneq S'$, clearly the intersection of $\boldsymbol{\xi}_i$ with Δ is empty. Each of the components of $\boldsymbol{\xi}_i$ with $I \subsetneq S'$ intersects Δ by breaking the off a 0-twig containing p_i but not x'. So the total contribution is $\boldsymbol{\xi}_i - \boldsymbol{\delta}_{ix'}$. □

Combining 1.4.4 and 1.5.8, we get

1.5.9 Key formula. *Let $\Delta = D(S', \beta' \mid S'', \beta'')$ be a boundary divisor with $p_i \in S'$. Then*

$$\rho_\Delta^* \overline{\boldsymbol{\psi}}_i = \begin{cases} \jmath_\Delta^* \overline{\boldsymbol{\psi}}_i & \text{in case } \beta' = \beta \\ \jmath_\Delta^* \overline{\boldsymbol{\psi}}_{x''} & \text{in case } \beta' = 0 \\ \jmath_\Delta^* \big(\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ix'}\big) & \text{in case } \beta' \neq 0, \beta' \neq \beta \end{cases} \qquad \square$$

1.6 Enumerative descendants

Definition. We define an *enumerative descendant* to be a top product of modified psi classes and evaluation classes, and agree on the following notation:

$$\langle\, \overline{\tau}_{m_1}(\gamma_1) \cdots \overline{\tau}_{m_n}(\gamma_n) \,\rangle_\beta := \int \overline{\boldsymbol{\psi}}_1^{m_1} \cup \nu_1^*(\gamma_1) \cup \cdots \cup \overline{\boldsymbol{\psi}}_n^{m_n} \cup \nu_n^*(\gamma_n) \cap [\overline{M}_{0,n}(X,\beta)].$$

1.6.1 Remark. Since the modified psi classes as well as the classes pulled back from X are E-classes, the projection formula immediately implies the following analogues of the string, dilaton and divisor equations:

$$\langle\, \overline{\tau}_{m_1}(\gamma_1) \cdots \overline{\tau}_{m_n}(\gamma_n) \overline{\tau}_0(T_0) \,\rangle_\beta = 0 \qquad (1.6.1.1)$$

$$\langle\, \overline{\tau}_{m_1}(\gamma_1) \cdots \overline{\tau}_{m_n}(\gamma_n) \overline{\tau}_1(T_0) \,\rangle_\beta = -2 \cdot \langle\, \overline{\tau}_{m_1}(\gamma_1) \cdots \overline{\tau}_{m_n}(\gamma_n) \,\rangle_\beta \qquad (1.6.1.2)$$

$$\langle\, \overline{\tau}_{m_1}(\gamma_1) \cdots \overline{\tau}_{m_n}(\gamma_n) \overline{\tau}_0(T_k) \,\rangle_\beta = \textstyle\int_\beta T_k \cdot \langle\, \overline{\tau}_{m_1}(\gamma_1) \cdots \overline{\tau}_{m_n}(\gamma_n) \,\rangle_\beta \qquad (1.6.1.3)$$

where in the last equation (the divisor equation) it is understood that $T_k \in A^1(X)$. Note that these equations are considerably simpler than the string, dilaton and divisor equation for the tautological psi classes.

Since the modified psi classes admit an expression in terms of boundary divisors (cf. 1.1.7.2 and 1.4.1) and we know how to restrict modified psi classes to the boundary (cf. 1.5.9), it is clear that topological recursion should work. It remains to handle the diagonal classes that appear. This step makes topological recursion much harder for modified psi classes than for the tautological ones.

The next result is a splitting lemma, i.e. a formula for the restriction to a boundary divisor of an almost-top product of modified psi classes and evaluation classes. To state it in a readable way we need some further notation.

Observe that the diagonal classes appearing as correction terms in 1.5.9 are all of the form $\boldsymbol{\delta}_{ix'}$ ($\boldsymbol{\delta}_{ix''}$ on the other twig). The effect of multiplying with such a class is to gather a lot of classes together at a single mark, in case the gluing mark x' , so let us introduce a shorthand notation for this. Given a set of marks B, and a product of classes $\prod_{i\in B} \overline{\tau}_{m_i}(\gamma_i)$ indexed by B, set

$$\gamma^B := \prod_{p_i \in B} \gamma_i, \qquad \text{and} \qquad \mathbf{m}_B := \sum_{p_i\in B} (m_i - 1).$$

In the definition of the integer $\mathbf{m}_B$, a priori there might be negative summands, but we are going to preclude this in the application. Now the one-pointed class of interest is $\overline{\tau}_{\mathbf{m}_B}(\gamma^B)$. With this notation,

1.6.2 Lemma. SPLITTING LEMMA. *Let $\Delta = D(S', \beta' | S'', \beta'')$ be a boundary divisor with β' and β'' non-zero. Suppose we are given an almost-top product $\overline{\tau}_{m_1}(\gamma_1)\cdots\overline{\tau}_{m_n}(\gamma_n)$, then the integral $\langle\, \Delta \cdot \overline{\tau}_{m_1}(\gamma_1)\cdots\overline{\tau}_{m_n}(\gamma_n)\,\rangle_\beta$ is equal to*

$$\sum_{e,f}\sum \Big(\langle \prod_{p_i\in A'} \overline{\tau}_{m_i}(\gamma_i)\cdot \overline{\tau}_{\mathbf{m}_{B'}}(\gamma^{B'} \cup T^e)\,\rangle_{\beta'}\Big) g^{ef} \Big(\langle \prod_{p_i\in A''} \overline{\tau}_{m_i}(\gamma_i)\cdot \overline{\tau}_{\mathbf{m}_{B''}}(\gamma^{B''} \cup T^f)\,\rangle_{\beta''}\Big)$$

The outer sum is over the splitting indices e and f running from 0 to r. The inner sum is over all partitions $A' \cup B' = S'$ such that $m_i > 0$ for all $p_i \in B'$, and over all partitions $A'' \cup B'' = S''$ such that $m_i > 0$ for all $p_i \in B''$.

Proof. The splitting indices come from the fact that the class of $\Delta \simeq \overline{M}' \times_X \overline{M}''$ in $\overline{M}' \times \overline{M}''$ is the pull-back of the diagonal in $X \times X$ along the product of the evaluation morphisms at the two gluing marks. This explains the factor

$$\sum_{e,f} \nu_{x'}^*(T_e)\, g^{ef}\, \nu_{x''}^*(T_f).$$

Now the cohomology classes at the gluing marks are not left at that, because of the appearance of the diagonal classes when the modified psi classes restrict to Δ. Let us explain what happens on the one-primed twig. Given $p_i \in S'$, the factor $\overline{\tau}_{m_i}(\gamma_i) = \overline{\boldsymbol{\psi}}_i^{m_i} \cup \nu_i^*(\gamma_i)$ restricts to Δ giving

$$(\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ix'})^{m_i} \cup \nu_i^*(\gamma_i) = \overline{\boldsymbol{\psi}}_i^{m_i} \cup \nu_i^*(\gamma_i) + \overline{\boldsymbol{\psi}}_i^{m_i-1} \cup \boldsymbol{\delta}_{ix'} \cup \nu_i^*(\gamma_i),$$

cf. 1.5.5. For each $p_i \in S'$ there is such a sum; expanding the product of all these sums, we get a sum over all 2-partitions $A' \cup B' = S'$, where the A'-part corresponds to taking the terms without a diagonal class involved, and the B'-part corresponds to the terms with a diagonal class. Now a diagonal class appears at all only at marks p_i where $m_i > 0$, so we must consider only partitions such that $p_i \in A'$ whenever $m_i = 0$. Now the effect of the diagonal class $\boldsymbol{\delta}_{ix'}$ is to move the classes $\overline{\boldsymbol{\psi}}_i^{m_i-1}\nu_i^*(\gamma_i)$ over to the gluing mark x'. To be more precise, use 1.5.3 to replace the indices i by x', and then push down forgetting p_i (cf. 1.5.2). Doing this for each of the marks in B', accumulates at x' the class $\overline{\tau}_{\mathbf{m}_B}(\gamma^B)$, and there was already a class $\nu_{x'}^*(T_e)$, thus totalling $\overline{\tau}_{\mathbf{m}_B}(\gamma^B \cup T_e)$, as claimed. This explains the contribution from the one-primed twig. Same arguments hold for the other twig. $\square$

1.6.3 Theorem. Topological recursion. *The following recursive relation holds on a space with at least three marks, say $\overline{M}_{0,S\cup\{p_1,p_2,p_3\}}(X,\beta)$.*

$$\begin{aligned}
\langle\, \overline{\tau}_{m_1+1}(\gamma_1)\overline{\tau}_{m_2}(\gamma_2)\overline{\tau}_{m_3}(\gamma_3)\cdot \prod_{p_i\in S}\overline{\tau}_{m_i}(\gamma_i)\,\rangle_\beta = &\\
+\ \langle\, \overline{\tau}_{m_1}(\gamma_1)\overline{\tau}_{m_2+m_3}(\gamma_2\cup\gamma_3)\cdot \prod_{p_i\in S}\overline{\tau}_{m_i}(\gamma_i)\,\rangle_\beta &\\
-\ \langle\, \overline{\tau}_{m_1+m_2}(\gamma_1\cup\gamma_2)\overline{\tau}_{m_3}(\gamma_3)\cdot \prod_{p_i\in S}\overline{\tau}_{m_i}(\gamma_i)\,\rangle_\beta &\\
-\ \langle\, \overline{\tau}_{m_1+m_3}(\gamma_1\cup\gamma_3)\overline{\tau}_{m_2}(\gamma_2)\cdot \prod_{p_i\in S}\overline{\tau}_{m_i}(\gamma_i)\,\rangle_\beta &\\
+\sum\sum\Big(\langle \prod_{i\in A'}\overline{\tau}_{m_i}(\gamma_i)\cdot\overline{\tau}_{\mathbf{m}_{B'}}(\gamma^{B'}\cup T^e)\,\rangle_{\beta'}\Big)g^{ef}\Big(\langle \prod_{i\in A''}\overline{\tau}_{m_i}(\gamma_i)\cdot\overline{\tau}_{\mathbf{m}_{B''}}(\gamma^{B''}\cup T^f)\,\rangle_{\beta''}\Big)&
\end{aligned}$$

The big outer sum is over all stable splittings $S'\cup S'' = S$, $\beta'+\beta''=\beta$, $\beta'>0$, $\beta''>0$, and over the splitting indices e,f running from 0 to r. The inner sum is over all partitions $A'\cup B' = S'\cup\{p_1\}$ such that $m_i>0$ for all $i\in B'$, and over all partitions $A''\cup B'' = S''\cup\{p_2,p_3\}$ such that $m_i>0$ for all $i\in B''$.

Proof. Write the first modified psi class as a sum of boundary divisors,

$$\overline{\boldsymbol{\psi}}_1 = (p_1 \mid p_2,p_3) - \boldsymbol{\xi}_1, \tag{1.6.3.1}$$

and compute the restriction of the remaining factors of the product to each of the irreducible components of this expression. Let us first consider the soft part. Since there are at least two marks on the contracting twig, say p_i and p_j we are in position to use the Remark 1.5.4: all the remaining factors are invariant under pull-back so we can substitute all indices i by j, and then compute the integral via a push-down, forgetting p_i. Now, unless p_i and p_j are the only marks on the 0-twig, we get zero push-down, cf. 1.5.2. So among the soft boundary divisors in the expression 1.6.3.1, only the ones with just two marks on the contracting twig contribute. From $(p_1 \mid p_2, p_3)$ we get D_{23} as well as D_{1k} for $p_k \in S$. Now all these last ones are also in $\boldsymbol{\xi}_1$. On the other hand, in $\boldsymbol{\xi}_1$ we find D_{12} and D_{13} which are not in $(1|2,3)$. So all together only three soft components contribute:

$$D_{23} - D_{12} - D_{13}.$$

The effect of restricting to such a boundary divisor, say D_{12} is to merge all the classes indexed by 1 and 2, so this explains the first three terms in the formula.

Now for the hard boundary: We are concerned with the sum of all boundary divisors $\Delta = D(S' \cup \{p_1\}, \beta' \mid S'' \cup \{p_2, p_3\}, \beta'')$ with $\beta' > 0$ and $\beta'' > 0$. (Thus explaining this summation in the formula.) For each of these boundary divisors the splitting lemma applies. □

1.6.4 Remark. While this recursion is not very pleasant to look at, it is well-suited for computer implementation, and in fact it is quite effective: each step reduces the number of modified psi classes by one, so the depth of recursion is equal to the number of modified psi classes.

Second, it is worth noticing that when applied to a specific enumerative problem, the terms appearing on the right hand side often can be given enumerative interpretation as well. This yields a recursion involving only *numbers* (and not just divisor classes). This idea is pursued in Chapter 3. In the next chapter, we restrict attention to degree-1 enumerative descendants, for which the recursion is particularly nice.

Chapter 2

The tangency quantum potential

2.1 Degree-1 enumerative descendants

Here we have a closer look at a particularly well-behaved class of enumerative descendants, namely those where the exponent of a modified psi class is at most 1. There are two reasons for this restriction: First, they are the ones needed to describe tangency conditions (cf. Chapter 3), and second, they reproduce themselves in a nice manner in the recursions, allowing for elegant version of the topological recursion relation 1.6.3, as we proceed to describe.

In this chapter and throughout we adopt the usual multi-index notation: for any vector $\mathbf{s} = (s_0, \ldots, s_r)$ of non-negative integers, set $\mathbf{s}! = s_0!s_1! \cdots s_r!$, and $|\mathbf{s}| = s_0 + \cdots + s_r$; if $\mathbf{y} = (y_0, y_1, \ldots, y_r)$ are formal variables, set $\mathbf{y}^{\mathbf{s}} = y_0^{s_0} y_1^{s_1} \cdots y_r^{s_r}$; and with $\mathbf{T} = (T_0, T_1, \ldots, T_r)$, set $\mathbf{T}^{\mathbf{s}} = T_0^{s_0} \cup T_1^{s_1} \cup \cdots \cup T_r^{s_r}$.

Let us introduce the following notation for these degree-1 enumerative descendants:

$$\langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta := \langle\ \prod_{k=0}^{r} (\overline{\tau}_0(T_k))^{a_k} (\overline{\tau}_1(T_k))^{b_k}\ \rangle_\beta,$$

where $\mathbf{a} = (a_0, \ldots, a_r)$ and $\mathbf{b} = (b_0, \ldots, b_r)$ are vectors of non-negative integers. The integral is zero unless $\sum_i \deg(T_i)(a_i + b_i + 1) = \dim \overline{M}_{0,|\mathbf{a}|+|\mathbf{b}|}(X, \beta)$.

Let us state the splitting Lemma 1.6.2 as it reads for degree-1 enumerative descendants.

2.1.1 Lemma. SPLITTING LEMMA. *Let* $\Delta = D(S', \beta' | S'', \beta'')$ *be a boundary divisor with* β' *and* β'' *non-zero. Suppose we are given an almost-top product* $\overline{\boldsymbol{\tau}}_0(\mathbf{a})\overline{\boldsymbol{\tau}}_1(\mathbf{b})$ *such that the marks corresponding to* $\mathbf{a}' \leq \mathbf{a}$ *and* $\mathbf{b}' \leq \mathbf{b}$ *belong to* S'*, and the remaining marks, corresponding to* $\mathbf{a}'' \leq \mathbf{a}$ *and* $\mathbf{b}'' \leq \mathbf{b}$*, belong to* S''*. Then*

$$\langle\, \Delta \cdot \overline{\boldsymbol{\tau}}_0(\mathbf{a})\overline{\boldsymbol{\tau}}_1(\mathbf{b}) \,\rangle_\beta = \sum_{e,f} \sum_{\mathbf{s}',\mathbf{s}''} \binom{\mathbf{b}'}{\mathbf{s}'}\binom{\mathbf{b}''}{\mathbf{s}''} \langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a}')\overline{\boldsymbol{\tau}}_1(\mathbf{b}'-\mathbf{s}')\overline{\tau}_0(\mathbf{T}^{\mathbf{s}'} \cup T_e) \,\rangle_{\beta'}$$
$$g^{ef} \;\; \langle\, \overline{\tau}_0(\mathbf{T}^{\mathbf{s}''} \cup T_f)\overline{\boldsymbol{\tau}}_0(\mathbf{a}'')\overline{\boldsymbol{\tau}}_1(\mathbf{b}''-\mathbf{s}'') \,\rangle_{\beta''}.$$

The outer sum is over the splitting indices e and f running from 0 to r, and the inner sum is over all vectors $\mathbf{s}'$ and $\mathbf{s}''$ of non-negative integers. The symbol $\overline{\boldsymbol{\tau}}_1(\mathbf{b}'-\mathbf{s}')$ makes sense only for $\mathbf{s}' \leq \mathbf{b}'$, but since there is a binomial coefficient $\binom{\mathbf{b}'}{\mathbf{s}'}$ in front of it, which is zero unless $\mathbf{s}' \leq \mathbf{b}'$ we will allow any $\mathbf{s}'$. The same remark applies of course to $\mathbf{s}''$.

Definition. The generating function corresponding to these degree-1 enumerative descendants is called the *tangency quantum potential*:

$$\Gamma(\mathbf{x}, \mathbf{y}) = \sum_{\beta > 0} \sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!} \langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a}) \; \overline{\boldsymbol{\tau}}_1(\mathbf{b}) \,\rangle_\beta$$

where $\mathbf{x} = (x_0, x_1, \ldots, x_r)$ and $\mathbf{y} = (y_0, y_1, \ldots, y_r)$ are formal parameters and the sum is over all effective (non-zero) $\beta \in A_1(X)$ and all pairs of vectors $\mathbf{a}, \mathbf{b}$ of non-negative integers. (The degree-zero case is considered and included below, in Section 2.6.)

The variables $\mathbf{x}$ are the usual formal variables from quantum cohomology, so when $\mathbf{y}$ is set to zero, then Γ is the usual (quantum part of the) genus-zero Gromov-Witten potential. (In FP-NOTES, the formal variables $\mathbf{x}$ were called $\mathbf{y}$; I apologise for any confusion caused by my choice of notation.)

The main results of this chapter (and of the thesis) are certain differential equations satisfied by the tangency quantum potential. To alleviate the proofs of these results, let us right away recall the interpretation of the derivatives of the potential, although these arguments are elementary.

2.1.2 Preliminary remarks. First, let lower index denote derivative with respect to that variable:

$$\Gamma_{x_i} := \frac{\partial}{\partial x_i}\Gamma.$$

This operation corresponds to incrementing the corresponding exponent in the invariants, in case a_k. Precisely,

Γ_{x_k} *is the generating function for the numbers* $\langle\, \overline{\tau}_0(T_k)\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta$,

in the sense that this number is the coefficient of $\frac{\mathbf{x}^\mathbf{a}}{\mathbf{a}!}\frac{\mathbf{y}^\mathbf{b}}{\mathbf{b}!}$ in Γ_{x_k}. Indeed, let $\mathbf{e}_k$ denote the k'th unit vector, then

$$\begin{aligned}\Gamma_{x_k}(\mathbf{x},\mathbf{y}) &= \sum_{\beta>0}\sum_{\substack{\mathbf{a},\mathbf{b}\\ a_k\geq 1}} \frac{\mathbf{x}^{\mathbf{a}-\mathbf{e}_k}}{(\mathbf{a}-\mathbf{e}_k)!}\frac{\mathbf{y}^\mathbf{b}}{\mathbf{b}}\langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta\\ &= \sum_{\beta>0}\sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^\mathbf{a}}{\mathbf{a}!}\frac{\mathbf{y}^\mathbf{b}}{\mathbf{b}!}\langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a}+\mathbf{e}_k)\ \overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta.\end{aligned}$$

Next, adopt the convention that if two x-variables are coupled in a parenthesis then the meaning is

$$\Gamma_{(x_ix_j)} := \sum_{k=0}^{r} g_{ij}^k\ \Gamma_{x_k}. \tag{2.1.2.1}$$

This function is the "directional derivative with respect to the product $T_i \cup T_j = \sum g_{ij}^k T_k$", cf. 2.2.1.1 below. Precisely,

$\Gamma_{(x_ix_j)}$ *is the generating function for* $\langle\, \overline{\tau}_0(T_i\cup T_j)\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta$,

in the sense above. This follows easily from the previous remark and the linearity of the integral.

Finally, for any vector $\mathbf{s}$, the potential

$\frac{\mathbf{y}^\mathbf{s}}{\mathbf{s}!}\Gamma$ *is the generating function for* $\binom{\mathbf{b}}{\mathbf{s}}\langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\overline{\boldsymbol{\tau}}_1(\mathbf{b}-\mathbf{s})\,\rangle_\beta$.

Indeed,

$$\begin{aligned}\frac{\mathbf{y}^\mathbf{s}}{\mathbf{s}!}\Gamma(\mathbf{x},\mathbf{y}) &= \sum_{\beta>0}\sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^\mathbf{a}}{\mathbf{a}!}\frac{\mathbf{y}^{\mathbf{b}+\mathbf{s}}}{\mathbf{b}!\ \mathbf{s}!}\langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta\\ &= \sum_{\beta>0}\sum_{\mathbf{a},\mathbf{b}\geq\mathbf{s}} \frac{\mathbf{x}^\mathbf{a}}{\mathbf{a}!}\frac{\mathbf{y}^{\mathbf{b}}}{(\mathbf{b}-\mathbf{s})!\ \mathbf{s}!}\langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b}-\mathbf{s})\,\rangle_\beta\\ &= \sum_{\beta>0}\sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^\mathbf{a}}{\mathbf{a}!}\frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!}\binom{\mathbf{b}}{\mathbf{s}}\langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b}-\mathbf{s})\,\rangle_\beta\end{aligned}$$

where in the last line we allow summing over all $\mathbf{b}$, since anyway the binomial coefficient $\binom{\mathbf{b}}{\mathbf{s}}$ is zero unless $\mathbf{b} \geq \mathbf{s}$.

2.2 Deformation of the Poincaré metric

While the usual quantum potential is based on the integral $\int_X$ and the Poincaré metric $g_{ij} = \int_X T_i \cup T_j$, the tangency quantum potential relates more naturally to a deformation of these structures, a certain "metric" with values in $\mathbb{Q}[[\mathbf{y}]]$.

2.2.1 The classical product in the Poincaré metric. The two important structures on $A = A^*(X)$ are the intersection product $\cup$, and the linear functional $\int_X : A \to \mathbb{Q}$. Set $g_{ij} = \int_X T_i \cup T_j$ and $g_{ijk} := \int_X T_i \cup T_j \cup T_k$. Let (g^{ij}) be the inverse matrix to (g_{ij}). It is used to raise indices as needed. In particular, with $g_{ij}^k = \sum_e g_{ije}\, g^{ek} = \int_X (T_i \cup T_j \cup \sum_e g^{ek} T_e)$, we get the multiplication formula

$$T_i \cup T_j = \sum_{k=0}^{r} g_{ij}^k T_k. \tag{2.2.1.1}$$

2.2.2 Intersection polynomials of X. Let $\phi(\mathbf{y}) \in \mathbb{Q}[[\mathbf{y}]]$ be the generating function for the top integrals on X,

$$\phi(\mathbf{y}) := \sum_{\mathbf{s}} \frac{\mathbf{y}^{\mathbf{s}}}{\mathbf{s}!} \int_X \mathbf{T}^{\mathbf{s}},$$

where the sum is over all vectors $\mathbf{s} = (s_0, s_1, \ldots, s_r)$ of non-negative integers. It is just the integral of $\exp(\sum y_i T_i) \in A[[\mathbf{y}]]$. Note that, there is a factor $\exp(y_0)$ and that the rest factor is in fact a polynomial in $y_1, \ldots, y_r$.

Now set

$$\phi_k := \tfrac{\partial}{\partial y_k}\, \phi = \sum_{\mathbf{s}} \frac{\mathbf{y}^{\mathbf{s}}}{\mathbf{s}!} \int_X \mathbf{T}^{\mathbf{s}} \cup T_k$$

and use the matrix (g^{ef}) to raise indices, setting

$$\phi^i = \sum_{j=0}^{r} g^{ij} \phi_j, \qquad \text{and} \qquad \phi^{ij} = \sum_k g^{ik} \phi_k^j. \tag{2.2.2.1}$$

Note in particular that

$$\phi_k^i = \sum_{\mathbf{s}} \frac{\mathbf{y}^{\mathbf{s}}}{\mathbf{s}!} \int_X \mathbf{T}^{\mathbf{s}} \cup T_k \cup \sum_{j=0}^{r} g^{ij} T_j. \tag{2.2.2.2}$$

2.2.3 Lemma. SPLITTING FORMULA. *Let* $\mathbf{y}'$ *and* $\mathbf{y}''$ *be independent formal variables. Then*

$$\sum_{e,f} \phi_e(\mathbf{y}')\, g^{ef}\, \phi_f(\mathbf{y}'') = \phi(\mathbf{y}' + \mathbf{y}'').$$

Proof. Write down the exponential series for $X \times X = X' \times X''$,

$$\exp\left(\textstyle\sum y_i' T_i' + \sum y_i'' T_i''\right)$$

and integrate over the diagonal Δ. Computing this integral via the Künneth formula for Δ and the projection formula yields the left hand side. On the other hand, computing the integral using the isomorphism $X \simeq \Delta$, yields the right hand side. □

2.2.4 The deformed metric. Instead of using the integral $\int_X : A \to \mathbb{Q}$, the new metric is based on the linear functional $\langle\ \rangle_X : A \to \mathbb{Q}[[\mathbf{y}]]$ defined as

$$\langle\, \alpha \,\rangle_X := \sum_{\mathbf{b}} \frac{(-2\mathbf{y})^{\mathbf{b}}}{\mathbf{b}!} \int_X \mathbf{T}^{\mathbf{b}} \cup \alpha.$$

It can be thought of as a deformation of $\int_X$ since we recover this functional upon setting the variables $\mathbf{y}$ equal to zero. Now define the new metric (γ_{ij}) by

$$\gamma_{ij} := \langle\, T_i \cup T_j \,\rangle_X = \phi_{ij}(-2\mathbf{y}),$$

and adopt the obvious notation $\gamma_{ijk} = \langle\, T_i \cup T_j \cup T_k \,\rangle_X = \phi_{ijk}(-2\mathbf{y})$. Let (γ^{ij}) denote the inverse matrix to (γ_{ij}). Then it follows readily from the splitting formula above that

$$\gamma^{ij} = \phi^{ij}(2\mathbf{y}) = \sum_{e,f} \phi^i_e g^{ef} \phi^j_f. \tag{2.2.4.1}$$

When the formal parameters of γ are reset, we recover the corresponding g, for example $\gamma^{ij}(\mathbf{0}) = g^{ij}$ and $\gamma^k_{ij}(\mathbf{0}) = g^k_{ij}$, so the intersection product can be written

$$T_i \cup T_j = \sum_{e,f} \gamma_{ije}\, \gamma^{ef}\, T_f,$$

since $\sum_e \phi_{ije}(-2\mathbf{y})\ \phi^{ef}(2\mathbf{y}) = \phi^f_{ij}(\mathbf{0}) = g^f_{ij}$.

2.2.5 Example. For $\mathbb{P}^r$ (with $h := c_1(\mathcal{O}(1))$ and basis $T_i = h^i$), we have $\sum_j g^{ij} T_j = T_{r-i}$, so it follows that $\phi^{ij} = \phi_{i+j-r}$. For example for $\mathbb{P}^2$ we get (omitting the factor $\exp(y_0)$) the polynomials

$$\phi^2 = \phi = y_2 + \tfrac{1}{2!} y_1^2, \quad \phi^1 = \phi_1 = y_1, \quad \text{and } \phi^0 = \phi_2 = 1.$$

Thus

$$(\gamma^{ij}) = \exp(2y_0) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2y_1 \\ 1 & 2y_1 & 2y_1^2 + 2y_2 \end{pmatrix}$$

2.3 Topological recursion

Now we can state the first important differential equation satisfied by the tangency quantum potential.

2.3.1 Theorem. *The tangency potential satisfies the differential equations*

$$\Gamma_{y_k x_i x_j} = \Gamma_{x_k (x_i x_j)} - \Gamma_{(x_k x_i) x_j} - \Gamma_{(x_k x_j) x_i} + \sum_{e,f} \Gamma_{x_k x_e} \ \gamma^{ef} \ \Gamma_{x_f x_i x_j}.$$

2.3.2 Remark. The differential equation determines all degree-1 enumerative descendants from the Gromov-Witten invariants. Indeed, the fact that the left hand side involves derivation with respect to y_k, while the right hand side does not, means that invariants with one or more modified psi classes are expressed in terms of invariants with fewer such classes, cf. the observations 2.1.2. The condition of the presence of the classes x_i and x_j is not serious for the application of the recursion: we can always use the divisor equation 1.6.1.3 to introduce two new classes.

Proof. The left hand side $\Gamma_{x_i x_j y_k}$ is the generating function for the invariants $\langle\, \overline{\tau}_1(T_k)\overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\ \overline{\boldsymbol{\tau}}_0(\mathbf{a})\overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta$, and let us agree on naming the marks such that the three first classes in this product correspond to p_1, p_2 and p_3. Let us compute this integral, writing out the first modified psi class as a sum of boundary divisors, as in 1.6.3.1, and restricting the remaining factors to each term in this sum. Taking out $\overline{\boldsymbol{\psi}}_1$ leaves a factor $\overline{\tau}_0(T_k)$ at the mark p_1. Since all the classes in the product are E-classes, among the soft boundary divisors in the expression $\overline{\boldsymbol{\psi}}_1 = (p_1 \mid p_2, p_3) - \boldsymbol{\xi}_1$, only the ones with just two marks on the degree-0 twig give any contribution, cf. the argument of 1.5.2. There are three such boundary divisors: D_{23} coming from $(p_1 \mid p_2, p_3)$, and $-D_{12}-D_{13}$ coming from $\boldsymbol{\xi}_1$. (Other contributions, like D_{14} are in both sums so they cancel out.) Restricting classes to such a boundary divisor, say D_{23}, is just to merge the two classes corresponding to p_2 and p_3, so we get $\langle\, \overline{\tau}_1(T_k)\overline{\tau}_0(T_i \cup T_j)\ \overline{\boldsymbol{\tau}}_0(\mathbf{a})\overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta$, whose generating function is exactly $\Gamma_{(x_i x_j)x_k}$. This explains the three linear terms on the right hand side of the differential equation.

The quadratic terms come from applying the splitting lemma 2.1.1 to each of the hard boundary divisors in the expression of $\overline{\boldsymbol{\psi}}_1$. Summing over all the components we get

$$\sum \binom{\mathbf{a}}{\mathbf{a}'}\binom{\mathbf{b}}{\mathbf{b}'} \sum_{p,q}\sum_{\mathbf{s}',\mathbf{s}''} \binom{\mathbf{b}'}{\mathbf{s}'} \langle\, \overline{\tau}_0(T_k)\overline{\boldsymbol{\tau}}_0(\mathbf{a}')\overline{\boldsymbol{\tau}}_1(\mathbf{b}'-\mathbf{s}')\overline{\tau}_0(\mathbf{T}^{\mathbf{s}'}\cup T_p)\,\rangle_{\beta'}$$
$$g^{pq}\binom{\mathbf{b}''}{\mathbf{s}''}\ \langle\, \overline{\tau}_0(\mathbf{T}^{\mathbf{s}''}\cup T_q)\overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\,\overline{\boldsymbol{\tau}}_0(\mathbf{a}'')\overline{\boldsymbol{\tau}}_1(\mathbf{b}''-\mathbf{s}'')\,\rangle_{\beta''}.$$

The big outer sum is over all positive partitions $\beta'+\beta''=\beta$, and over all partitions $\mathbf{a}'+\mathbf{a}''=\mathbf{a}$ and $\mathbf{b}'+\mathbf{b}''=\mathbf{b}$. The inner two summations are those from the splitting lemma, but note that we have chosen the splitting indices to be called p and q.

Now the outer sum, with the two binomial coefficients $\binom{\mathbf{a}}{\mathbf{a}'}\binom{\mathbf{b}}{\mathbf{b}'}$ corresponds to a product of two generating functions, so it remains to interpret the number $\sum_p\sum_{\mathbf{s}'}\binom{\mathbf{b}'}{\mathbf{s}'}\langle\, \overline{\tau}_0(T_k)\overline{\boldsymbol{\tau}}_0(\mathbf{a}')\overline{\boldsymbol{\tau}}_1(\mathbf{b}'-\mathbf{s}')\overline{\tau}_0(\mathbf{T}^{\mathbf{s}'}\cup T_p)\,\rangle_{\beta'}$ from the one-primed twig, and the corresponding number on the other twig. To this end, write out the factor $\overline{\tau}_0(T_p\cup\mathbf{T}^{\mathbf{s}'})$ in terms of the basis. If $T_p\cup\mathbf{T}^{\mathbf{s}'}=\sum_e m_e T_e$, then the coefficient m_e is given as the projection onto the span of T_e, which is $\int_X T_p\cup\mathbf{T}^{\mathbf{s}'}\cup\sum_f g^{ef}T_f$. So the number is

$$\sum_{p,e}\sum_{\mathbf{s}'}\binom{\mathbf{b}'}{\mathbf{s}'}\langle\, \overline{\tau}_0(T_k)\overline{\boldsymbol{\tau}}_0(\mathbf{a}')\overline{\boldsymbol{\tau}}_1(\mathbf{b}'-\mathbf{s}')\overline{\tau}_0(T_e)\,\rangle_{\beta'}\ \Big(\int_X T_p\cup\mathbf{T}^{\mathbf{s}'}\cup\sum_f g^{ef}T_f\Big).$$

According to the preliminary remarks 2.1.2, the corresponding generating function is

$$\sum_{p,e}\sum_{\mathbf{s}'}\frac{\mathbf{y}^{\mathbf{s}'}}{\mathbf{s}'!}\,\Gamma_{x_k x_e}(\mathbf{x},\mathbf{y})\ \Big(\int_X T_p\cup\mathbf{T}^{\mathbf{s}'}\cup\sum_f g^{ef}T_f\Big)=\sum_{p,e}\Gamma_{x_k x_e}(\mathbf{x},\mathbf{y})\ \phi_p^e(\mathbf{y}),$$

by 2.2.2.2. Similarly, the generating function corresponding to the two-primed twig is

$$\sum_{q,f}\Gamma_{x_i x_j x_f}(\mathbf{x},\mathbf{y})\ \phi_q^f(\mathbf{y}).$$

Multiplying these two expressions, the proof is then completed by observing that $\sum_{p,q} \phi_p^e \, g^{pq} \, \phi_q^f = \gamma^{ef}$, cf. 2.2.4.1. $\square$

2.3.3 Integration of the differential equation. In the particular differential equation

$$\Gamma_{y_k x_k x_k} = -\Gamma_{(x_k x_k) x_k} + \sum_{e,f} \Gamma_{x_k x_e} \, \gamma^{ef} \, \Gamma_{x_f x_k x_k},$$

all the terms are total derivatives with respect to x_k, so we actually get a better differential equation

$$\Gamma_{x_k y_k} = -\Gamma_{(x_k x_k)} + \tfrac{1}{2} \sum_{e,f} \Gamma_{x_k x_e} \, \gamma^{ef} \, \Gamma_{x_f x_k}.$$

(Since all terms are exponential in x_k, the integration constant must be zero.) This is most useful in the case where $T_k \in A^1(X)$.

2.3.4 Example. Continuing Example 2.2.5, for $X = \mathbb{P}^2$ and $k = 1$, this differential equation reads

$$\Gamma_{y_1 x_1} = -\Gamma_{x_2} + \tfrac{1}{2}\Gamma^2_{x_1 x_1} + 2y_1 \Gamma_{x_1 x_1} \Gamma_{x_2 x_1} + (y_1^2 + y_2)\Gamma^2_{x_1 x_2}$$

This equation was first found by Rahul Pandharipande [39].

2.3.5 Remark. Since the topological recursion works for enumerative descendants of any degree, cf. 1.6.3, a differential equation similar to 2.3.1 should exist also in this full generality. Unfortunately I have not mastered the combinatorics involved in finding the generalisation of the symbols γ^{ef} to this situation.

I intend to take up these questions at a later occasion.

2.4 Tangency WDVV equations

2.4.1 Theorem. *For simplicity, set $\Gamma_i = \Gamma_{x_i}$. Then the following form of the WDVV equations hold for the tangency quantum potential:*

$$\Gamma_{(ij)k\ell} + \Gamma_{ij(k\ell)} + \sum_{e,f} \Gamma_{ije} \, \gamma^{ef} \, \Gamma_{fk\ell} = \Gamma_{(jk)i\ell} + \Gamma_{jk(i\ell)} + \sum_{e,f} \Gamma_{jke} \, \gamma^{ef} \, \Gamma_{fi\ell}.$$

2.4.2 Remark. In the special case $X = \mathbb{P}^2$, it turns out the only non-trivial relation is the one with $i = j = 1$ and $k = \ell = 2$. The equation then reads

$$\Gamma_{222} = \exp(2y_0)(\Gamma_{112}^2 - \Gamma_{111}\Gamma_{122} + 2y_1\big(\Gamma_{122}\Gamma_{112} - \Gamma_{111}\Gamma_{222}) + (2y_1^2 + 2y_2)(\Gamma_{122}^2 - \Gamma_{112}\Gamma_{222})\big).$$

This equation was first found by Lars Ernström and Gary Kennedy [13], cf. Remark 2.5.2 below, while the special case of $y_0 = y_2 = 0$ goes back to Di Francesco-Itzykson [9]. Setting also $y_1 = 0$ we are back to the formula of Maxim Kontsevich [32].

Proof. Let us show that the coefficients of $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$ agree on the two sides of the equation. Consider the almost-top product $\overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\overline{\tau}_0(T_k)\overline{\tau}_0(T_\ell)\,\boldsymbol{\overline{\tau}}_0(\mathbf{a})\boldsymbol{\overline{\tau}}_1(\mathbf{b})$, and let $p_1, \dots, p_4$ be the marks corresponding to the first four factors. Now integrate the product over each side of the fundamental equivalence

$$(p_1, p_2 \mid p_3, p_4) = (p_2, p_3 \mid p_1, p_4),$$

Here $(p_1, p_2 \mid p_3, p_4)$ denotes the sum of all boundary divisors having p_1 and p_2 on one twig and p_3 and p_4 on the other.

On the left hand side of the equation, let us first consider the contribution from the soft boundary divisors in the sum, say having $\beta' = 0$. Then the only possible distribution of the marks giving contribution is putting all the spare marks on the right hand twig, so we are left with the single irreducible boundary divisor D_{12}. Now according to 1.5.4, the effect of multiplication with this divisor is to merge the two classes $\overline{\tau}_0(T_i)$ and $\overline{\tau}_0(T_j)$, so in the end we get

$$\langle\, \overline{\tau}_0(T_i \cup T_j)\overline{\tau}_0(T_k)\overline{\tau}_0(T_\ell)\boldsymbol{\overline{\tau}}_0(\mathbf{a})\boldsymbol{\overline{\tau}}_1(\mathbf{b})\,\rangle_\beta.$$

Similarly, the case $\beta'' = 0$ gives $\langle\, \overline{\tau}_0(T_k \cup T_\ell)\overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\boldsymbol{\overline{\tau}}_0(\mathbf{a})\boldsymbol{\overline{\tau}}_1(\mathbf{b})\,\rangle_\beta$, and the generating function for these two invariants are exactly the two linear terms on the left hand side of the equation.

Now for the hard boundary divisors in the linear equivalence. To each irreducible component on the left hand side, we apply the splitting lemma 2.1.1, getting all together

$$\sum \tbinom{\mathbf{a}}{\mathbf{a}'}\tbinom{\mathbf{b}}{\mathbf{b}'} \sum_{p,q}\sum_{\mathbf{s}',\mathbf{s}''} \tbinom{\mathbf{b}'}{\mathbf{s}'}\,\langle\, \overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\boldsymbol{\overline{\tau}}_0(\mathbf{a}')\boldsymbol{\overline{\tau}}_1(\mathbf{b}'-\mathbf{s}')\overline{\tau}_0(\mathbf{T}^{\mathbf{s}'} \cup T_p)\,\rangle_{\beta'}$$
$$g^{pq}\,\tbinom{\mathbf{b}''}{\mathbf{s}''}\,\langle\, \overline{\tau}_0(\mathbf{T}^{\mathbf{s}''} \cup T_q)\overline{\tau}_0(T_k)\overline{\tau}_0(T_\ell)\,\boldsymbol{\overline{\tau}}_0(\mathbf{a}'')\boldsymbol{\overline{\tau}}_1(\mathbf{b}''-\mathbf{s}'')\,\rangle_{\beta''}$$

The big outer sum is over all positive partitions $\beta' + \beta'' = \beta$, and over all partitions $\mathbf{a}' + \mathbf{a}'' = \mathbf{a}$ and $\mathbf{b}' + \mathbf{b}'' = \mathbf{b}$. The inner two summations are those from the splitting lemma, but note that we have chosen the splitting indices to be called p and q.

Seeing that this is exactly the coefficient of $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$ of the left hand side of the equation follows from argument identical to the conclusion of the proof of 2.3.1. □

2.5 The tangency quantum product

Definition. The *tangency quantum product* is the product on $A[[\mathbf{x}, \mathbf{y}]]$ defined by the rule

$$T_i * T_j := T_i \cup T_j + \sum_{e,f} \Gamma_{ije}\, \gamma^{ef}\, T_f.$$

Observe that this product specialises to the usual quantum product upon setting the formal variables $\mathbf{y}$ to zero.

2.5.1 Theorem. *The tangency quantum product is associative.*

Proof. It is enough to check on the generators; let us write out $(T_i * T_j) * T_k$. By definition,

$$\begin{aligned}(T_i * T_j) * T_k = T_i \cup T_j \cup T_k &+ \sum_{\ell,m} \Gamma_{(ij)k\ell}\, \gamma^{\ell m}\, T_m \\ &+ \sum_{e,f} \Gamma_{ije}\, \gamma^{ef}\, (T_f \cup T_k) + \sum_{e,f} \sum_{\ell,m} \Gamma_{ije}\, \gamma^{ef}\, \Gamma_{fk\ell}\, \gamma^{\ell m}\, T_m.\end{aligned}$$

Now the third term can also be written

$$\begin{aligned}\sum_{e,f} \Gamma_{ije}\, \gamma^{ef}\, (T_f \cup T_k) &= \sum_{e,f,m} \Gamma_{ije}\, \gamma^{ef}\, g_{fk}^{m}\, T_m \\ &= \sum_{e,m} \Gamma_{ije}\, \gamma_k^{em}\, T_m \\ &= \sum_{e,\ell,m} \Gamma_{ije}\, g_{k\ell}^{e}\, \gamma^{\ell m}\, T_m \\ &= \sum_{\ell,m} \Gamma_{ij(k\ell)}\, \gamma^{\ell m}\, T_m\end{aligned}$$

using the properties of the classical structure constants g_{ij}^k, listed in 2.2.2.1 and 2.1.2.1. Now the left hand side reads

$$T_i \cup T_j \cup T_k + \sum_{\ell,m} \Gamma_{(ij)k\ell}\, \gamma^{\ell m}\, T_m + \sum_{\ell,m} \Gamma_{ij(k\ell)}\, \gamma^{\ell m}\, T_m + \sum_{\ell,m}\sum_{e,f} \Gamma_{ije}\, \gamma^{ef}\, \Gamma_{fk\ell}\, \gamma^{\ell m}\, T_m.$$

Similarly the right hand side becomes

$$T_i \cup T_j \cup T_k + \sum_{\ell,m} \Gamma_{(jk)i\ell}\, \gamma^{\ell m}\, T_m + \sum_{\ell,m} \Gamma_{jk(i\ell)}\, \gamma^{\ell m}\, T_m + \sum_{\ell,m}\sum_{e,f} \Gamma_{jke}\, \gamma^{ef}\, \Gamma_{fi\ell}\, \gamma^{\ell m}\, T_m$$

The equality of these two expressions is immediate from the previous theorem. □

2.5.2 Remark. In the special case $X = \mathbb{P}^2$, this product (or a simple change of coordinates of it) was previously constructed via ad hoc methods by Lars Ernström and Gary Kennedy [13], who also gave a tour de force proof of its associativity. Their construction relies on the space of stable lifts, and seems to be peculiar to the projective plane.

2.6 Frobenius structure

2.6.1 The classical potential for X is the generating function for the triple top products

$$\sum_{i,j,k} \frac{x_i x_j x_k}{6} \int_X T_i \cup T_j \cup T_k = \sum_{|\mathbf{a}|=3} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \int_X \mathbf{T}^{\mathbf{a}}.$$

By construction, its third derivatives are just g_{ijk}, the structure constants for the cup multiplication (in the Poincaré metric). In usual quantum cohomology, this potential is reinterpreted as the $\beta = 0$ part of the Gromov-Witten potential: a quantum correction (the $\beta > 0$ part) is added to the classical potential, in such a way that the third derivatives of this sum are the structure constants of a new associative product, the quantum product.

2.6.2 "The tangency classical potential". The tangency quantum product relates to $\langle\ \rangle_X$ exactly as the usual quantum product relates to $\int_X$. So let us introduce a potential

$$\begin{aligned}\Phi^0(\mathbf{x},\mathbf{y}) := \sum_{i,j,k} \frac{x_i x_j x_k}{6} \langle\, T_i \cup T_j \cup T_k \,\rangle_X &= \sum_{|\mathbf{a}|=3} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \langle\, \mathbf{T}^{\mathbf{a}} \,\rangle_X \\ &= \sum_{|\mathbf{a}|=3} \sum_{\mathbf{b}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \frac{(-2\mathbf{y})^{\mathbf{b}}}{\mathbf{b}!} \int_X \mathbf{T}^{\mathbf{a}+\mathbf{b}}\end{aligned}$$

which to fit into the picture could be called the tangency classical potential, although it is neither classical, nor has it anything particular to do with tangency. By construction, its third derivatives are γ_{ijk}, the structure constants of the cup multiplication, but this time in the deformed metric:

$$T_i \cup T_j = \sum_{e,f} \Phi^0_{ije}\; \gamma^{ef}\; T_f,$$

since $\sum_e \phi_{ije}(-2\mathbf{y})\; \phi^{ef}(2\mathbf{y}) = \phi_{ij}^f(\mathbf{0}) = g_{ij}^f$.

2.6.3 The tangency potential, including $\beta = 0$. Now the tangency quantum product is obtained by introducing a bigger potential

$$\Phi(\mathbf{x},\mathbf{y}) = \Phi^0(\mathbf{x},\mathbf{y}) + \Gamma(\mathbf{x},\mathbf{y}),$$

and using the third derivatives $\gamma_{ijk} + \Gamma_{x_i x_j x_k}$ as structure constants, defining

$$\begin{aligned}T_i * T_j &:= T_i \cup T_j + \sum_{e,f} \Gamma_{ije}\; \gamma^{ef}\; T_f \\ &= \sum_{e,f} \Phi_{ije}\; \gamma^{ef}\; T_f,\end{aligned}$$

and by Theorem 2.5.1, this product is associative. With the introduction of this potential, the WDVV equation 2.4.1 takes the usual simpler form

$$\sum_{e,f} \Phi_{ije}\; \gamma^{ef}\; \Phi_{fk\ell} = \sum_{e,f} \Phi_{jke}\; \gamma^{ef}\; \Phi_{fi\ell}.$$

2.6.4 Interpretation of the degree-zero term. Writing down this potential $\Phi = \Phi^0 + \Gamma$ calls for an interpretation of Φ^0 in terms of some top products on the degree-zero spaces. Unfortunately there is no way to define the modified psi class on $\overline{M}_{0,n}$, if we want it to satisfy the two rules: (1) it should be a

1-pointed E-class, and (2) it should satisfy the push-down formula $\pi_*\overline{\boldsymbol{\psi}} = -2$, to give the dilaton equation. Indeed, the minimal space of definition for a 1-pointed class should be a 1-pointed space, and such spaces do not exist in degree zero.

The best one can do is to define the class on a 4-pointed space and then pull it back to the hierarchy lying over this space, but this definition depends on the choice of the three extra marks. Note that on $\overline{M}_{0,4} \simeq \mathbb{P}^1$, there is only one choice for the modified psi class, namely the class of degree -2. For example, set $\overline{\boldsymbol{\psi}}_1 := \boldsymbol{\psi}_1 - \Delta$, where Δ denotes the sum of the three boundary divisors.

Take a moduli space $\overline{M}_{0,\{p_1,p_2,p_3\}\cup S}(X,0)$ with three distinguished marks. For each of the other marks $p_i \in S$, define a modified psi class $\overline{\boldsymbol{\psi}}_i$ as the pull-back from $\overline{M}_{0,\{p_1,p_2,p_3,p_i\}}(X,0)$ of the class $\overline{\boldsymbol{\psi}}_i = \boldsymbol{\psi}_i - \Delta$ as above. Note that the degree of this divisor in $\overline{M}_{0,4}$ is -2, and more generally, the modified psi classes in the overlying spaces enjoy the push-down property $\pi_*\overline{\boldsymbol{\psi}}_i = -2$.

Now consider the top integral $\langle\, \overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\overline{\tau}_0(T_k)\overline{\boldsymbol{\tau}}_0(\mathbf{a})\overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_0$, defined just as usual — the first three classes correspond to the three distinguished marks. Identifying $\overline{M}_{0,n}(X,0)$ with $\overline{M}_{0,n}\times X$, all the evaluation morphisms are just the projection p to X, so the integrand has a factor $p^*(T_i \cup T_j \cup T_k \cup \mathbf{T}^{\mathbf{a}+\mathbf{b}})$. The remaining factors are modified psi classes from $\overline{M}_{0,n}$. Since each of them is alone on its mark, we can push them down one by one, arriving at a factor $(-2)^{\mathbf{b}}$. The remaining integral $\int 1$ is zero for dimension reasons, unless we have come down to just $\overline{M}_{0,3}$, which means $\mathbf{a} = 0$. We conclude

$$\langle\, \overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\overline{\tau}_0(T_k)\,\overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_0 = (-2)^{\mathbf{b}} \int_X T_i \cup T_j \cup T_k \cup \mathbf{T}^{\mathbf{b}}$$

Now consider the generating function for these numbers,

$$\begin{aligned} \sum_{\mathbf{b}} \frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!}\,\langle\, \overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\overline{\tau}_0(T_k)\,\overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_0 \quad &= \quad \sum_{\mathbf{b}} \frac{(-2\mathbf{y})^{\mathbf{b}}}{\mathbf{b}!} \int_X T_i \cup T_j \cup T_k \cup \mathbf{T}^{\mathbf{b}} \\ &= \quad \phi_{ijk}(-2\mathbf{y}) = \gamma_{ijk}. \end{aligned}$$

This shows that at least the third derivatives of Φ^0 have the interpretation as top products on degree-0 moduli spaces, and after all it is the third derivatives that are really interesting.

Once we know that the structure constants of the tangency quantum product are third derivatives of the single potential Φ, we are in position to given an interpretation in terms of Frobenius manifolds.

2.6.5 Formal Frobenius manifolds. For convenience let us recall (from Manin [38], Ch. III, 1.1.2) the definition of a formal Frobenius manifold (over k). Let k be a commutative $\mathbb{Q}$-algebra. Let H be a free k-module of finite rank, with generators $T_0, \ldots, T_r$, and let $g : H \otimes H \to k$ denote a symmetric non-degenerate bi-linear pairing (non-degenerate in the sense that the induced homomorphism $g' : H \to H^t$ to the dual module is an isomorphism. Let $K = k[[H^t]]$ be the completed symmetric algebra of H^t. In other words, if $\mathbf{x} = (x_0, \ldots, x_r)$ are coordinates on H then $K = k[[\mathbf{x}]]$. Now the structure of a formal Frobenius manifold on (H, g) is given by a potential $\Phi \in K$ (defined up to quadratic terms) satisfying WDVV. In other words, the multiplication $T_i * T_j := \sum_{e,f} \Phi_{ije} g^{ef} T_f$ makes $H \otimes_k K$ into an associative K-algebra.

2.6.6 Example. The cohomology space $A^*(X)$, with the Poincaré metric g_{ij} and the usual quantum potential, constitutes a formal Frobenius manifold over $\mathbb{Q}$.

The results of this chapter readily imply:

2.6.7 Corollary. *The cohomology* $\mathbb{Q}[[\mathbf{y}]]$*-module* $H = A^*(X)[[\mathbf{y}]]$ *with bilinear non-degenerate pairing* $\gamma : H \otimes H \to \mathbb{Q}[[\mathbf{y}]]$*, equipped with the tangency quantum potential* $\Phi \in \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ *constitutes a formal Frobenius manifold over* $\mathbb{Q}[[\mathbf{y}]]$. $\square$

Another view-point, carefully prepared throughout this chapter, is to consider this formal Frobenius manifold as a deformation over $\mathbb{Q}[[\mathbf{y}]]$ of the formal Frobenius manifold of usual quantum cohomology, cf. Example 2.6.6. While the underlying space is trivially deformed under this deformation, the metric and the potential vary non-trivially.

Chapter 3

Characteristic numbers

The motivation for studying modified psi classes comes from enumerative geometry: This chapter shows that tangency conditions can be naturally expressed in terms of modified psi classes, and that characteristic numbers (of rational curves) are degree-1 enumerative descendants. The central idea is the use of pointed conditions.

3.1 Tangency conditions

The first step is to compute the class of the cycle in moduli space defined by the condition of the map being tangent to a given hypersurface at a specified mark. Thanks to the presence of the mark, such a condition can then be combined with incidence conditions to get the condition of being tangent to the hypersurface along a subvariety in it.

Let $H \subset X$ be a very ample hypersurface, given as the zero scheme of a section of a line bundle, $f \in H^0(X, L)$. We consider first the case where there is only one mark. The mark of a map $\mu : C \to X$ lands in H if and only if $\mu^* f$ vanishes at the mark. The locus of such maps, (or more precisely the class of this cycle) will be called the *eta class:*

$$\boldsymbol{\eta}_i := c_1(\nu_i^* L),$$

(eta for evaluation, or eta for H).

Definition. A map $\mu : C \to X$ with a single mark is *tangent* to H at the mark if and only if $\mu^* f$ vanishes at the mark with multiplicity 2, i.e., when also the first derivative vanishes. Let $\Theta_1(H)$ denote the locus of maps tangent to H at the mark. (Shortly we will see that the reduced scheme structure is the natural one.)

In the presence of more marks, we must make the following reservation: in this case, the boundary divisor $\boldsymbol{\xi}_1$ is non-empty, so there exist maps whose source curve is reducible with the first mark on a degree 0 twig, and such that the mark (and thus the twig) maps to a point in H. For such a map, $\mu^* f$ vanishes identically along the contracting twig and thus to any order. But we do not in general want to consider such maps tangent.

Note that these maps have their source collapsed under the forgetful morphism $\hat{\pi}_1$ that forgets all marks but the first, so a technically convenient way to characterise pointed tangency in the case of more marks is this:

Definition. A map $\mu : C \to X$ is tangent to H at the first mark if its image under $\hat{\pi}_1$ is tangent to H at the mark. Let $\Theta_1(H)$ be the locus of such maps. Accordingly,

$$\Theta_1(H) = \hat{\pi}_1^* \Theta_1(H), \tag{3.1.0.1}$$

where the Theta cycle on the right hand side lives on the space with only one mark (compare 1.4.3).

3.1.1 Proposition. *In a space $\overline{M}_{0,n}(X, \beta)$ whose general member is an immersion, the locus $\Theta_i(H)$ is the zero scheme of a regular section of a rank-two vector bundle E, and its class is*

$$\Theta_i := c_2(E) = \boldsymbol{\eta}_i(\boldsymbol{\eta}_i + \overline{\boldsymbol{\psi}}_i).$$

See 5.2.15 for another construction of Θ.

3.1.2 Remark. In his Ph.D. thesis [17], Andreas Gathmann remarked that off the boundary, the locus of pointed tangency is the zero scheme of a regular section, and that its class is $\boldsymbol{\eta}_i(\boldsymbol{\eta}_i + \boldsymbol{\psi}_i)$ modulo a correction term supported at $\boldsymbol{\eta}_i \boldsymbol{\xi}_i$. (This is enough to handle questions involving a single tangency condition.) Proposition 3.1.1 gives the exact class (the proof follows on p. 63). Note that the boundary correction is built into the definition of the modified psi class.

Since we are going to intersect various Theta cycles, we will need a slightly more general set-up.

Definition. A one-marked map $\mu : C \to \mathbb{P}^r$ has *critical mark* if its differential vanishes at the mark. The locus of such maps is denoted Σ_1. As in the definition of Θ_1, we don't want to consider the maps in $\boldsymbol{\xi}_1$ as having critical mark, so in a situation with more marks, we define a map to have critical first mark if its image under $\hat{\pi}_1$ has critical mark. It follows that

$$\Sigma_1 = \hat{\pi}_1^* \Sigma_1.$$

Note that the locus Σ_i includes the boundary cycle Π_i consisting generically of maps whose source has got three twigs, the middle one of which carries mark p_i and is of degree 0. This class is studied more in Section 4.1. The other component(s) are those whose generic map is irreducible but at some isolated points fails to be an immersion. In the case of $\mathbb{P}^2$, this is the locus of cusps, and we will pay more attention to this in Chapter 4 and compute its class.

3.1.3 Families of maps. Given a flat family of stable maps

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\mu"] \arrow[d, shift left, "\pi"] & X \\ S, \arrow[u, shift left, "\sigma_1"] \arrow[ur, "\nu_1"'] & \end{tikzcd}$$

consider the following conditions:

(A) S is equidimensional and reduced.

(B) The locus of maps with reducible source is of pure codimension 1 in S (or it is empty).

(C) For each mark i, the locus $\Sigma_i \subset S$ has codimension at least 2.

Clearly, $\overline{M}_{0,n}(\mathbb{P}^r, d)$ satisfies all three conditions. For more general target space X, it can be necessary to make some restrictions on the degree β. For example, for $X = \mathbb{P}^1$, no maps of degree $d \geq 2$ are immersions, so the condition of critical mark is only codimension 1. Variants of this phenomenon occur in higher dimension, for example for maps to $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(m, 0)$.

If the general map is an immersion, then the non-immersions occur in codimension at least 1, and then there is yet another codimension to force the mark into the locus of critical points. So a sufficient condition to satisfy (C) is that the general map is an immersion.

3.1.4 Proposition. *In any family satisfying* (A), (B) *and* (C), *for general H, the scheme $\Theta_1(H)$ is of codimension 2 and satisfies* (A) *and* (B) *(or it may be empty).*

Proof. Let $\hat{\pi}_1$ denote the morphism that forgets all marks but the first. Since $\Theta_1(H) = \hat{\pi}_1^* \Theta_1(H)$, and since the properties are preserved under flat pull-back, it is enough to prove the proposition in the case of only one mark.

Consider the pull-back of f along μ, and its first jet relative to π, (i.e. we derive only in vertical direction). The vanishing of this section

$$\partial_\pi^1 \mu^* f \in J_\pi^1 \mu^* L$$

defines the locus of maps tangent to H. Pulling back along σ_1 we get only those maps tangent to H at the mark, so set-theoretically $\Theta_1(H)$ is the zero locus of the section

$$\varphi := \sigma_1^* \partial_\pi^1 (\mu^* f)$$

of the rank-2 vector bundle

$$E := \sigma_1^* J_\pi^1 (\mu^* L).$$

(Note that π is not a smooth morphism, so a priori the sheaf $J_\pi^1(\mu^* L)$ might not be locally free. But π *is* smooth in a neighbourhood of the image of σ_1; therefore E is indeed a vector bundle.)

Now since L is very ample, it separates points and infinitely near points. Therefore, off Σ_1, the vector bundle E is generated by global sections (namely the value and the value of the derivative), so by Kleiman's Bertini theorem ([27], Remark 6 and 7), for a general H, the zero-scheme $Z(\varphi) \smallsetminus \Sigma_1$ is reduced of pure codimension 2 (or else it is empty).

Now $\Theta_1(H)$ is contained in the divisor $\boldsymbol{\eta}_1(H)$ of maps whose mark lands in H; this divisor moves in a linear system without base points, so the intersection $\Sigma_1 \cap \boldsymbol{\eta}_1(H)$, and hence the intersection $\Sigma_1 \cap Z(\varphi)$, is of codimension (at least) 3 in S. It follows that the whole of $Z(\varphi)$ is in fact reduced of pure codimension 2, so $\Theta_1(H) = Z(\varphi)$ satisfies (A).

To see that $\Theta_1(H)$ satisfies (B), let $B \subset S$ denote the locus of maps with reducible source, and apply the Kleiman-Bertini to $B \smallsetminus \Sigma_1$. It follows that for general H, the intersection $\Theta_1(H) \cap B \subset S$ is of codimension 3, away from Σ_1. But anyway $\Theta_1(H) \cap \Sigma_1 \subset S$ is of codimension 3, so the dimensionality holds everywhere. $\square$

3.1.5 Remark. The last dimension reduction argument of the proof above holds for any subvariety $D \subset S$ of codimension at least 1: For general H, the intersection $\Theta_1(H) \cap D$ is of codimension at least 3 in S.

3.1.6 Remark. Note that $\overline{M}_{0,n}(X, \beta)$ is Cohen-Macaulay. Being locally the quotient of a smooth variety by a finite group, its Cohen-Macaulay-ness follows from the following algebraic result, valid in characteristic zero: *If R is a Cohen-Macaulay ring, acted upon by a finite group G, then the ring of invariants R^G is also Cohen-Macaulay.* (See Bruns-Herzog [5], Cor. 6.4.6.)

Proof of Proposition 3.1.1. Suppose first there is only one mark, and continue the notation of the proof above, with $S = \overline{M}_{0,1}(X, \beta)$. Since $Z(\varphi)$ is of expected codimension and $\overline{M}_{0,1}(X, \beta)$ is Cohen-Macaulay, the section $\varphi \in H^0(S, E)$ is regular, and $[Z(\varphi)] = c_2(E) \cap [\overline{M}_{0,1}(X, \beta)]$. This top Chern class is easily computed via the fundamental jet bundle exact sequence (pulled back along σ_1)

$$0 \longrightarrow \sigma_1^*\big(\mu^*L \otimes \omega_\pi\big) \longrightarrow \sigma_1^*\big(J^1_\pi \mu^*L\big) \longrightarrow \sigma_1^*\mu^*L \longrightarrow 0;$$

the last term is just ν_1^*L, with first Chern class $\boldsymbol{\eta}_1$, and the first term is also a line bundle, with first Chern class $\boldsymbol{\eta}_1 + \boldsymbol{\psi}_1$. Since $\boldsymbol{\psi}_1 = \overline{\boldsymbol{\psi}}_1$ on a space with just one mark, this proves the proposition in case $n = 1$.

In the case of more marks, take the vector bundle to be the pull-back of E along the forgetful morphism $\hat{\pi}_1$. Since the classes Θ_1, $\overline{\boldsymbol{\psi}}_1$ and $\boldsymbol{\eta}_1$ are preserved under this pull-back, the result follows. □

3.1.7 Remark. Invoking the Cohen-Macaulay-ness of $\overline{M}_{0,n}(X, \beta)$ is not strictly necessary to obtain the class of Θ_1; in fact the formula $\Theta_1 = \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)$ holds in any family satisfying (A), (B) and (C). To see this, recall (from Fulton [14], 14.1) that there is a localised top Chern class $\mathbf{Z}(\varphi) \in A_*(Z(\varphi))$ whose image in $A_*(S)$ is $c_2(E) \cap [S]$; when $Z(\varphi)$ has expected codimension, with cycle $[Z(\varphi)] =: \sum m_i[D_i]$, then $\mathbf{Z}(\varphi) = \sum e_i[D_i]$, and $1 \leq e_i \leq m_i$ ([14], Example 14.4.4). In our case, $Z(\varphi)$ is of correct dimension and is furthermore *reduced*, hence $[Z(\varphi)] = \mathbf{Z}(\varphi) = c_2(E) \cap [S]$.

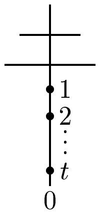
3.1.8 Remark. It should be noted that the results above do not exclude the case that all maps tangent to H are in fact contained in H. This happens for example in $\overline{M}_{0,1}(\mathbb{P}^r, 1)$ when asking such a linear stable map to be tangent to a hyperplane in $\mathbb{P}^r$. Note however that the image of the mark is not fixed by this condition; it can be any point of the map. Therefore one degree of freedom is spent with this free mark, and when imposing other conditions until reaching the dimension of the space, there will be one degree of freedom missing, the one that was spent with the free mark, and in the end no solutions will survive.

3.2 Enumerative significance

When X is a homogeneous space the Gromov-Witten invariants are enumerative, in the sense that they are answers to questions like "how many irreducible rational curves are incident to certain subvarieties in X".

In this section it is shown that when the general map of $\overline{M}_{0,n}(X,\beta)$ is an immersion, then top intersections of Theta classes (with distinct marks) together with evaluation classes have enumerative significance, responding questions like "how many rational curves are incident to specified subvarieties and tangent to given hypersurfaces in X" (possibly with further specification of where the tangency is required to occur). The key result is that for general choices of hypersurfaces H, the corresponding top scheme theoretic intersection is reduced and (at most) zero dimensional.

However, the transversality argument has an interesting extra twist due to the phenomenon of *illusory excess*: For any $3 \leq t \leq r$, the intersection cycle $\Theta_1 \cdots \Theta_t$ (which ought to be of codimension $2t$) has excess along the locus of maps with source



where the vertical twig has degree 0 and maps to the intersection of the t hypersurfaces. This locus has codimension only $t+2$. However this is illusory excess in the sense that the excessive freedom is just the freedom of marks moving on a contracting twig. The corresponding image curves move much less. Indeed, we can forget $t-1$ marks without destabilising, so in the space with no marks, the bad locus gets codimension $t+1$, while the image of the cycle $\Theta_1 \cdots \Theta_t$ has codimension t. If the total intersection is of top degree, a dimension reduction argument then shows that the bad curves do not occur at all! The proof of the Lemma below shows how to cope with the phenomenon.

3.2.1 Proposition. ENUMERATIVE SIGNIFICANCE. *Suppose the general map of $\overline{M}_{0,n}(X,\beta)$ is an immersion, and let $Z \subset \overline{M}_{0,n}(X,\beta)$ denote the scheme theoretic intersection of Theta cycles and evaluation cycles, such that for each mark there is at most one Theta, and such that the total codimension of the cycles equals* $\dim \overline{M}_{0,n}(X,\beta)$. *Then for general hypersurfaces H, the intersection Z is reduced of dimension 0 (or it may be empty).*

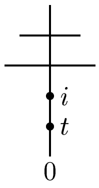
Furthermore, the corresponding finitely many maps are immersions with irreducible source (so in particular they have no automorphisms).

The proof will follow easily from the following result:

3.2.2 Lemma. *In a $2t$-parameter family satisfying* (A), (B) *and* (C), *for general hypersurfaces H_i, the scheme-theoretic intersection $\Theta_1(H_1) \cap \cdots \cap \Theta_t(H_t)$ is reduced of dimension 0 (or it may be empty).*

Proof of the Lemma. Induction on t. If $t = 1$ then the result follows from Proposition 3.1.4. Suppose the lemma holds for families of dimension $2t-2$ and consider a family S of dimension $2t$. For short, set $\Theta_i = \Theta_i(H_i)$.

Again by the Proposition 3.1.4, the variety Θ_t, with the induced family, has dimension $2t-2$ and satisfies (A) and (B). However, it will not in general satisfy (C), because the locus P of maps with source



and with the vertical contracting twig mapping to H_t, has codimension 3 in S. These maps are tangent to H_t and have critical i'th mark.

But we can circumvent this obstacle by forgetting the t'th mark: Since P has at least four special points on the contracting twig, its image under π_t retains the codimension 3 with respect to the whole space, and therefore it has codimension 2 with respect to $\pi_t(\Theta_t)$.

$$\begin{tikzcd}
P \arrow[r, phantom, "\overset{\text{codim } 1}{\subset}"] \arrow[d] & \Theta_t \arrow[r, phantom, "\overset{\text{codim } 2}{\subset}"] \arrow[d] & S \\
\pi_t(P) \arrow[r, phantom, "\overset{\text{codim } 2}{\subset}"] & \pi_t(\Theta_t) &
\end{tikzcd}$$

So $\pi_t(\Theta_t)$ satisfies (C), and since $\Theta_t \to \pi_t(\Theta_t)$ is generically one-to-one, the image also satisfies (A) and (B).

By induction, for general choices of hypersurfaces $H_1, \ldots, H_{t-1}$, the intersection $\Theta_1 \cap \cdots \cap \Theta_{t-1}$ in the space $\pi_t(\Theta_t)$ is reduced of dimension zero, and by the dimension reduction argument 3.1.5 all these finitely many points correspond to simple tangencies. Now in an open subset around these points, the morphism $\Theta_t \to \pi_t(\Theta_t)$ is immersive and one-to-one, so we conclude that also the intersection upstairs $\Theta_1 \cap \cdots \cap \Theta_{t-1} \cap \Theta_t$ in the space S is reduced. $\square$

Proof of Proposition 3.2.1. By assumption, the space $\overline{M}_{0,n}(X, \beta)$ satisfies (A), (B) and (C). The intersection of all the cycles pulled back via evaluation morphisms also satisfy the three conditions, by the results of usual Gromov-Witten theory. We have arrived at a $2t$-parameter family with t Theta conditions, so the lemma applies.

As to the further properties, they follow from the dimension reduction argument: In each step, the loci of maps with reducible source, critical marks, or non-simple tangency have codimension at least 1, so when we come down to dimension zero they no longer occur. $\square$

Since the intersection is reduced of dimension zero, the number of points in it is equal to the top product of the corresponding classes. The enumerative meaning is this. Each mark i corresponds to a condition; the factor $\Theta_i \nu_i^*(\gamma_i)$ corresponds to the condition of being tangent to a given hyperplane H_i along its intersection with a subvariety of class γ_i. This understood, the Proposition implies that

3.2.3 Corollary. *The top products of Theta classes (with distinct marks) and evaluation classes count what they are expected to.*

3.2.4 Example. THE PROJECTIVE PLANE. The case $X = \mathbb{P}^2$ has been studied with various techniques. Rahul Pandharipande [42] used unmarked conditions: Let $\boldsymbol{\iota}$ be the divisor of maps passing through a given point, and let $\boldsymbol{\tau}$ be the divisor of maps tangent to a given line. The relation to this point of view is easily explained: Let $\pi_i : \overline{M}_{0,n+1}(\mathbb{P}^2, d) \to \overline{M}_{0,n}(\mathbb{P}^2, d)$ be the morphism that forgets p_i. Then

$$\begin{aligned} \pi_{i*}\Theta_i &= \boldsymbol{\tau} \\ \pi_{i*}(\boldsymbol{\eta}_i^2) &= \boldsymbol{\iota}. \end{aligned}$$

Therefore, for $a + b = 3d - 1$ we have

$$\int \boldsymbol{\eta}_1^2 \cdots \boldsymbol{\eta}_a^2 \cdot \Theta_{a+1} \cdots \Theta_{a+b} \cap [\overline{M}_{0,a+b}(\mathbb{P}^2, d)] = \int \boldsymbol{\iota}^a \boldsymbol{\tau}^b \cap [\overline{M}_{0,0}(\mathbb{P}^2, d)],$$

by the projection formula. These are the simple characteristic numbers $N_d(a, b)$ of rational plane curves passing through a points and tangent to b lines.

The condition of being tangent to a given line at a specified point is simply $\Xi_i := \Theta_i \boldsymbol{\eta}_i$. Let $N_d(a, b, c)$ denote the number of rational curves passing through a points, tangent to b lines, and tangent to c lines at specified points. Lars Ernström and Gary Kennedy [12], [13] constructed a space of stable lifts, and gave algorithms for computing its Gromov-Witten invariants, which include these characteristic numbers.

By the corollary above, the characteristic number $N_d(a, b, c)$ is the top intersection

$$\int_{\overline{M}_{0,a+b+c}(\mathbb{P}^2,d)} \boldsymbol{\eta}_1^2 \cdots \boldsymbol{\eta}_a^2 \cdot \Theta_{a+1} \cdots \Theta_{a+b} \cdot \Xi_{a+b+1} \cdots \Xi_{a+b+c}.$$

In 3.3.8, effective recursive formulae are given for these numbers, in form of differential equations for their generating function.

3.3 Differential equations for characteristic numbers

Classically, characteristic numbers (of curves in $\mathbb{P}^r$, and in particular $\mathbb{P}^2$) means the numbers of curves of given genus and degree which satisfy conditions of incidence to specified linear subvarieties, and conditions of tangency to a hyperplane H (optionally, one can also specify a linear subvariety of H where the tangency then is required to occur). The interest in these particular numbers came from the fact that more general enumerative problems often can be solved by simple combinatorics, once one knows the characteristic numbers.

Here we are concerned exclusively with rational curves.

The generalisation to other homogeneous varieties involves the choice of what subvarieties one should impose conditions with respect to. Whatever the choice, the results of the preceding sections show that the numbers can be computed. However it is desirable to have formulae for the numbers itself, or for their generating functions, without mention of moduli spaces or cohomology classes. While closed formulae are not available, the best solutions come in the form of recursions, or partial differential equations for the generating function.

These recursions naturally come in two steps: one step determines those numbers involving only incidence conditions, i.e. the Gromov-Witten invariants. These recursions come from associativity relations. This step is not the main concern of the present work, as it has been well treated elsewhere — see for example Kresch [35].

The other step consists in expressing the numbers involving also tangency conditions in terms of the Gromov-Witten invariants. This is the subject of this section; it is shown how simple coordinate changes translate the differential equations for the tangency potential into equations for the characteristic numbers themselves.

3.3.1 Characteristic numbers. Let p be the Picard number of X, such that the classes $T_1, \ldots, T_p$ are all divisor classes. For simplicity, we allow as incidence conditions only those of mapping the mark into a given subvariety of class Poincaré dual to a basis element T_i, $i = p+1, \ldots, r$ (codimension 2 and up). We denote this condition Ω_i, and use the same symbol for its class in the moduli space:

$$\Omega_i = \overline{\tau}_0(T_i) \qquad \text{for } i = p+1, \ldots, r.$$

Suppose we have chosen t types of conditions, Θ_j, of being tangent to certain hypersurfaces (at a specified mark) (possibly along specified loci). Note that the indices j here have nothing to do with the indexing of the marks. Precisely, we are considering conditions whose class can be expressed (cf. Section 3.1) as linear combinations

$$\Theta_j \;=\; \sum_{i=0}^{r} \rho_{ij}\overline{\tau}_0(T_i) + \sum_{i=0}^{r} \sigma_{ij}\overline{\tau}_1(T_i), \qquad \text{for } j = 1, \ldots, t.$$

Now define characteristic numbers $N_\beta(a_{p+1}, \ldots, a_r, b_1, \ldots, b_t)$ as the number of rational curves of class β satisfying a_i conditions of type Ω_i, for $i = p+1, \ldots, r$, and further b_j conditions of type Θ_j, for $j = 1, \ldots, t$, (provided the codimensions of all the imposed conditions add up to $\dim \overline{M}_{0,n}(X, \beta)$).

3.3.2 Generating functions. Consider first the generating function for the characteristic numbers in degree β,

$$\sum_{\substack{a_{p+1},\dots,a_r\\ b_1,\dots,b_t}} \frac{u_{p+1}^{a_{p+1}}}{a_{p+1}!}\cdots\frac{u_r^{a_r}}{a_r!}\cdot\frac{v_1^{b_1}}{b_1!}\cdots\frac{v_t^{b_t}}{b_t!}N_\beta(a_{p+1},\dots,a_r,b_1,\dots,b_t),$$

where $u_{p+1},\dots,u_r$ and $\mathbf{v}=(v_1,\dots,v_t)$ are formal independent variables. This is admittedly an awkward notation, but it will be justified when we sum over the degree β. To this end, introduce partial degrees $d_i := \int_\beta T_i$, for $i=1,\dots,p$. Now the generating function is $G(u_1,\dots,u_p,\ u_{p+1},\dots,u_r,\ \mathbf{v}) =$

$$\begin{aligned} G(\mathbf{u},\mathbf{v}) &:= \sum_{\beta>0}\exp\Big(\sum_{i=1}^{p} d_i u_i\Big)\sum_{\substack{a_{p+1},\dots,a_r\\ b_1,\dots,b_t}} \frac{u_{p+1}^{a_{p+1}}}{a_{p+1}!}\cdots\frac{u_r^{a_r}}{a_r!}\cdot\frac{\mathbf{v}^{\mathbf{b}}}{\mathbf{b}!}N_\beta(a_{p+1},\dots,a_r,b_1,\dots,b_t),\\ &= \sum_{\beta>0}\sum_{\mathbf{a},\mathbf{b}}\frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!}\frac{\mathbf{v}^{\mathbf{b}}}{\mathbf{b}!}d_1^{a_1}\cdots d_p^{a_p}\,N_\beta(a_{p+1},\dots,a_r,b_1,\dots,b_t). \end{aligned}$$

3.3.3 Lemma. *The generating function for the characteristic numbers is related to the tangency quantum potential by*

$$G(\mathbf{u},\mathbf{v}) = \Gamma(\mathbf{x},\mathbf{y}),$$

subject to the change of variables

$$\begin{aligned} x_0 &= 0, \quad x_i = u_i + \sum_{j=1}^{t}\rho_{ij}v_j \qquad \textit{for } i=1,\dots,r\\ y_0 &= 0, \quad y_i = \sum_{j=1}^{t}\sigma_{ij}v_j \qquad \textit{for } i=1,\dots,r \end{aligned}$$

which is just the dual to the change of variables relating the conditions Ω_j and Θ_j to the basis $\overline{\boldsymbol{\tau}}_0$ and $\overline{\boldsymbol{\tau}}_1$.

Proof. In fact this is a purely formal property: Since

$$\begin{aligned} \Gamma &= \exp\Big(\sum(x_i\overline{\tau}_0(T_i)+y_i\overline{\tau}_1(T_i))\Big) \quad \text{and}\\ G &= \exp\Big(\sum(u_j\Omega_j+v_j\Theta_j)\Big), \end{aligned}$$

and each Ω_j and Θ_j is expressible as linear combination of the basis elements $\overline{\tau}_0(T_i),\overline{\tau}_1(T_i)$, then simply inverting the rôle of the symbols, such that what before were symbols having a meaning now become formal variables, and vice versa, the result follows. $\square$

Proof. As an exercise, let us work through the proof by hand, restricting to the case where there is only one sort of tangency condition Θ with corresponding variable v. Let us first write out what G is, expressing Ω_j and Θ in terms of the basis $\overline{\tau}_0(T_i), \overline{\tau}_1(T_i)$:

$$\sum_{\beta>0}\sum_{\mathbf{a},b} \frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!}\frac{v^b}{b!} d_1^{a_1}\cdots d_p^{a_p}\, N_\beta(a_{p+1},\ldots,a_r,\, b) = \sum_{\beta>0}\sum_{\mathbf{a},b} \frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!}\frac{v^b}{b!} \langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a})\cdot\Theta^b\,\rangle_\beta,$$

by the divisor equation. (For the sake of correctness it must be pointed out that in the sum on the left hand side we are not summing over a_0, since it does not appear in the invariants. On the right hand side the string equation takes care: only $a_0 = 0$ gives contribution.)

Now expand

$$\begin{aligned}\Theta^b &= \Big(\sum_i \rho_i\overline{\tau}_0(T_i) + \sigma_i\overline{\tau}_1(T_i)\Big)^b \\ &= \sum_{\substack{\mathbf{m},\mathbf{n}\\ |\mathbf{m}|+|\mathbf{n}|=b}} \boldsymbol{\rho}^{\mathbf{m}}\boldsymbol{\sigma}^{\mathbf{n}}\frac{b!}{\mathbf{m}!\mathbf{n}!}\,\overline{\boldsymbol{\tau}}_0(\mathbf{m})\overline{\boldsymbol{\tau}}_1(\mathbf{n})\end{aligned}$$

thus getting

$$G(\mathbf{u},\mathbf{v}) = \sum_{\beta>0}\sum_{\mathbf{a},\mathbf{m},\mathbf{n}} \frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!}\, v^b\, \frac{\boldsymbol{\rho}^{\mathbf{m}}}{\mathbf{m}!}\frac{\boldsymbol{\sigma}^{\mathbf{n}}}{\mathbf{n}!}\ \langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{a}+\mathbf{m})\overline{\boldsymbol{\tau}}_1(\mathbf{n})\,\rangle_\beta$$

where $b = |\mathbf{m}| + |\mathbf{n}|$.

On the other hand, substituting the $\mathbf{u}$ and v into $\mathbf{x}$ and $\mathbf{y}$ in Γ, according to the specified rules, we get:

$$\Gamma(\mathbf{x},\mathbf{y}) = \sum_{\beta>0}\sum_{\mathbf{m},\mathbf{n}} \frac{(\mathbf{u}+\boldsymbol{\rho}v)^{\mathbf{m}}}{\mathbf{m}!}\frac{(\boldsymbol{\sigma}v)^{\mathbf{n}}}{\mathbf{n}!}\ \langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{m})\overline{\boldsymbol{\tau}}_1(\mathbf{n})\,\rangle_\beta,$$

and expanding $(\mathbf{u}+\boldsymbol{\rho}v)^{\mathbf{m}} = \sum \binom{\mathbf{m}}{\mathbf{a}}\mathbf{u}^{\mathbf{a}}\boldsymbol{\rho}^{\mathbf{m}-\mathbf{a}}v^{|\mathbf{m}|-|\mathbf{a}|}$, we get

$$\Gamma(\mathbf{x},\mathbf{y}) = \sum_{\beta>0}\sum_{\substack{\mathbf{m},\mathbf{n}\\ \mathbf{a}\leq\mathbf{m}}} \frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!}\, v^{|\mathbf{n}|+|\mathbf{m}|-|\mathbf{a}|}\ \frac{\boldsymbol{\rho}^{\mathbf{m}-\mathbf{a}}}{(\mathbf{m}-\mathbf{a})!}\frac{\boldsymbol{\sigma}^{\mathbf{n}}}{\mathbf{n}!}\ \langle\, \overline{\boldsymbol{\tau}}_0(\mathbf{m})\overline{\boldsymbol{\tau}}_1(\mathbf{n})\,\rangle_\beta$$

which is easily seen to be equal to $G(\mathbf{u}, v)$ by a change of summation variable $\mathbf{m} \rightsquigarrow \mathbf{m}+\mathbf{a}$. $\square$

3.3.4 Remark. This result does not depend on the assumption that the incidence conditions Ω_j span $A^*(X)$ (from degree 2 and up). This assumption is only needed in the following result, in order to translate the differential equation back to an equation for G:

3.3.5 Proposition. *The following differential equations hold for the generating function of the characteristic numbers.*

$$\begin{aligned} G_{v_k u_i u_j} &= \sum_{q=0}^{r} \rho_{qk} G_{u_q u_i u_j} \\ &+ \sum_{q=0}^{r} \sigma_{qk} \Big(G_{u_q (u_i u_j)} - G_{(u_q u_i) u_j} - G_{(u_q u_j) u_i} + \sum_{e,f} G_{u_q u_e}\, \gamma^{ef}(\mathbf{v})\, G_{u_f u_i u_j} \Big). \end{aligned}$$

Here $(\gamma^{ef}(\mathbf{v}))$ *denotes the matrix* (γ^{ef}) *with* y_i *substituted by* $\sum_j \sigma_{ij} v_j$.

Proof. By the chain rule:

$$\begin{aligned} G_{v_k u_i u_j} &= \tfrac{\partial}{\partial v_k} \Gamma_{x_i x_j} = \sum_{q=0}^{r} \rho_{qk} \Gamma_{x_q x_i x_j} + \sum_{q=0}^{r} \sigma_{qk} \Gamma_{y_q x_i x_j} \\ &= \sum_{q=0}^{r} \rho_{qk} \Gamma_{x_q x_i x_j} + \sum_{q=0}^{r} \sigma_{qk} \Big(\Gamma_{x_q (x_i x_j)} - \Gamma_{(x_q x_i) x_j} - \Gamma_{(x_q x_j) x_i} + \sum_{e,f} \Gamma_{x_q x_e} \gamma^{ef} \Gamma_{x_f x_i x_j} \Big). \end{aligned}$$

□

3.3.6 WDVV for characteristic numbers. Given the change of variables relating G and Γ, it is straight forward to derive WDVV for G. Indeed, since there are no y_i-derivatives involved in the equation, it follows easily from the chain rule that the corresponding equation for G is the same, just substituting u_i for x_i and using $\gamma^{ef}(\mathbf{v})$ instead of γ^{ef}.

3.3.7 Example. PROJECTIVE SPACE. Consider $X = \mathbb{P}^r$ with $h := c_1(\mathcal{O}(1))$ and the natural cohomology basis $T_i = h^i$. Consider the conditions $\Theta_j = \overline{\tau}_0(T_{j+1}) + \overline{\tau}_1(T_j)$ of being tangent to a hyperplane H along a specified codimension-j plane contained in H ($j = 1, \ldots, r$), together with the conditions $\Omega_j = \overline{\tau}_0(T_j)$ of being incident to a codimension-j plane ($j = 2, \ldots, r$). Let $N_d(\mathbf{a}, \mathbf{b})$ denote the number of rational curves of degree d satisfying $\mathbf{a}$ such incident conditions and $\mathbf{b}$ such tangency conditions. Then the corresponding generating function $G(\mathbf{u},\mathbf{v})$ (as defined in 3.3.2) is related to the tangency quantum potential by

$$G(\mathbf{u},\mathbf{v}) = \Gamma(\mathbf{x},\mathbf{y})$$

with $x_i = u_i + v_{i-1}$ and $y_i = v_i$. By Proposition 3.3.5, the differential equations for G are

$$G_{v_k u_1 u_1} = G_{u_1 u_1 u_{k+1}} + G_{u_2 u_k} - 2\, G_{u_{k+1} u_1} + \sum_{e,f} G_{u_k u_e}\, \gamma^{ef}(\mathbf{v})\, G_{u_f u_1 u_1}.$$

Here the matrix $(\gamma^{ef}(v))$ is obtained from $(\gamma^{ij}) = (\phi^{ij}(2\mathbf{y}))$ by substituting v_i for y_i. Let us have a look at the particular case of $\mathbb{P}^2$:

3.3.8 Example. THE PROJECTIVE PLANE. The matrix (γ^{ef}) for $X = \mathbb{P}^2$ was given in Example 2.2.5. There are two differential equations for G in this case. The first one reads, in its integrated form (cf. 2.3.3),

$$G_{v_1 u_1} = G_{u_1 u_2} - G_{u_2} + \tfrac{1}{2} G^2_{u_1 u_1} + 2 v_1 G_{u_1 u_1} G_{u_1 u_2} + (v_1^2 + v_2) G^2_{u_1 u_2}. \quad (3.3.8.1)$$

Setting $v_2 = 0$, corresponding to excluding the condition of tangency at specified point, gives an equation that determines the simple characteristic numbers $N_d(a,b)$ from the Gromov-Witten invariants. This differential equation was first found by Rahul Pandharipande [39], and later, with different methods, by Ravi Vakil [46].

The second equation,

$$\begin{aligned} G_{v_2 u_1 u_1} = G_{u_2 u_2} + G_{u_2 u_1} G_{u_1 u_1 u_1} + 2 v_1 G_{u_2 u_2} G_{u_1 u_1 u_1} + 2 v_1 G_{u_2 u_1} G_{u_2 u_1 u_1} \\ + (2 v_1^2 + 2 v_2) G_{u_2 u_2} G_{u_2 u_1 u_1}. \end{aligned}$$

can be simplified considerably if we set $v_1 = 0$ — that is, considering only characteristic numbers $N_d(a,0,c)$ of point conditions and point conditions with tangent direction. Then it reads

$$G_{u_1 u_1 v_2} = G_{u_2 u_2} + G_{u_2 u_1} G_{u_1 u_1 u_1} + 2 v_2 G_{u_2 u_2} G_{u_1 u_1 u_2} \pmod{v_1}. \quad (3.3.8.2)$$

This equation, together with the first one, constitute a very transparent and effective way to compute the characteristic numbers $N_d(a,b,c)$, whose definition was given in Example 3.2.4.

Considering the WVDD equations for G in this case (cf. 3.3.6), it turns out the only non-trivial relation is the one with $i = j = 1$ and $k = \ell = 2$. The relation then reads

$$G_{222} = G_{112}^2 - G_{111}G_{122} + 2v_1\big(G_{122}G_{112} - G_{111}G_{222}\big) + (2v_1^2 + 2v_2)\big(G_{122}^2 - G_{112}G_{222}\big),$$

where e.g. G_{222} is short for $G_{u_2u_2u_2}$. This equation was first found by Lars Ernström and Gary Kennedy ([13], 6.3) (cf. Remark 2.5.2). Note that this equation is not sufficient to determine all the tangency numbers from the Gromov-Witten invariants alone: only N_d's with at least three incidence conditions are determined.

3.3.9 Example. Extracting recursive formulae. This is a mere exercise in multiplying power series and extracting coefficients, cf. also Remarks 2.1.2. First note that derivative of G with respect to a variable corresponds to incrementing the number of corresponding conditions, and if the variable belongs to a divisor class condition, by the divisor equation this derivation corresponds to a factor d in front of the N_d. So for example, continuing with $X = \mathbb{P}^2$, the (a_2, b_1, b_2, d)'th coefficient of $G_{v_1u_1}$ is $d \cdot N_d(a_2, b_1 + 1, b_2)$. A single factor v_j in front of the generating function corresponds to decrementing the number of the corresponding tangency conditions and having a factor b_j in front of the characteristic number. For higher powers of v_j, this rule must be applied consecutively, such that a factor v_j^k corresponds to decrementing the number of the corresponding tangency condition by k and having a factor $b_j!/(b_j - k)!$ in front of the N_d. So for example $v_1^2 G_{u_1u_2}$ corresponds to $d\,b(b-1)\,N_d(a_2+1, b_1-2, b_2)$. And then finally, the coefficients of a product of two functions involves a sum over all 2-partitions of all the variables involved, with a binomial coefficient for each variable.

Here is the recursive relation corresponding to 3.3.8.1. In order to comply to the usual notation we write $N_d(a, b, c)$ instead of $N_d(a_2, b_1, b_2)$:

$$\begin{aligned}
d \cdot N_d(a, b+1, c) \;=\; & d \cdot N_d(a+1, b, c) - N_d(a+1, b, c) \\
& + \sum \tbinom{a}{a'}\tbinom{b}{b'}\tbinom{c}{c'}\; \tfrac{1}{2} d'^2 N_{d'}(a', b', c')\; d''^2 N_{d''}(a'', b'', c'') \\
& + \sum \tbinom{a}{a'}\tbinom{b}{b'}\tbinom{c}{c'}\; 2d'^2 N_{d'}(a', b'-1, c')\; d'' N_{d''}(a''+1, b'', c'') \\
& + \sum \tbinom{a}{a'}\tbinom{b}{b'}\tbinom{c}{c'}\; d'b'(b'-1) N_{d'}(a'+1, b'-2, c')\; d'' N_{d''}(a''+1, b'', c'') \\
& + \sum \tbinom{a}{a'}\tbinom{b}{b'}\tbinom{c}{c'}\; d'c' N_{d'}(a'+1, b', c'-1)\; d'' N_{d''}(a''+1, b'', c'').
\end{aligned}$$

Each sum is over positive partitions $d' + d'' = d$ and non-negative partitions $a' + a'' = a$, $b' + b'' = b$, and $c' + c'' = c$. The six terms corresponds exactly to the six terms in 3.3.8.1. This is a quite casual way to write the sums, since most of the terms in each summation are in fact zero due to incompatibility of a', b', c' with the degree d'. A neater organisation of the terms in this formula can be found in Vakil [46](§2.3.) (for the case $c = 0$).

3.3.10 Example. THE QUADRIC SURFACE $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let T_0 be the fundamental class; let T_3 be the class of a point; and let T_1 and T_2 be the hyperplane classes pulled back from the two factors. The canonical class is $-2T_1 - 2T_2$, so the moduli space $\overline{M}_{0,0}(X, (m, n))$ of rational curves in X of bi-degree (m, n) is of dimension $2m + 2n - 1$.

The deformed metric is given by the matrix

$$(\gamma^{ef}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2y_1 \\ 0 & 1 & 0 & 2y_2 \\ 1 & 2y_1 & 2y_2 & 2y_3 + 4y_1y_2 \end{pmatrix}$$

The only reasonable incidence condition to consider is that of passing through a general point; this condition is $\Omega_3 = \overline{\tau}_0(T_3)$. For the tangency conditions, we need a very ample class, so consider the condition Θ of being tangent to a general curve of bi-degree $(1, 1)$. It follows readily from the discussion in the first section that the class is $\Theta = 2\overline{\tau}_0(T_3) + \overline{\tau}_1(T_1) + \overline{\tau}_1(T_2)$. Consider finally the condition of being tangent to such a curve at a specified point. This class is $\Xi = \overline{\tau}_1(T_3)$. The two change-of-variables matrices are

$$\rho = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the first columns correspond to Θ and the second to Ξ.

Let $N_{(m,n)}(a_3, b_{12}, b_3)$ denote the characteristic numbers of rational curves of bi-degree (m, n) satisfying a_3 conditions of type Ω_3, b_{12} conditions of type Θ and b_3 conditions of type Ξ. Let $G(u_1, u_2, u_3, v_{12}, v_3)$ be the corresponding generating function (u_1 and u_2 being the formal variables corresponding to the partial degrees m and n). Then $G(u_1, u_2, u_3, v_{12}, v_3) = \Gamma(x_1, x_2, x_3, y_1, y_2, y_3)$, where $x_1 = u_1$, $x_2 = u_2$, $x_3 = u_3 + 2v_{12}$; $y_1 = v_{12}$, $y_2 = v_{12}$, $y_3 = v_3$.

Proposition 3.3.5 gives the differential equation

$$G_{v_{12}u_1u_1} = 2G_{u_3u_1u_1} - 2G_{u_3u_1} + \sum_{e,f} G_{u_1u_e}\,\gamma^{ef}(\mathbf{v})\,G_{u_fu_1u_1} + \sum_{e,f} G_{u_2u_e}\,\gamma^{ef}(\mathbf{v})\,G_{u_fu_1u_1}$$

where

$$(\gamma^{ef}(\mathbf{v})) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2v_{12} \\ 0 & 1 & 0 & 2v_{12} \\ 1 & 2v_{12} & 2v_{12} & 4v_{12}^2 + 2v_3 \end{pmatrix}.$$

But note that it is not possible to integrate the equation. Also there is an annoying doubling of all the terms, with one term for u_1 and another for u_2. In fact, a much nicer equation can be obtained by taking derivative with respect to $u_{12} := u_1 + u_2$, the formal variable corresponding to $T_1 + T_2$. That equation reads (in its integrated version):

$$\begin{aligned} G_{v_{12}u_{12}} &= 2G_{u_3u_{12}} - 2G_{u_3} + \tfrac{1}{2}\sum_{e,f} G_{u_{12}u_e}\,\gamma^{ef}(\mathbf{v})\,G_{u_fu_{12}} \\ &= 2G_{u_3u_{12}} - 2G_{u_3} + G_{u_{12}u_1}G_{u_2u_{12}} \\ &\quad +2v_{12}G_{u_{12}u_{12}}G_{u_3u_{12}} + (2v_{12}^2 + v_3)G_{u_{12}u_3}^2. \end{aligned} \tag{3.3.10.1}$$

Extracting terms from this differential equation (as in Example 3.3.9) yields a recursion which decrements the number of Θ-conditions, and thus determines the numbers $N_{(m,n)}(a,b,c)$ from those of type $N_{(m,n)}(a,0,c)$. Note that derivative with respect to u_{12} corresponds to a factor $m+n$ in front of the characteristic number.

We also need an equation for decrementing the number of Ξ-conditions, i.e. involving derivation with respect to v_3. Again the nicest one is that with u_{12} derivatives:

$$\begin{aligned} G_{v_3u_{12}u_{12}} &= 2G_{u_3u_3} + G_{u_3u_1}G_{u_2u_{12}u_{12}} + G_{u_3u_2}G_{u_1u_{12}u_{12}} \\ &\quad +2v_{12}G_{u_3u_{12}}G_{u_3u_{12}u_{12}} + 2v_{12}G_{u_3u_3}G_{u_{12}u_{12}u_{12}} \\ &\quad +(4v_{12}^2 + 2v_3)G_{u_3u_3}G_{u_3u_{12}u_{12}} \end{aligned} \tag{3.3.10.2}$$

This equation will be important also in Example 4.4.8. For the present purposes it is enough to look at the simplified version obtained setting $v_{12} = 0$, corresponding to considering only characteristic numbers of type $N_{(m,n)}(a,0,c)$.

Then it reads

$$\begin{aligned} G_{v_3 u_{12} u_{12}} &= 2G_{u_3u_3} + G_{u_3u_1}G_{u_2u_{12}u_{12}} + G_{u_3u_2}G_{u_1u_{12}u_{12}} \\ &\quad +2v_3\, G_{u_3u_3}G_{u_3u_{12}u_{12}} \pmod{v_{12}}. \end{aligned}$$

This equation together with 3.3.10.1 constitute a transparent algorithm for computing the characteristic numbers from the Gromov-Witten invariants. (These in turn are computed using associativity relations.) A maple implementation of the recurrences produced the numbers tabulated in the Appendix to this chapter.

3.3.11 Example. THE GRASSMANNIAN $\mathrm{Gr}(2,4)$. Let $X = \mathrm{Gr}(2,4)$, the Grassmannian of lines in $\mathbb{P}^3$. Take the basis of Schubert varieties, such that T_0 is the fundamental class, T_1 is the hyperplane section (under the Plücker embedding $\mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}^5$), T_2 and T_3 are the two Schubert varieties of dimension 2, T_4 is dual to T_1 (it's a line), and T_5 is the class of a point. The intersection matrix is

$$(g_{ij}) = (g^{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The intersection polynomial ϕ is

$$\phi = e^{y_0}\left(y_5 + y_1y_4 + \tfrac{1}{2}y_2^2 + \tfrac{1}{2}y_3^2 + \tfrac{1}{2}y_1^2y_2 + \tfrac{1}{2}y_1^2y_3 + \tfrac{1}{12}y_1^4\right).$$

Let $N_d(b, a_2, a_3, a_4, a_5)$ be the number of rational curves of class $d \cdot T_4$ that are tangent to b subvarieties of class T_1 and incident to a_i subvarieties of class T_i, for $i = 2,3,4,5$. The tangency condition is of class $\Theta = \overline{\tau}_0(T_2) + \overline{\tau}_0(T_3) + \overline{\tau}_1(T_1)$. Let $G(v, u_1, \ldots, u_5)$ denote the corresponding generating function, as in 3.3.2. By Proposition 3.3.5, the differential equation for G is

$$G_{vu_1} = G_{u_1u_2} + G_{u_1u_3} - G_{u_2} - G_{u_3} + \tfrac{1}{2}\sum_{e,f} G_{u_1u_e}\, \gamma^{ef}(v)\, G_{u_fu_1}$$

which is the integrated form, cf. 2.3.3. Here the matrix is

$$(\gamma^{ef}(v)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2v \\ 0 & 0 & 1 & 0 & 2v & 2v^2 \\ 0 & 0 & 0 & 1 & 2v & 2v^2 \\ 0 & 1 & 2v & 2v & 4v^2 & \frac{8}{3}v^3 \\ 1 & 2v & 2v^2 & 2v^2 & \frac{8}{3}v^3 & \frac{4}{3}v^4 \end{pmatrix}$$

Extracting terms from the differential equation, as in the previous examples, yields a recursion for the characteristic numbers, with two linear and eleven quadratic terms. The numbers in degree 2 and 3 are listed in the Appendix to this chapter. They were computed via a maple implementation of the recurrence.

3.A Appendix: Numbers

Here we list the characteristic numbers in low degree of $\mathbb{P}^1 \times \mathbb{P}^1$, as described in Example 3.3.10, and of $\mathrm{Gr}(2,4)$ as described in Example 3.3.11.

Characteristic numbers of rational curves in $\mathbb{P}^1 \times \mathbb{P}^1$

Let $N_{(m,n)}(a,b,c)$ be the number of rational curves of bi-degree (m,n) tangent to b curves of bi-degree $(1,1)$, tangent to c such curves at specified points, and furthermore passing through a specified points, $(a+b+2c = 2m+2n-1)$.

Since the two generators are interchangeable, we have $N_{(m,n)}(a,b,c) = N_{(n,m)}(a,b,c)$, so only numbers with $m \geq n$ are listed.

Curves of bi-degree $(m,0)$ are not treated, since such curves are not immersions, except for $m=1$. In this case there are two characteristic numbers: $N_{(1,0)}(1,0,0) = 1$, (one rule through a point) and $N_{(1,0)}(0,1,0) = 0$, (no rules tangent to a $(1,1)$-curve).

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(1,1)$:

	$c=0$	$c=1$
$b=0$	1	1
$b=1$	2	2
$b=2$	4	
$b=3$	8	

$a = 3 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(2,1)$:

	$c=0$	$c=1$	$c=2$
$b=0$	1	1	1
$b=1$	4	4	4
$b=2$	16	14	
$b=3$	56	44	
$b=4$	176		
$b=5$	512		

$a = 5 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(3,1)$:

	$c=0$	$c=1$	$c=2$	$c=3$
$b=0$	1	1	1	1
$b=1$	6	6	6	6
$b=2$	36	34	32	
$b=3$	200	176	152	
$b=4$	1020	804		
$b=5$	4632	3272		
$b=6$	18832			
$b=7$	69792			

$a = 7 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(4,1)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$
$b=0$	1	1	1	1	1
$b=1$	8	8	8	8	8
$b=2$	64	62	60	58	
$b=3$	488	452	416	380	
$b=4$	3496	3040	2608		
$b=5$	23072	18592	14592		
$b=6$	138976	101536			
$b=7$	752768	496064			
$b=8$	3667456				
$b=9$	16243712				

$a = 9 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(5,1)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$
$b=0$	1	1	1	1	1	1
$b=1$	10	10	10	10	10	10
$b=2$	100	98	96	94	92	
$b=3$	968	920	872	824	776	
$b=4$	8948	8156	7388	6644		
$b=5$	77912	67432	57672	48632		
$b=6$	632560	513640	409120			
$b=7$	4733920	3566656	2607712			
$b=8$	32394256	22306640				
$b=9$	200734496	125512352				
$b=10$	1124673344					
$b=11$	5734390400					

$a = 11 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(6,1)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$	$c=6$
$b=0$	1	1	1	1	1	1	1
$b=1$	12	12	12	12	12	12	12
$b=2$	144	142	140	138	136	134	
$b=3$	1688	1628	1568	1508	1448	1388	
$b=4$	19104	17880	16680	15504	14352		
$b=5$	206592	186272	166912	148512	131072		
$b=6$	2116384	1823824	1555744	1311424			
$b=7$	20359488	16631904	13374720	10557696			
$b=8$	182449536	139956736	104950016				
$b=9$	1510097408	1077470720	744611840				
$b=10$	11463765504	7520193024					
$b=11$	79236384768	47473200128					
$b=12$	497549940736						
$b=13$	2849483661312						

$a = 13 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(2,2)$:

	$c=0$	$c=1$	$c=2$	$c=3$
$b=0$	12	10	8	6
$b=1$	56	44	32	24
$b=2$	248	176	120	
$b=3$	1008	656	424	
$b=4$	3808	2296		
$b=5$	13504	7600		
$b=6$	45312			
$b=7$	144896			

$a = 7 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(3,2)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$
$b=0$	96	72	52	36	24
$b=1$	528	376	256	168	112
$b=2$	2776	1852	1184	740	
$b=3$	13776	8580	5144	3084	
$b=4$	64332	37316	21068		
$b=5$	282272	152864	81664		
$b=6$	1166976	592392			
$b=7$	4566432	2181904			
$b=8$	16995712				
$b=9$	60455424				

$a = 9 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(4, 2)$:

	$c = 0$	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
$b = 0$	640	448	304	200	128	80
$b = 1$	4096	2752	1792	1136	704	440
$b = 2$	25312	16224	10096	6128	3664	
$b = 3$	149792	91280	54032	31264	18064	
$b = 4$	845024	487712	273536	150688		
$b = 5$	4526720	2466336	1307648	687040		
$b = 6$	22966912	11785024	5908864			
$b = 7$	110226304	53256448	25301632			
$b = 8$	500829184	228172800				
$b = 9$	2159618048	929967104				
$b = 10$	8867409920					
$b = 11$	34798014464					

$a = 11 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(5, 2)$:

	$c = 0$	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$	$c = 6$
$b = 0$	3840	2560	1664	1056	656	400	240
$b = 1$	28160	18176	11456	7072	4288	2560	1536
$b = 2$	200960	125120	76192	45520	26752	15552	
$b = 3$	1386240	830144	486592	279984	158656	89696	
$b = 4$	9200128	5282848	2969200	1638032	892064		
$b = 5$	58511680	32113120	17239808	9085472	4751552		
$b = 6$	355393440	185810992	94951232	47704464			
$b = 7$	2055577280	1020667280	495267104	237196720			
$b = 8$	11297269520	5314781648	2446881424				
$b = 9$	58926853056	26237633216	11468451520				
$b = 10$	291756079360	122985315040					
$b = 11$	1373145397120	548701238464					
$b = 12$	6157856052480						
$b = 13$	26388768081920						

$a = 13 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(3,3)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$
$b=0$	3510	2270	1414	846	486	270
$b=1$	20220	12428	7324	4140	2268	1260
$b=2$	111448	64668	35936	19236	10152	
$b=3$	584496	319264	167312	85248	43632	
$b=4$	2911608	1496200	741608	361368		
$b=5$	13778000	6672240	3138928	1468752		
$b=6$	62053920	28395056	12722368			
$b=7$	266707776	115662656	49508032			
$b=8$	1097153632	452247648				
$b=9$	4332747456	1702110400				
$b=10$	16473279360					
$b=11$	60465708800					

$a = 11 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(4,3)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$	$c=6$
$b=0$	87544	51280	29056	15912	8424	4320	2160
$b=1$	542832	304224	164656	86112	43632	21600	10800
$b=2$	3240336	1731736	893376	445824	216528	104040	
$b=3$	18562976	9441668	4637576	2209116	1031088	483060	
$b=4$	101886348	49263204	23035244	10488132	4718412		
$b=5$	535354608	245977040	109571472	47780784	20780208		
$b=6$	2692810624	1176155848	499821232	209217624			
$b=7$	12974288352	5392647312	2190202432	882019728			
$b=8$	59952866112	23748833536	9236533632				
$b=9$	266135952896	100650153728	37559141888				
$b=10$	1137085798656	411326250240					
$b=11$	4685557843968	1624170692096					
$b=12$	18659621361664						
$b=13$	71961386348544						

$a = 13 - b - 2c$

Characteristic numbers of rational curves in $\mathrm{Gr}(2,4)$

Let $N_d(b, a_2, a_3, a_4, a_5)$ be the number of rational curves of class $d \cdot T_4$ that are tangent to b subvarieties of class T_1 and incident to a_i subvarieties of class T_i, for $i = 2,3,4,5$. Since the two Schubert classes T_2 and T_3 are interchangeable from an intersection theoretic view-point, interchanging a_2 and a_3 does not affect the characteristic number. This redundance has been eliminated in the following listings. The numbers with $d = 1$ and $b > 0$ are all zero, and have not been listed either.

Numbers with $d = 2$:

1 tangency condition (41 such numbers):

$N_2(1,0,0,1,2) = 2$	$N_2(1,2,2,2,0) = 22$	$N_2(1,4,4,0,0) = 84$
$N_2(1,0,0,4,0) = 6$	$N_2(1,3,0,1,1) = 4$	$N_2(1,5,0,0,1) = 2$
$N_2(1,1,0,2,1) = 4$	$N_2(1,3,1,2,0) = 18$	$N_2(1,5,1,1,0) = 18$
$N_2(1,1,1,0,2) = 2$	$N_2(1,3,2,0,1) = 10$	$N_2(1,5,3,0,0) = 68$
$N_2(1,1,1,3,0) = 12$	$N_2(1,3,3,1,0) = 42$	$N_2(1,6,0,1,0) = 6$
$N_2(1,2,0,0,2) = 2$	$N_2(1,4,0,2,0) = 10$	$N_2(1,6,2,0,0) = 36$
$N_2(1,2,0,3,0) = 10$	$N_2(1,4,1,0,1) = 6$	$N_2(1,7,1,0,0) = 12$
$N_2(1,2,1,1,1) = 6$	$N_2(1,4,2,1,0) = 34$	$N_2(1,8,0,0,0) = 4$

2 tangency conditions (31 such numbers):

$N_2(2,0,0,2,1) = 8$	$N_2(2,2,2,0,1) = 20$	$N_2(2,4,3,0,0) = 136$
$N_2(2,1,0,0,2) = 4$	$N_2(2,3,0,2,0) = 20$	$N_2(2,5,0,1,0) = 12$
$N_2(2,1,0,3,0) = 20$	$N_2(2,3,1,0,1) = 12$	$N_2(2,5,2,0,0) = 72$
$N_2(2,1,1,1,1) = 12$	$N_2(2,3,2,1,0) = 68$	$N_2(2,6,1,0,0) = 24$
$N_2(2,2,0,1,1) = 8$	$N_2(2,4,0,0,1) = 4$	$N_2(2,7,0,0,0) = 8$
$N_2(2,2,1,2,0) = 36$	$N_2(2,4,1,1,0) = 36$	

3 tangency conditions (23 such numbers):

$N_2(3,0,0,0,2) = 8$	$N_2(3,2,0,2,0) = 40$	$N_2(3,3,3,0,0) = 208$
$N_2(3,0,0,3,0) = 32$	$N_2(3,2,1,0,1) = 24$	$N_2(3,4,0,1,0) = 24$
$N_2(3,0,6,0,0) = 16$	$N_2(3,2,2,1,0) = 104$	$N_2(3,4,2,0,0) = 144$
$N_2(3,1,0,1,1) = 16$	$N_2(3,3,0,0,1) = 8$	$N_2(3,5,1,0,0) = 48$
$N_2(3,1,1,2,0) = 56$	$N_2(3,3,1,1,0) = 72$	$N_2(3,6,0,0,0) = 16$

4 tangency conditions (16 such numbers):

$N_2(4,0,0,1,1) = 24$	$N_2(4,2,0,0,1) = 16$	$N_2(4,3,2,0,0) = 224$
$N_2(4,1,0,2,0) = 64$	$N_2(4,2,1,1,0) = 112$	$N_2(4,4,1,0,0) = 96$
$N_2(4,1,1,0,1) = 32$	$N_2(4,3,0,1,0) = 48$	$N_2(4,5,0,0,0) = 32$

5 tangency conditions (11 such numbers):

$N_2(5,0,0,2,0) = 80$	$N_2(5,2,0,1,0) = 80$	$N_2(5,3,1,0,0) = 160$
$N_2(5,1,0,0,1) = 24$	$N_2(5,2,2,0,0) = 256$	$N_2(5,4,0,0,0) = 64$
$N_2(5,1,1,1,0) = 128$		

6, 7, 8, and 9 tangency conditions (7+4+2+1 such numbers):

$N_2(6,0,0,0,1) = 24$	$N_2(6,3,0,0,0) = 112$	$N_2(7,2,0,0,0) = 160$
$N_2(6,1,0,1,0) = 104$	$N_2(7,0,0,1,0) = 104$	$N_2(8,1,0,0,0) = 184$
$N_2(6,2,1,0,0) = 208$	$N_2(7,1,1,0,0) = 208$	$N_2(9,0,0,0,0) = 184$

Numbers with $d = 3$:

1 tangency condition (102 such numbers):

$N_3(1,0,0,0,4) = 8$	$N_3(1,3,3,3,0) = 3048$	$N_3(1,6,3,0,1) = 3122$
$N_3(1,0,0,3,2) = 20$	$N_3(1,4,0,1,2) = 72$	$N_3(1,6,4,1,0) = 23662$
$N_3(1,0,0,6,0) = 124$	$N_3(1,4,0,4,0) = 752$	$N_3(1,6,6,0,0) = 79876$
$N_3(1,1,0,1,3) = 16$	$N_3(1,4,1,2,1) = 444$	$N_3(1,7,0,1,1) = 336$
$N_3(1,1,0,4,1) = 78$	$N_3(1,4,2,0,2) = 240$	$N_3(1,7,1,2,0) = 3516$
$N_3(1,1,1,2,2) = 52$	$N_3(1,4,2,3,0) = 2780$	$N_3(1,7,2,0,1) = 1978$
$N_3(1,1,1,5,0) = 372$	$N_3(1,4,3,1,1) = 1416$	$N_3(1,7,3,1,0) = 17608$
$N_3(1,2,0,2,2) = 48$	$N_3(1,4,4,2,0) = 8772$	$N_3(1,7,5,0,0) = 72706$
$N_3(1,2,0,5,0) = 342$	$N_3(1,5,0,2,1) = 284$	$N_3(1,8,0,2,0) = 1572$
$N_3(1,2,1,0,3) = 28$	$N_3(1,5,1,0,2) = 168$	$N_3(1,8,1,0,1) = 926$
$N_3(1,2,1,3,1) = 212$	$N_3(1,5,1,3,0) = 2084$	$N_3(1,8,2,1,0) = 10426$
$N_3(1,2,2,1,2) = 116$	$N_3(1,5,2,1,1) = 1152$	$N_3(1,8,4,0,0) = 54256$
$N_3(1,2,2,4,0) = 1076$	$N_3(1,5,3,2,0) = 7984$	$N_3(1,9,0,0,1) = 310$
$N_3(1,3,0,0,3) = 20$	$N_3(1,5,4,0,1) = 3850$	$N_3(1,9,1,1,0) = 4748$
$N_3(1,3,0,3,1) = 178$	$N_3(1,5,5,1,0) = 26012$	$N_3(1,9,3,0,0) = 32398$
$N_3(1,3,1,1,2) = 104$	$N_3(1,6,0,0,2) = 80$	$N_3(1,10,0,1,0) = 1634$
$N_3(1,3,1,4,0) = 986$	$N_3(1,6,0,3,0) = 1240$	$N_3(1,10,2,0,0) = 15092$
$N_3(1,3,2,2,1) = 542$	$N_3(1,6,1,1,1) = 732$	$N_3(1,11,1,0,0) = 5462$
$N_3(1,3,3,0,2) = 268$	$N_3(1,6,2,2,0) = 5948$	$N_3(1,12,0,0,0) = 1552$

2 tangency conditions (83 such numbers):

$N_3(2,0,0,1,3) = 48$
$N_3(2,0,0,4,1) = 200$
$N_3(2,1,0,2,2) = 136$
$N_3(2,1,0,5,0) = 876$
$N_3(2,1,1,0,3) = 80$
$N_3(2,1,1,3,1) = 544$
$N_3(2,2,0,0,3) = 64$
$N_3(2,2,0,3,1) = 492$
$N_3(2,2,1,1,2) = 288$
$N_3(2,2,1,4,0) = 2524$
$N_3(2,2,2,2,1) = 1360$
$N_3(2,3,0,1,2) = 216$
$N_3(2,3,0,4,0) = 2052$
$N_3(2,3,1,2,1) = 1212$
$N_3(2,3,2,0,2) = 640$
$N_3(2,3,2,3,0) = 7040$
$N_3(2,3,3,1,1) = 3472$
$N_3(2,4,0,2,1) = 824$
$N_3(2,4,1,0,2) = 488$
$N_3(2,4,1,3,0) = 5644$
$N_3(2,4,2,1,1) = 3084$
$N_3(2,4,3,2,0) = 19968$
$N_3(2,4,4,0,1) = 9272$
$N_3(2,5,0,0,2) = 248$
$N_3(2,5,0,3,0) = 3524$
$N_3(2,5,1,1,1) = 2084$
$N_3(2,5,2,2,0) = 15928$
$N_3(2,5,3,0,1) = 8220$
$N_3(2,5,4,1,0) = 58556$
$N_3(2,6,0,1,1) = 1004$
$N_3(2,6,1,2,0) = 9872$
$N_3(2,6,2,0,1) = 5528$
$N_3(2,6,3,1,0) = 46656$
$N_3(2,6,5,0,0) = 178468$
$N_3(2,7,0,2,0) = 4584$
$N_3(2,7,1,0,1) = 2700$
$N_3(2,7,2,1,0) = 28920$
$N_3(2,7,4,0,0) = 142508$
$N_3(2,8,0,0,1) = 936$
$N_3(2,8,1,1,0) = 13620$
$N_3(2,8,3,0,0) = 88948$
$N_3(2,9,0,1,0) = 4836$
$N_3(2,9,2,0,0) = 42716$
$N_3(2,10,1,0,0) = 15916$
$N_3(2,11,0,0,0) = 4676$

3 tangency conditions (67 such numbers):

$N_3(3,0,0,2,2) = 352$
$N_3(3,0,0,5,0) = 2048$
$N_3(3,1,0,0,3) = 192$
$N_3(3,1,0,3,1) = 1256$
$N_3(3,1,1,1,2) = 720$
$N_3(3,1,1,4,0) = 5888$
$N_3(3,2,0,1,2) = 608$
$N_3(3,2,0,4,0) = 5216$
$N_3(3,2,1,2,1) = 3040$
$N_3(3,2,2,0,2) = 1536$
$N_3(3,2,2,3,0) = 16192$
$N_3(3,3,0,2,1) = 2256$
$N_3(3,3,1,0,2) = 1312$
$N_3(3,3,1,3,0) = 14208$
$N_3(3,3,2,1,1) = 7560$
$N_3(3,3,3,2,0) = 45280$
$N_3(3,4,0,0,2) = 736$
$N_3(3,4,0,3,0) = 9504$
$N_3(3,4,1,1,1) = 5568$
$N_3(3,4,2,2,0) = 39584$
$N_3(3,4,3,0,1) = 19792$
$N_3(3,4,4,1,0) = 131344$
$N_3(3,5,0,1,1) = 2880$
$N_3(3,5,1,2,0) = 26240$
$N_3(3,5,2,0,1) = 14480$
$N_3(3,5,3,1,0) = 114688$
$N_3(3,5,5,0,0) = 397424$
$N_3(3,6,0,2,0) = 12864$
$N_3(3,6,1,0,1) = 7536$
$N_3(3,6,2,1,0) = 75888$
$N_3(3,6,4,0,0) = 347424$
$N_3(3,7,0,0,1) = 2736$
$N_3(3,7,1,1,0) = 37536$
$N_3(3,7,3,0,0) = 231024$
$N_3(3,8,0,1,0) = 13872$
$N_3(3,8,2,0,0) = 116064$
$N_3(3,9,1,0,0) = 44880$
$N_3(3,10,0,0,0) = 13728$

4 tangency conditions (53 such numbers):

$N_3(4,0,0,0,3) = 512$	$N_3(4,3,0,3,0) = 23984$	$N_3(4,5,2,1,0) = 185824$
$N_3(4,0,0,3,1) = 2912$	$N_3(4,3,1,1,1) = 13728$	$N_3(4,5,4,0,0) = 770912$
$N_3(4,1,0,1,2) = 1568$	$N_3(4,3,2,2,0) = 89568$	$N_3(4,6,0,0,1) = 7680$
$N_3(4,1,0,4,0) = 12160$	$N_3(4,3,3,0,1) = 42400$	$N_3(4,6,1,1,0) = 98304$
$N_3(4,1,1,2,1) = 6848$	$N_3(4,4,0,1,1) = 7808$	$N_3(4,6,3,0,0) = 560192$
$N_3(4,2,0,2,1) = 5728$	$N_3(4,4,1,2,0) = 65120$	$N_3(4,7,0,1,0) = 38304$
$N_3(4,2,1,0,2) = 3200$	$N_3(4,4,2,0,1) = 34944$	$N_3(4,7,2,0,0) = 299712$
$N_3(4,2,1,3,0) = 32656$	$N_3(4,4,3,1,0) = 256448$	$N_3(4,8,1,0,0) = 121632$
$N_3(4,2,2,1,1) = 16544$	$N_3(4,5,0,2,0) = 34336$	$N_3(4,9,0,0,0) = 39072$
$N_3(4,3,0,0,2) = 2048$	$N_3(4,5,1,0,1) = 19840$	

5 tangency conditions (41 such numbers):

$N_3(5,0,0,1,2) = 3616$	$N_3(5,2,2,2,0) = 177920$	$N_3(5,4,4,0,0) = 1496704$
$N_3(5,0,0,4,0) = 25344$	$N_3(5,3,0,1,1) = 19712$	$N_3(5,5,0,0,1) = 20480$
$N_3(5,1,0,2,1) = 13232$	$N_3(5,3,1,2,0) = 148288$	$N_3(5,5,1,1,0) = 241792$
$N_3(5,1,1,0,2) = 6880$	$N_3(5,3,2,0,1) = 75760$	$N_3(5,5,3,0,0) = 1245056$
$N_3(5,1,1,3,0) = 66368$	$N_3(5,3,3,1,0) = 501696$	$N_3(5,6,0,1,0) = 100992$
$N_3(5,2,0,0,2) = 5248$	$N_3(5,4,0,2,0) = 86144$	$N_3(5,6,2,0,0) = 727296$
$N_3(5,2,0,3,0) = 55744$	$N_3(5,4,1,0,1) = 48512$	$N_3(5,7,1,0,0) = 314496$
$N_3(5,2,1,1,1) = 30544$	$N_3(5,4,2,1,0) = 416896$	$N_3(5,8,0,0,0) = 107136$

6 tangency conditions (31 such numbers):

$N_3(6,0,0,2,1) = 26848$	$N_3(6,2,2,0,1) = 139296$	$N_3(6,4,3,0,0) = 2443584$
$N_3(6,1,0,0,2) = 12128$	$N_3(6,3,0,2,0) = 200096$	$N_3(6,5,0,1,0) = 251776$
$N_3(6,1,0,3,0) = 116064$	$N_3(6,3,1,0,1) = 107728$	$N_3(6,5,2,0,0) = 1631168$
$N_3(6,1,1,1,1) = 58464$	$N_3(6,3,2,1,0) = 826304$	$N_3(6,6,1,0,0) = 768768$
$N_3(6,2,0,1,1) = 45488$	$N_3(6,4,0,0,1) = 51200$	$N_3(6,7,0,0,0) = 281088$
$N_3(6,2,1,2,0) = 299744$	$N_3(6,4,1,1,0) = 549440$	

7 tangency conditions (23 such numbers):

$N_3(7,0,0,0,2) = 24256$	$N_3(7,2,1,0,1) = 207168$	$N_3(7,4,0,1,0) = 585152$
$N_3(7,0,0,3,0) = 212128$	$N_3(7,2,2,1,0) = 1398240$	$N_3(7,4,2,0,0) = 3272384$
$N_3(7,1,0,1,1) = 92608$	$N_3(7,3,0,0,1) = 117696$	$N_3(7,5,1,0,0) = 1751744$
$N_3(7,1,1,2,0) = 522464$	$N_3(7,3,1,1,0) = 1117696$	$N_3(7,6,0,0,0) = 699904$
$N_3(7,2,0,2,0) = 418752$	$N_3(7,3,3,0,0) = 4085536$	

8 tangency conditions (16 such numbers):

$N_3(8,0,0,1,1) = 161840$	$N_3(8,2,0,0,1) = 239616$	$N_3(8,3,2,0,0) = 5683568$
$N_3(8,1,0,2,0) = 771504$	$N_3(8,2,1,1,0) = 1974544$	$N_3(8,4,1,0,0) = 3622144$
$N_3(8,1,1,0,1) = 329872$	$N_3(8,3,0,1,0) = 1235520$	$N_3(8,5,0,0,0) = 1634432$

9, 10, 11, 12 and 13 tangency conditions (11+7+4+2+1 such numbers):

$N_3(9,0,0,2,0) = 1237472$	$N_3(9,4,0,0,0) = 3504384$	$N_3(11,0,0,1,0) = 5306112$
$N_3(9,1,0,0,1) = 416720$	$N_3(10,0,0,0,1) = 601760$	$N_3(11,1,1,0,0) = 14180096$
$N_3(9,1,1,1,0) = 2967968$	$N_3(10,1,0,1,0) = 3762304$	$N_3(11,2,0,0,0) = 11438656$
$N_3(9,2,0,1,0) = 2310128$	$N_3(10,2,1,0,0) = 10417792$	$N_3(12,1,0,0,0) = 17079168$
$N_3(9,2,2,0,0) = 8359264$	$N_3(10,3,0,0,0) = 6740192$	$N_3(13,0,0,0,0) = 22772224$
$N_3(9,3,1,0,0) = 6605904$		

Chapter 4

Curves with cusp or prescribed triple contact

This chapter introduces a slight generalisation of the degree-1 enumerative descendants which allows for treating problems concerning cuspidal curves, as well as curves with a single specified triple contact. The slightly enriched degree-1 enumerative descendants are those where in addition to the classes $\overline{\tau}_0$ and $\overline{\tau}_1$ we allow a single class which can have a square of a modified psi class or it can have a factor which is a certain codimension-2 boundary class that arises naturally in many applications.

The first two sections treat these new classes and establish differential equations for their potentials, and as such these two sections are natural prolongations of Chapter 1 and 2. The next two sections apply these concepts to the enumerative geometry of cuspidal curves and curves with prescribed triple contact, and these two sections are continuation of the matter of Chapter 3.

4.1 Codimension-2 boundary classes

4.1.1 Pi classes. Let Π_i denote the sum of all codimension-2 boundary cycles whose middle twig has degree 0 and carries the mark p_i, while the two other twigs have positive degree. Since the middle twig has got two other special points, in addition to the mark, this configuration is stable without need of further marks. Therefore, in contrast to the similarly defined divisors $\boldsymbol{\xi}_i$, the class Π_i is an E-class.

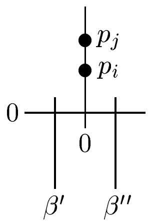
Let Π_{ij} denote the sum of all codimension-2 boundary cycles whose middle twig has degree 0 and carries the marks p_i and p_j, while the two other twig have positive degree. It is clearly a (2-pointed) E-class.

4.1.2 Remark. Since each irreducible component of Π_{ij} has got at least four special points on the middle twig, it pushes down to zero when forgetting p_i or p_j. Therefore, any top product where Π_{ij} is up against E-classes belonging to other marks is zero. Note however the following lemma. For the same reason, in a top product where Π_i is up against E-classes belonging to other marks, only the components where p_i is alone on the middle twig give any contribution.

4.1.3 Lemma.

$$\boldsymbol{\delta}_{ij}\Pi_{ij} = \boldsymbol{\delta}_{ij}\Pi_i = \boldsymbol{\delta}_{ij}\Pi_j$$

Indeed, the only components of Π_i that intersect $\boldsymbol{\delta}_{ij}$ are those where also p_j is on the middle twig, and these components are exactly the components of Π_{ij}. Similarly for Π_j. In conclusion, each of the three cycles is the locus of maps with source of type



4.1.4 Lemma. *Let α_i be a 1-pointed E-class. Then*

$$\Pi_{ij}\alpha_i = \Pi_{ij}\alpha_j.$$

Proof. This is just an adaptation for the proof of the corresponding property 1.5.3 for diagonal classes. Note however that the diagonal classes enjoy the property up against 2-pointed as well as 1-pointed E-classes, (including the case $\alpha_{ij} = \Pi_{ik}$), whereas the Π_{ij} class in question (as much as the proof allows) only enjoys the property up against 1-pointed E-classes.

Considering Π_{ij} as a subscheme of $\overline{M}_{0,n}(X,\beta)$ (with inclusion ι) we have a commutative diagram

$$\begin{tikzcd} & \Pi_{ij} \arrow[dl, "\hat{\tau}_i"'] \arrow[dr, "\hat{\tau}_j"] & \\ \overline{M}_{\{p_i\}} & \simeq & \overline{M}_{\{p_j\}} \end{tikzcd}$$

where $\overline{M}_{\{p_k\}}$ is short for $\overline{M}_{0,\{p_k\}}(X,\beta)$, the horizontal isomorphism is change of label $p_i \leftrightarrow p_j$, and $\hat{\tau}_i$ is the restriction of $\hat{\pi}_i$ to Π_{ij}. Indeed, when all marks except p_i are forgotten, an element in Π_{ij} retains exactly three special points on the contracting twig, and the position of three marks on a contracting twig is immaterial. So whether we keep p_i or p_j we end up with the same result.

Now

$$\begin{array}{c} \Pi_{ij}\alpha_i = \iota^*\alpha_i = \iota^*\hat{\pi}_j^*\alpha_i = \hat{\tau}_j^*\alpha_i \\ \qquad\qquad\qquad\qquad\qquad\quad \shortparallel \\ \Pi_{ij}\alpha_j = \iota^*\alpha_j = \iota^*\hat{\pi}_i^*\alpha_j = \hat{\tau}_i^*\alpha_j \end{array}$$

□

4.1.5 Lemma. *Let Δ be an irreducible component of Π_i, then*

$$\overline{\boldsymbol{\psi}}_i \cdot \Delta = 0.$$

Proof. This follows from an argument similar to that of 1.3.3 and 1.4.4: Let x' and x'' denote the two attachment points on the middle twig of Δ. Now restrict $\overline{\boldsymbol{\psi}}_i = \boldsymbol{\psi}_i - \boldsymbol{\xi}_i$ to the middle twig: the tautological psi class restricts to give the corresponding psi class of the mark of the middle twig, while restricting $\boldsymbol{\xi}_i$ corresponds to breaking off a twig of degree zero containing p_i but not x' nor x''. In other words, the restriction of $\boldsymbol{\xi}_i$ to the middle twig is $(p_i \mid x', x'') = \boldsymbol{\psi}_i$, so altogether the restriction of $\overline{\boldsymbol{\psi}}_i$ is zero. □

4.2 Slightly enriched degree-1 enumerative descendants

Let P^k denote the generating function for top products of classes of type $\overline{\tau}_0(T_i)$ and $\overline{\tau}_1(T_j)$ and a single factor (say at the first mark) of type $\Pi_1 \cup \nu_1^*(T_k)$. Precisely

$$P^k(\mathbf{x},\mathbf{y}) := \sum_{\beta>0}\sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}\frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!} \langle\, \Pi_1 \cup \nu_1^*(T_k)\ \overline{\boldsymbol{\tau}}_0(\mathbf{a})\ \overline{\boldsymbol{\tau}}_1(\mathbf{b})\,\rangle_\beta.$$

(In view of Lemma 4.1.5, there is no reason for allowing also marks combining Π with $\overline{\boldsymbol{\psi}}$.)

For each fixed k consider the matrix $\gamma_k^{ij} = \phi_k^{ij}(2\mathbf{y})$. (Note that this is *not* the k'th derivative of the matrix γ^{ij}.)

4.2.1 Proposition. *The following differential equation relates P^k to Γ:*

$$P^k = \tfrac{1}{2} \sum_{e,f} \Gamma_{x_e}\, \gamma_k^{ef}\, \Gamma_{x_f}.$$

Proof. By Remark 4.1.2, we need only consider components of Π_1 where p_1 is alone on the middle twig. Each such component Δ is the image of a birational morphism from a triple fibred product $\overline{M}' \times_X \overline{M}_{0,3}(X,0) \times_X \overline{M}''$ (with obvious notation similar to that of 1.1.5). Therefore there is the following sort of splitting lemma:

$$\langle\, \Delta \cdot \nu_1^*(T_k) \cdot \boldsymbol{\overline{\tau}}_0(\mathbf{a})\boldsymbol{\overline{\tau}}_1(\mathbf{b})\,\rangle_\beta = \sum_{\substack{p',q'\\p'',q''}} \sum_{\mathbf{s}',\mathbf{s}''} \binom{\mathbf{b}'}{\mathbf{s}'}\binom{\mathbf{b}''}{\mathbf{s}''} \langle\, \boldsymbol{\overline{\tau}}_0(\mathbf{a}')\boldsymbol{\overline{\tau}}_1(\mathbf{b}'-\mathbf{s}')\overline{\tau}_0(\mathbf{T}^{\mathbf{s}'} \cup T_{p'})\,\rangle_{\beta'}$$
$$g^{p'q'}\; g_{q'kp''}\; g^{p''q''}\;\; \langle\, \overline{\tau}_0(\mathbf{T}^{\mathbf{s}''} \cup T_{q''})\boldsymbol{\overline{\tau}}_0(\mathbf{a}'')\boldsymbol{\overline{\tau}}_1(\mathbf{b}''-\mathbf{s}'')\,\rangle_{\beta''}.$$

Using this, the result follows easily, noting that $\sum \phi^e_{p'} \left(g^{p'q'}\; g_{q'kp''}\; g^{p''q''}\right) \phi^f_{q''} = \sum \phi^e_{p'}\, g_k^{p'q''}\; \phi^f_{q''} = \gamma_k^{ef}$.

Note the presence of the factor $\frac{1}{2}$, due to the fact that all the components of Π appear twice in the sum, depending on which of the two outer twigs we consider to be the one-primed and which is two-primed. In the very special case where p_1 is the only mark in play, there is no repetition in the sum since nothing distinguishes the two twigs, but this very symmetry means that the morphism from the fibred product is actually two-to-one, so in this case we divide by two for this reason. □

4.2.2 Example. For $X = \mathbb{P}^2$ (continuing Example 2.2.5), we have

$$(\gamma_1^{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2y_1 \end{pmatrix}$$

and thus, $P^1 = \Gamma_1\Gamma_2 + y_1\Gamma_2\Gamma_2$. The second derivative will be needed in the applications, so let us write it down:

$$P^1_{x_1x_1} = \Gamma_{x_1x_1x_1}\Gamma_{x_2} + 2\Gamma_{x_1x_1}\Gamma_{x_2x_1} + \Gamma_{x_1}\Gamma_{x_1x_1x_2} + 2y_1\Gamma_{x_1x_1x_2}\Gamma_{x_2} + 2y_1\Gamma_{x_1x_2}\Gamma_{x_1x_2}.$$

4.2.3 Example. Continuing Example 3.3.10, consider $X = \mathbb{P}^1 \times \mathbb{P}^1$. As in that example, it is advantageous to consider derivative with respect to $x_{12} = x_1 + x_2$, and $y_{12} = y_1 y_2$. Now the matrix $\gamma^{ef}_{(12)} = \gamma^{ef}_1 + \gamma^{ef}_2$ is

$$(\gamma^{ef}_{(12)}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2y_1 + 2y_2 \end{pmatrix},$$

so the potential is

$$\begin{aligned} P^{(12)} &= \tfrac{1}{2}\big(2\Gamma_{x_3}\Gamma_{x_1} + 2\Gamma_{x_3}\Gamma_{x_2} + (2y_1 + 2y_2)\Gamma_{x_3}\Gamma_{x_3}\big) \\ &= \Gamma_{x_3}\Gamma_{x_{12}} + (y_1 + y_2)\Gamma_{x_3}\Gamma_{x_3}. \end{aligned}$$

The double derivative with respect to x_{12} is the one needed in the applications, so let us write it down:

$$\begin{aligned} P^{(12)}_{x_{12}x_{12}} &= \Gamma_{x_{12}}\Gamma_{x_3x_{12}x_{12}} + \Gamma_{x_3}\Gamma_{x_{12}x_{12}x_{12}} + (2y_1 + 2y_2)\Gamma_{x_3}\Gamma_{x_3x_{12}x_{12}} \\ &\quad +2\Gamma_{x_3x_{12}}\Gamma_{x_{12}x_{12}} + (2y_1 + 2y_2)\Gamma_{x_{12}x_3}\Gamma_{x_3x_{12}} \\ &= G_{u_{12}}G_{u_3u_{12}u_{12}} + G_{u_3}G_{u_{12}u_{12}u_{12}} + 4v_{12}G_{u_3}G_{u_3u_{12}u_{12}} \\ &\quad +2G_{u_3u_{12}}G_{u_{12}u_{12}} + 4v_{12}G_{u_{12}u_3}G_{u_3u_{12}} \end{aligned}$$

Here the potential G and the u and v variables are those defined in 3.3.10.

A single modified psi class square

As noted, there should exist a differential equation for more general enumerative descendants. One generalisation is possible without leaving the framework of the deformed metric. It is allowing a single quadratic modified psi class: Let $Q^k(\mathbf{x}, \mathbf{y})$ denote the corresponding generating function

$$Q^k(\mathbf{x}, \mathbf{y}) := \sum_{\beta > 0} \sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!} \langle \overline{\boldsymbol{\tau}}_0(\mathbf{a}) \; \overline{\boldsymbol{\tau}}_1(\mathbf{b}) \; \overline{\tau}_2(T_k) \rangle_\beta.$$

To relate this potential to the tangency quantum potential, one more notational convention is in place, extending the notation introduced in 2.1.2.1: Coupling a y-variable with an x-variable in a parenthesis is given the following meaning:

$$\Gamma_{(y_i x_j)} := \sum_{k=0}^{r} g^k_{ij} \; \Gamma_{y_k}.$$

4.2.4 Proposition. *There is the following differential equation relating the potential Q^k with the tangency quantum potential:*

$$Q^k_{x_i x_j} = \Gamma_{(x_i x_j) y_k} - \Gamma_{(y_k x_i) x_j} - \Gamma_{(y_k x_j) x_i} + \sum_{e,f} \left(\Gamma_{y_k x_e} + \Gamma_{(x_k x_e)} \right) \gamma^{ef} \, \Gamma_{x_f x_i x_j}.$$

Proof. The proof follows exactly the same lines as the proof of Theorem 2.3.1. Let the mark p_1 correspond to the class $\overline{\tau}_2(T_k)$, and let p_2 and p_3 correspond to the extra classes $\overline{\tau}_0(T_i)$ and $\overline{\tau}_0(T_j)$ corresponding to the derivative. Take out one of the two modified psi classes at p_1 and write it as sum of boundary divisors. This leaves a class $\overline{\tau}_1(T_k)$ on the first mark. The three linear terms now arise exactly as in 2.3.1, with a derivative with respect to y_k instead of x_k. As to the quadratic term, it splits up in two, because the factor $\overline{\boldsymbol{\psi}}_1 \cdot \nu_1^*(T_k)$ restricts to give $\overline{\boldsymbol{\psi}}_1 \cdot \nu_1^*(T_k) + \boldsymbol{\delta}_{1x'} \cdot \nu_{x'}^*(T_k)$, sending the evaluation class $\nu_1^*(T_k)$ over to the gluing mark x'. This explains the factor $\left(\Gamma_{y_k x_e} + \Gamma_{(x_k x_e)}\right)$ in the quadratic term. $\square$

Observe that $\sum \Gamma_{x_e} \gamma_k^{ef} = \sum \Gamma_{(x_k x_m)} \gamma^{mf}$, so the last quadratic term is very similar to the terms of $P^k_{x_i x_j}$.

4.2.5 Corollary. *For $k = 0$, there is a much simpler equation:*

$$Q^0 = -\tfrac{1}{2} \sum_{e,f} \Gamma_{x_e} \, \gamma^{ef} \, \Gamma_{x_f}$$

Proof. After using the dilaton equation $\Gamma_{y_0} = -2\Gamma$ twice, the equation of the proposition reads

$$Q^0_{x_i x_i} = -2\Gamma_{(x_i x_i)} - 2\Gamma_{y_i x_i} - \sum_{e,f} \Gamma_{x_e} \, \gamma^{ef} \, \Gamma_{x_f x_i x_i}.$$

Now apply the differential equation 2.3.3 to the second term and simplify, ending up with

$$Q^0_{x_i x_i} = -\sum_{e,f} \Gamma_{x_i x_e} \, \gamma^{ef} \, \Gamma_{x_f x_i} - \sum_{e,f} \Gamma_{x_e} \, \gamma^{ef} \, \Gamma_{x_f x_i x_i}.$$

This identity can be integrated twice with respect to x_i, as in 2.3.3, getting the result. $\square$

4.2.6 Remark. We have $P^0 + Q^0 = 0$.

In fact, more generally, $-\Pi_1$ and $\overline{\boldsymbol{\psi}}_1^2$ on one-pointed space $\overline{M}_{0,1}(X,\beta)$ have the same push-down in $\overline{M}_{0,0}(X,\beta)$. Indeed, generally Π_1 pushes down to give the hard boundary, but in $\overline{M}_{0,0}(X,\beta)$ the hard boundary is the whole boundary. On the other hand, the push-down of $\overline{\boldsymbol{\psi}}_1^2 = \boldsymbol{\psi}_1^2$ is the kappa class by definition (see Arbarello-Cornalba [2]), and according to Pandharipande [42], Lemma 2.1.2, the kappa class is minus the boundary. (That proof treats $\mathbb{P}^r$ but it carries over directly to the present case.)

4.2.7 Example. For $X = \mathbb{P}^2$, there are three such equations:

$$Q^0 = -\tfrac{1}{2}\Gamma_{x_1}^2 - 2y_1\Gamma_{x_1}\Gamma_{x_2} - (y_2 + y_1^2)\Gamma_{x_2}^2$$

$$\begin{aligned} Q^1_{x_1x_1} = {}& \Gamma_{y_1x_2} - 2\Gamma_{y_2x_1} + \Gamma_{y_1x_1}\Gamma_{x_1x_1x_1} + 2y_1\Gamma_{y_1x_1}\Gamma_{x_2x_1x_1} + 2y_1\Gamma_{y_1x_2}\Gamma_{x_1x_1x_1} \\ & + (2y_1^2 + 2y_2)\Gamma_{y_1x_2}\Gamma_{x_2x_1x_1} + \Gamma_{x_2}\Gamma_{x_1x_1x_1} + \Gamma_{x_1}\Gamma_{x_2x_1x_1} + 2y_1\Gamma_{x_2}\Gamma_{x_2x_1x_1} \end{aligned}$$

$$\begin{aligned} Q^2_{x_1x_1} = {}& \Gamma_{y_2x_2} + \Gamma_{y_2x_1}\Gamma_{x_1x_1x_1} + 2y_1\Gamma_{y_2x_1}\Gamma_{x_2x_1x_1} + 2y_1\Gamma_{y_2x_2}\Gamma_{x_1x_1x_1} \\ & + (2y_2 + 2y_1^2)\Gamma_{y_2x_2}\Gamma_{x_2x_1x_1} + \Gamma_{x_2}\Gamma_{x_2x_1x_1} \end{aligned}$$

which will be of importance in the treatment of cusps in 4.3.2.

4.2.8 Example. Continuing Example 3.3.10 and 4.2.3, consider $X = \mathbb{P}^1 \times \mathbb{P}^1$. Again we could write down the four potentials Q^0 Q^1 Q^2 and Q^3 and their various derivatives. Instead let us just consider the directional derivative with respect to $x_{12} := x_1 + x_2$, the formal variable corresponding to $T_1 + T_2$. In other words, set $\Gamma_{x_{12}} = \Gamma_{x_1} + \Gamma_{x_2}$ and $Q^{(12)}_{x_{12}x_{12}} = Q^1_{x_1x_1} + 2Q^1_{x_1x_2} + Q^1_{x_2x_2} + Q^2_{x_1x_1} + 2Q^2_{x_1x_2} + Q^2_{x_2x_2}$. While of course it is easy to establish differential equations for this potential by linearity, it is more rewarding getting the differential equation directly, by arguments analogous to those of the proof of 2.3.1,

$$\begin{aligned}
Q^{(12)}_{x_{12}x_{12}} &= \Gamma_{(x_{12}x_{12})y_{12}} - 2\Gamma_{(y_{12}x_{12})x_{12}} + \sum_{e,f} \left(\Gamma_{y_{12}x_e} + \Gamma_{(x_{12}x_e)}\right) \gamma^{ef}\, \Gamma_{x_f x_{12}x_{12}} \\
&= 2\Gamma_{y_{12}x_3} - 4\Gamma_{y_3x_{12}} + \Gamma_{y_{12}x_1}\Gamma_{x_2x_{12}x_{12}} + \Gamma_{y_{12}x_2}\Gamma_{x_1x_{12}x_{12}} \\
&\quad +2y_1\Gamma_{y_{12}x_1}\Gamma_{x_3x_{12}x_{12}} + 2y_2\Gamma_{y_{12}x_2}\Gamma_{x_3x_{12}x_{12}} + 2y_1\Gamma_{y_{12}x_3}\Gamma_{x_1x_{12}x_{12}} \\
&\quad +2y_2\Gamma_{y_{12}x_3}\Gamma_{x_2x_{12}x_{12}} + (4y_1y_2 + 2y_3)\Gamma_{y_{12}x_3}\Gamma_{x_3x_{12}x_{12}} \\
&\quad +\Gamma_{x_{12}}\Gamma_{x_3x_{12}x_{12}} + \Gamma_{x_3}\Gamma_{x_{12}x_{12}x_{12}} + (2y_1 + 2y_2)\Gamma_{x_3}\Gamma_{x_3x_{12}x_{12}} \\
&= 2G_{v_{12}u_3} - 4G_{v_3u_{12}} + G_{v_{12}u_1}G_{u_2u_{12}u_{12}} + G_{v_{12}u_2}G_{u_1u_{12}u_{12}} \\
&\quad +2v_{12}G_{v_{12}u_{12}}G_{u_3u_{12}u_{12}} + 2v_{12}G_{v_{12}u_3}G_{u_{12}u_{12}u_{12}} \\
&\quad +(4v_{12}^2 + 2v_3)G_{v_{12}u_3}G_{u_3u_{12}u_{12}} - 2G_{v_3u_{12}u_{12}} \\
&\quad +G_{u_{12}}G_{u_3u_{12}u_{12}} + G_{u_3}G_{u_{12}u_{12}u_{12}} + 4v_{12}G_{u_3}G_{u_3u_{12}u_{12}}.
\end{aligned}$$

Again the meaning of G and the variables $u_1, u_2, u_{12}, u_3, v_{12}, v_3$ is that introduced in 3.3.10.

4.3 Characteristic numbers of cuspidal curves

In this section we are concerned with plane curves, in which case the question of cuspidal curves naturally imposes itself. In 4.3.11 we briefly discuss to what extent the techniques apply also to $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\boldsymbol{\eta}_i := c_1(\nu_i^*(h))$. The symbols Ω, Θ and Ξ are used for the conditions of passing through a point, being tangent to a given line, and being tangent to a given line at a given point, respectively, and we also use the symbols to denote the class of the corresponding cycles,

$$\begin{aligned}
\Omega &= \boldsymbol{\eta}_1^2 \\
\Theta &= \boldsymbol{\eta}_1^2 + \boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 \\
\Xi &= \boldsymbol{\eta}_1^2\overline{\boldsymbol{\psi}}_1.
\end{aligned}$$

4.3.1 The locus of marked cusp. We place ourselves in the space $\overline{M}_{0,1}(\mathbb{P}^2, d)$ of maps with one mark p_1: Off the boundary, consider the locus of maps such that p_1 is a critical point, i.e. the differential vanishes at p_1. Since most maps are immersions, the locus of non-immersions is of codimension 1. Requiring further that the mark is critical gives codimension 2. We are interested in the closure of this locus, denoted $\mathrm{K} = \mathrm{K}_1$, which is the locus of maps having a cusp at p_1. In spaces with more marks, the locus of marked cusp is the pull-back of the one in $\overline{M}_{0,1}(\mathbb{P}^2, d)$.

4.3.2 Proposition. *The class of this marked cusp locus is:*

$$\mathrm{K}_1 = 3\boldsymbol{\eta}_1^2 + 3\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 - \Pi_1$$

Proof. Consider the tautological family $\overline{M}_{0,2}(\mathbb{P}^2, d)$ with tautological map ν_2. We consider first the vanishing of the section

$$\sigma_1^* T_\pi \longrightarrow \sigma_1^* \nu_2^* T_{\mathbb{P}^2}.$$

Note that T_π is not locally free, since the morphism π is not smooth, but when we restrict to the image of σ_1 it *is* smooth, since the section is not allowed to pass through any singular point of the fibres.

The class is the second Chern class of $\sigma_1^*\nu_2^*T\mathbb{P}^2 \otimes (\sigma_1^*T_\pi)^{-1}$. Now $c\big(\mu^*(T_{\mathbb{P}^2})\big) = 1 + 3\boldsymbol{\eta}_1 + 3\boldsymbol{\eta}_1^2$ and $c((\sigma_1^*T_\pi)^{-1}) = 1 + \overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2$, so the class is

$$3\boldsymbol{\eta}_1^2 + 3\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2.$$

(Since we are on a space with just one mark, we have $\overline{\boldsymbol{\psi}}_1 = \boldsymbol{\psi}_1$. In a situation with more marks, the right locus is the pull-back of this one, so it's better to use the modified psi classes.) The vanishing of this section is the locus Σ_1 of critical mark.

Now the section also vanishes along Π_1 (indeed the differential vanishes identically along the middle twig of a curve in Π_1) and we do not want to consider that a cusp since it is not in general the limit of a honest cusp. Now we claim that the section vanishes along Π_1 to multiplicity 1. To establish this multiplicity we resort to an argument involving an unstable family: take a family of stable maps $\mathfrak{X} \to B \times \mathbb{P}^2$ without marks, and make sure it intersects the boundary. Put a section on the family that transversally intersects the locus of nodes of the fibres. Now perform the differential vanishing argument in this situation. Since the section passes through the singular locus, it is better not to use the relative tangent bundle and instead look at the vector bundle homomorphism

$$\sigma_1^* T_{\mathfrak{X}} \to \sigma_1^* \overline{\mu}^* T_{\mathbb{P}^2 \times B}.$$

Where this homomorphism drops rank from 2 to 1 we've got a cusp. Pulling it back along the section σ_1 we get the locus of maps where it happens at the mark. Now let $I \subset \mathcal{O}_{\mathfrak{X}}$ denote the ideal sheaf of the locus $N \subset \mathfrak{X}$ of singular points of the fibres. Then the exact sequence

$$0 \to \pi^*\Omega_B \to \Omega_{\mathfrak{X}} \to I \otimes \omega_\pi \to 0$$

yields the relation of total Chern classes

$$c(T_{\mathfrak{X}}) = \pi^* c(T_B)\big(1 - K_\pi + [N]\big),$$

and thus $\sigma_1^* c(T_{\mathfrak{X}}) = c(T_B)\big(1 - \overline{\boldsymbol{\psi}}_1 + \sigma_1^*[N]\big)$ and finally $c(T_B)/\sigma_1^* c(T_{\mathfrak{X}}) = 1 + \overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 - \sigma_1^*[N]$.

Now according to Porteous' formula, the rank dropping locus has class equal to the degree-2 part of the total Chern class

$$\begin{aligned} \sigma_1^* \overline{\mu}^* c(T_{\mathbb{P}^2 \times B})/\sigma_1^* c(T_{\mathfrak{X}}) &= \sigma_1^* \mu^* c(T_{\mathbb{P}^2}) \cdot c(T_B)/\sigma_1^* c(T_{\mathfrak{X}}) \\ &= (1 + 3\boldsymbol{\eta}_1 + 3\boldsymbol{\eta}_1^2)(1 + \overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 - \sigma_1^*[N]). \end{aligned}$$

The degree 2 part of this is just $3\boldsymbol{\eta}_1^2 + 3\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 - \sigma_1^*[N]$. Now it only remains to notice that the locus $\sigma_1^*[N] \subset B$ of the unstable family is the same as Π_1 of the stabilised family. □

See 5.1.4 for another construction.

4.3.3 Remark. For $d = 1$, the locus is empty, so in this case Porteous' formula yields the relation $3\boldsymbol{\eta}^2 + 3\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 = 0$. Under the natural identification of $\overline{M}_{0,1}(\mathbb{P}^2, 1)$ with the incidence variety $I \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2$ of points and lines in $\mathbb{P}^2$, this relation is equivalent to the well-known relation $h^2 + \check{h}^2 = h\check{h}$.

For $d = 2$, the multiple-covers occur already in codimension 1. On the other hand, there are no birational maps in degree 2 with a critical point, so for $d = 2$ the locus K consists of all the double covers such that the mark is one of the ramification points.

For $d \geq 3$, the locus of multiple-covers is of codimension at least 2, so the extra condition of having the mark as one of the ramification points prevents these curves from contributing. So in this case the locus K consists generically of birational maps.

4.3.4 Example. Consider the codimension-3 condition of the marked cusp mapping to a given line. The class Kl_1 of this condition is obtained simply by cutting with $\boldsymbol{\eta}_1$:

$$\mathrm{Kl}_1 = 3\boldsymbol{\eta}_1^2\overline{\boldsymbol{\psi}}_1 + \boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1^2 - \boldsymbol{\eta}_1\Pi_1.$$

Similarly, the locus of maps with marked cusp mapping to a specified point is

$$\mathrm{Kp}_1 = \boldsymbol{\eta}_1^2 \overline{\boldsymbol{\psi}}_1^2 - \boldsymbol{\eta}_1^2 \Pi_1.$$

See also 5.1.3 for another construction.

4.3.5 Tangency conditions in cuspidal environment. Suppose we are inside the locus K_1 and want to impose the condition of being tangent to a given line at mark p_2. In families where the general member is cuspidal, the usual description of the tangency class fails: the vanishing of the jet section includes the locus of maps such that the p_1-cusp maps to the given line, and the two marks have come together. So we are talking about the cycle $\boldsymbol{\delta}_{12} \cdot \boldsymbol{\eta}_1 \cdot \mathrm{K}_1$. Now an easy 1-parameter family argument (given in the Appendix to Chapter 5) shows this bad component appears with multiplicity 1, so in cuspidal environment (cusp at p_1) the class of tangency at p_2 is

$$\Theta_2' = \boldsymbol{\eta}_2(\boldsymbol{\eta}_2 + \overline{\boldsymbol{\psi}}_2 - \boldsymbol{\delta}_{12}). \tag{4.3.5.1}$$

Note that if we are even inside the locus Kp_1 of p_1-cusp mapping to a given point Q, then the correction term is zero: geometrically because it is not possible for p_1 and p_2 to come together since p_1 maps to Q which by generality assumption is off the given line L. In terms of the properties of the diagonal class: its effect is to merge the factor $\boldsymbol{\eta}_2$ over to the first mark where there is already a factor $\boldsymbol{\eta}_1^2$ present — the result is zero for dimension reasons.

Similarly in cuspidal environment, the class of p_2-tangency to a given line at a specified point is

$$\Xi_2' = \boldsymbol{\eta}_2^2(\overline{\boldsymbol{\psi}}_2 - \boldsymbol{\delta}_{12}).$$

Since the family K_1 satisfies the three conditions 3.1.3 except for its cusp, these cuspidal tangency conditions can now be applied iteratively as in 3.2.1, and their top products are the characteristic numbers for cuspidal plane curves. These numbers were first computed by Lars Ernström and Gary Kennedy [12], using the space of stable lifts.

Using the generating functions for the slightly enriched degree-1 enumerative descendants, and their relation to the tangency potential, it is straight forward to derive differential equations determining these characteristic numbers:

4.3.6 Lemma. *Let $CP_d(a, b, c)$ denote the characteristic numbers of cuspidal curves with fixed cusp, subject to a incidence conditions, b tangent conditions,* *and c flag conditions, as usual. Let $KP(u_1, u_2, v_1, v_2)$ denote the corresponding generating function, with u_1 being the formal variable corresponding to the degree d and the three other variables corresponding to the three types of conditions, as in 3.3.8. Then*

$$KP_{u_1u_1} = G_{v_2u_2} + G_{v_2u_1}G_{u_1u_1u_1} + 2v_1G_{v_2u_1}G_{u_2u_1u_1} + 2v_1G_{v_2u_2}G_{u_1u_1u_1} + (2v_1^2 + 2v_2)G_{v_2u_2}G_{u_2u_1u_1} - G^2_{u_1u_2}.$$

Proof. Since $\mathrm{Kp}_1 = \boldsymbol{\eta}_1^2(\overline{\boldsymbol{\psi}}_1^2 - \Pi_1)$, there is the following relation:

$$KP(u_1, u_2, v_1, v_2) = (Q^2 - P^2)(x_1, x_2, y_1, y_2),$$

where $x_1 = u_1$, $x_2 = u_2 + v_1$, $y_1 = v_1$, and $y_2 = v_2$. Now the result follows from 4.2.7 and 4.2.2 and a trivial application of the chain rule. □

4.3.7 Lemma. *Let $CL_d(a, b, c)$ denote the characteristic numbers of cuspidal curves with the cusp at a specified line, and subject further to a incidence conditions, b tangent conditions, and c flag conditions. Let $KL(u_1, u_2, v_1, v_2)$ denote the corresponding generating function, then*

$$KL_{u_1u_1} = -v_1KP_{u_1u_1} + 2G_{u_1u_1v_2} + G_{v_1u_2} - 2G_{v_2u_1} + G_{v_1u_1}G_{u_1u_1u_1} + 2v_1G_{v_1u_1}G_{u_2u_1u_1} + 2v_1G_{v_1u_2}G_{u_1u_1u_1} + (2v_1^2 + 2v_2)G_{v_1u_2}G_{u_2u_1u_1} - 2G_{u_1u_1}G_{u_2u_1} - 2v_1G^2_{u_1u_2}.$$

Proof. The formula for tangency is $\Theta_2' = \boldsymbol{\eta}_2(\boldsymbol{\eta}_2 + \overline{\boldsymbol{\psi}}_2 - \boldsymbol{\delta}_{12})$. There are b such factors, and the first step is to is to expand that product. For dimension reasons there is room for at most one correction factor in each term (since each correction terms switches an eta class over to the first mark, where there is already such an eta factor), so there are these two terms in the expansion:

$$\begin{aligned} CL_d(a, b, c) &= \mathrm{Kl}\ \Omega^a\ \Theta'^b\ \Xi^c \\ &= \mathrm{Kl}\ \Omega^a\ \Theta^b\ \Xi^c - b\mathrm{Kp}\ \Omega^a\ \Theta^{b-1}\ \Xi^c \end{aligned}$$

The last term explains $-v_1KP_{u_1u_1}$ in the formula. The first term is now expanded (and differentiating twice):

$$d^2\mathrm{Kl}\ \Omega^a\ \Theta^b\ \Xi^c = 3d^2\boldsymbol{\eta}_1^2\overline{\boldsymbol{\psi}}_1\ \Omega^a\ \Theta^b\ \Xi^c + d^2\ (\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1^2 - \boldsymbol{\eta}_1\Pi_1)\ \Omega^a\ \Theta^b\ \Xi^c$$

The last term is sent to the formulae 4.2.8 and 4.2.3 which account for the remaining terms in the formula. When writing things in terms of G instead of

Γ, the chain rule enters in a non-trivial way, introducing five new terms; using 3.3.10.1 backwards shows that these five terms are exactly $-G_{u_1u_1v_2}$. On the other hand, the first term is $3d^2\boldsymbol{\eta}_1^2\overline{\boldsymbol{\psi}}_1\ \Omega^a\ \Theta^b\ \Xi^c = 3d^2N_d(a,b,c+1)$, which accounts for another $3G_{u_1u_1v_2}$, thus establishing the formula. □

Finally, the formula for cuspidal curves, without conditions imposed on the cusp:

4.3.8 Lemma. *Let $C_d(a,b,c)$ denote the characteristic numbers of cuspidal curves subject to a incidence conditions, b tangent conditions, and c flag conditions. Let $K(u_1,u_2,v_1,v_2)$ denote the corresponding generating function, then*

$$K = 3G_{v_1} - G_{u_1}^2 - 4v_1G_{u_1}G_{u_2} - (2v_1^2+2v_2)G_{u_2}^2 - v_1KL - (\tfrac{1}{2}v_1^2+v_2)KP.$$

Proof. This time there are expansions to be done for both $\Theta_2' = \boldsymbol{\eta}_2(\boldsymbol{\eta}_2+\overline{\boldsymbol{\psi}}_2-\boldsymbol{\delta}_{12})$ and $\Xi_2' = \boldsymbol{\eta}_2^2(\overline{\boldsymbol{\psi}}_2-\boldsymbol{\delta}_{12})$. We get then

$$\begin{aligned} C_d(a,b,c) &= \mathrm{K}\ \Omega^a\ \Theta'^b\ \Xi'^c \\ &= \mathrm{K}\ \Omega^a\ \Theta^b\ \Xi^c - b\mathrm{Kl}\ \Omega^a\ \Theta^{b-1}\ \Xi^c + \tbinom{b}{2}\mathrm{Kp}\ \Omega^a\ \Theta^{b-2}\ \Xi^c - c\mathrm{Kp}\ \Omega^a\ \Theta^b\ \Xi^{c-1} \\ &= \mathrm{K}\ \Omega^a\ \Theta^b\ \Xi^c - bCL_d(a,b-1,c) - \tbinom{b}{2}CP_d(a,b-2,c) - cCP_d(a,b,c-1). \end{aligned}$$

Here the first term is expanded to

$$\mathrm{K}\ \Omega^a\ \Theta^b\ \Xi^c = 3N_d(a,b+1,c) + (\overline{\boldsymbol{\psi}}_1^2 - \Pi_1)\ \Omega^a\ \Theta^b\ \Xi^c,$$

and this last term is taken care of by 4.2.7 and 4.2.2 (and 4.2.6). □

The differential equations of these three lemmas are very similar to the recursions used in Ernström and Kennedy [12] (found with completely different methods), and I believe they are equivalent (modulo equations 3.3.8.1 and 3.3.8.2), but I have not been able to identify all the terms of their recursion. (Needless to say, I have checked that the numbers agree as far as they are listed in [12]. . .)

4.3.9 Remark. Setting $v_1 = v_2 = 0$ (corresponding to considering only incidence conditions) and then differentiating with respect to u_1 yields

$$\begin{aligned} K_{u_1} &= 3G_{u_1v_1} - \tfrac{\partial}{\partial u_1}G_{u_1}^2 \quad (\mathrm{mod}\ (v_1,v_2)) \\ &= 3\big(G_{u_1u_2} - G_{u_2} + \tfrac{1}{2}G_{u_1u_1}^2\big) - \tfrac{\partial}{\partial u_1}G_{u_1}^2 \quad (\mathrm{mod}\ (v_1,v_2)), \end{aligned}$$

which is algebraically equivalent to the formula of proposition 5 in Pandharipande [42].

Cuspidal curves in $\mathbb{P}^1 \times \mathbb{P}^1$

4.3.10 Differential equations. It is a question of formal manipulations to do the same for $\mathbb{P}^1 \times \mathbb{P}^1$. The tangent bundle is

$$c(T) = 1 + 2(T_1 + T_2) + 4T_3$$

so the locus of cusp at mark p_1 is

$$4\nu_1^*(T_3) + 2(\nu_1^*(T_1) + \nu_1^*(T_2))\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 - \Pi_1.$$

Let KP be the potential corresponding to cusp mapping to a specified point, (and further a conditions of passing through a point, b conditions of being tangent to a $(1,1)$-curve, and c conditions of tangenciating such a curve at a specified point). Then

$$\begin{aligned} KP_{u_1u_1} = {} & G_{v_3u_1}G_{u_2u_1u_1} + G_{v_3u_2}G_{u_1u_1u_1} + 2v_{12}G_{v_3u_{12}}G_{u_3u_1u_1} \\ & + 2v_{12}G_{v_3u_3}G_{u_{12}u_1u_1} + (4v_{12}^2 + 2v_3)G_{v_3u_3}G_{u_3u_1u_1} - G_{u_1u_3}G_{u_1u_3}. \end{aligned}$$

Let KL be the generating function for such characteristic numbers, but with the cusp mapping to a specified $(1,1)$-curve. Then

$$\begin{aligned} KL_{u_{12}u_{12}} &= 4G_{v_3u_{12}u_{12}} - 2v_{12}KP_{u_{12}u_{12}} + Q^{(12)}_{u_{12}u_{12}} - P^{(12)}_{u_{12}u_{12}} \\ &= 2G_{v_3u_{12}u_{12}} - 2v_{12}KP_{u_{12}u_{12}} + 2G_{v_{12}u_3} - 4G_{v_3u_{12}} \\ &\quad +G_{v_{12}u_1}G_{u_2u_{12}u_{12}} + G_{v_{12}u_2}G_{u_1u_{12}u_{12}} \\ &\quad +2v_{12}G_{v_{12}u_{12}}G_{u_3u_{12}u_{12}} + 2v_{12}G_{v_{12}u_3}G_{u_{12}u_{12}u_{12}} \\ &\quad +(4v_{12}^2 + 2v_3)G_{v_{12}u_3}G_{u_3u_{12}u_{12}} \\ &\quad -2G_{u_3u_{12}}G_{u_{12}u_{12}} - 4v_{12}G_{u_{12}u_3}G_{u_3u_{12}} \end{aligned}$$

And finally, let K be the generating function for the characteristic numbers of cuspidal curves in $\mathbb{P}^1 \times \mathbb{P}^1$, with the cusp varying freely. Then

$$\begin{aligned} K &= 2G_{v_{12}} - v_{12}KL - (v_{12}^2 + v_3)KP \\ &\quad -2G_{u_1}G_{u_2} - 4v_{12}G_{u_{12}}G_{u_3} - (4v_{12}^2 + 2v_3)G_{u_3}G_{u_3} \end{aligned}$$

4.3.11 Poor enumerative significance. Unfortunately, these characteristic numbers for $\mathbb{P}^1 \times \mathbb{P}^1$ count also *reducible* curves one of whose twigs is a multiple cover of one of the generators. In fact, already the locus K_1 is not irreducible:

it has a component for each boundary divisor corresponding to degree splitting $(m,n) = (0,n') + (m,n'')$. For each of these divisors, the one-primed twig is always a multiple-cover, and then forcing the mark to a ramification point produces the "cusp" already in codimension 2. The other ramification points can then satisfy tangency conditions, giving contribution in the characteristic number.

Only in the case where all the conditions are incidence conditions can we exclude these contributions. Indeed the multiple-cover twig varies in a family of dimension $2n'-1$, and one condition is given by the requirement that a ramification point is the cusp. Then it is left with $2n'-2 \geq 2$ conditions, which is impossible to fulfil if they are all incidence conditions.

Let us just give the equation giving the pure incidence conditions.

4.3.12 Cuspidal curves in $\mathbb{P}^1 \times \mathbb{P}^1$ subject to incidence conditions. Let $C_{(m,n)}$ denote the number of cuspidal curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree (m,n) that pass through $2m+2n-2$ general points. Let K denote the corresponding potential (obtained setting $v_{12} = v_3 = 0$ in the potential defined in 4.3.10), then

$$K_{u_{12}} = 4G_{u_{12}u_3} - 4G_{u_3} + 2G_{u_{12}u_1}G_{u_2u_{12}} - 2\tfrac{\partial}{\partial u_{12}}(G_{u_1}G_{u_2}) \pmod{(v_{12}, v_3)}.$$

Here $u_{12} = u_1 + u_2$ denotes the formal variable dual to $T_1 + T_2$. The corresponding recursion relation is not too ugly: Set $d = m+n$ and let d' and d'' be defined similarly in the sum:

$$C_{(m,n)} = \frac{4(d-1)}{d} N_{(m,n)} + \sum \binom{2d-2}{2d'-1} \frac{(m'n''+n'm'')(d'd''-d)}{d} N_{(m',n')}N_{(m'',n'')},$$

where the sum is over degree splittings $m' + m'' = m$ and $n' + n'' = n$.

Here are some numbers.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$m=2$	24	288	2304	15360	92160
$m=3$		14880	435168	9628128	179808960
$m=4$			37398528	2178662400	100057743360
$m=5$				295935717504	29449981140480
$m=6$					5876933469987840

Whenever m or n is 0 or 1, we have $C_{(m,n)} = 0$.

4.4 Characteristic numbers of curves with a prescribed triple contact

As in Chapter 3, let $H \subset X$ be a general, very ample hypersurface, given as the zero scheme of a section f of a line bundle L, and set $\boldsymbol{\eta}_i := c_1(\nu_i^* L)$.

4.4.1 Components mapping into H. Denote by $\mathrm{I}_i = \mathrm{I}_i(H)$ (the closure of) the locus of maps such that p_i is on a non-contracting twig that maps entirely into H. Note that we define I_i only on set-theoretic level and not as a cycle. In general, this locus has irreducible components in various codimensions. The locus of *irreducible* such maps $\mathrm{I}_i \subset M_{0,n}(X, \beta)$ is the zero scheme of a regular section of the vector bundle $\pi_* \nu_{n+1}^* L$, so the codimension is $\dim H^0(\mathbb{P}^1, \mu^* L)$ (or it is empty).

However, commonly the component consisting of irreducible such maps is not the one of lowest codimension: For example, if $X = \mathbb{P}^r$ and H is a hypersurface of degree z, then the locus $\mathrm{I}_i \subset M_{0,n}(\mathbb{P}^r, d)$ is of codimension $dz + 1$, while the locus of reducible such maps having p_i on a twig of degree 1 is of codimension $z + 2$. So for $d \geq 2$ or $z \geq 2$, the locus I_i is always of codimension at least 3. The only situation in which we need to worry about codimension 2, is $d = z = 1$: in this case $\operatorname{codim} \mathrm{I}_i = 2$, and of course these maps are irreducible.

4.4.2 Enumerative irrelevance of I_i. Note first of all that I_i is an E-class (or more precisely an E-locus), since we are not considering the case of p_i belonging to a twig contracting to a point in H. In other words, we have excluded $\boldsymbol{\eta}_i \boldsymbol{\xi}_i$.

Now in the locus I_i, the mark p_i has not been fixed: it can be any point of the twig mapping into H, and since this twig is not just a contracting twig, different choices of where to put the mark on the twig are not isomorphic. The consequence of this observation is that I_i has zero push-down under forgetting p_i (whatever the multiplicities attributed to the components of it). This is turn implies (cf. also 4.1.2) that any top product where I_i is up against E-classes belonging to other marks is zero.

With these remarks we are in position to describe the locus of maps having a triple contact with H, using a jet bundle argument similar to that of 3.1.1.

4.4.3 Families of maps. Given a flat family of stable maps $\mathfrak{X} \to X \times S$ satisfying the conditions (A), (B) and (C) as on page 61, and assume furthermore that

(D) The general map in the locus $\Theta_1 \subset S$ is only simply tangent to H.

(E) The general map in the locus $\mathrm{K}_1 \subset S$ in only a simple cusp.

The easiest example of a family violating condition (D) is the one mentioned above: $\overline{M}_{0,1}(\mathbb{P}^r, 1)$ when $H \subset \mathbb{P}^r$ is a hyperplane. In this case any map tangent to H is actually contained in H, cf. also the Remark 3.1.8

4.4.4 Proposition. *In a family as above let* F_i *denote the closure of the locus of irreducible immersions having a triple contact to* H *at the mark* p_i. *Then its class is*

$$\mathrm{F}_i = \boldsymbol{\eta}_i(\boldsymbol{\eta}_i + \overline{\boldsymbol{\psi}}_i)(\boldsymbol{\eta}_i + 2\overline{\boldsymbol{\psi}}_i) - [\mathrm{I}_i]_3 - \boldsymbol{\eta}_i\Pi_i,$$

where $[\mathrm{I}_i]_3$ *denotes the codimension-3 part of* I_i.

Proof. The proof follows the same lines as the proof of 3.1.1 on page 63. We can assume there is only one mark p_1. The zero scheme of the section $\sigma_1^*\partial_\pi^2\mu^*f$ of the bundle of second jets, $\sigma_1^* J_\pi^2\mu^*L$ is of class

$$c_3(\sigma_1^* J_\pi^2\mu^*L) = \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)(\boldsymbol{\eta}_1 + 2\overline{\boldsymbol{\psi}}_1),$$

as it follows from the standard jet short exact sequences. It is the codimension-3 locus of maps such that the pull-back of f vanishes to order 3 at p_1. If it happens isolated at p_1, then we are in F_1. (Assumption (E) excludes the possibility of a deep cusp mapping to H: deep critical mark occurs only in codimension 3, and then it costs one further codimension to put this cusp on H.) If μ^*f vanishes identically on the twig carrying p_1 then *either* this twig is of degree zero, and then we are at $\boldsymbol{\eta}_1\Pi_1$ (or in higher codimension); *or* the twig is of positive degree, and that means it maps entirely inside H and then we are in the locus I_1. (By the condition (D) no component of I_1 can have codimension lower than 3. There may be many components of higher codimension but since a section of a rank-3 vector bundle cannot vanish isolated along such a component, we conclude that these loci are in fact already contained in one of the codimension-3 loci, which must then be F_1. In other words, such maps are limits of honest triple contacts.)

It remains to establish multiplicity 1 at $\boldsymbol{\eta}_1\Pi_1$. This is accomplished by a 1-parameter family argument, given in the Appendix to Chapter 5, on page 148. (Since I_i was defined only as a set, and since we have argued that it plays no rôle in enumerative applications, we are not concerned with questions of multiplicity along it.) □

4.4.5 Remark. Note that we are not excluding the possibility that all maps in F_1 are in fact contained in H. This happens for example in $\overline{M}_{0,1}(\mathbb{P}^3, 1)$ when $H \subset \mathbb{P}^3$ is a quadric surface, or in the very similar situation of $\overline{M}_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))$ when $H \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of bi-degree $(1,1)$.

This locus with its induced family satisfies the three conditions (A), (B), and (C), so that imposing further tangency conditions work without further corrections, so we are in position to compute the characteristic numbers of rational curves having a triple contact with H.

4.4.6 Example. THE PROJECTIVE PLANE. Fix a general curve $H \subset \mathbb{P}^2$ of degree z. Let $NF^z_d(a,b,c)$ denote the number of rational curves of degree $d \geq 2$ that have triple contact with H, are tangent to b lines, tangent to c lines at specified points, and furthermore pass through a specified points, $(a + b + 2c = 2d - 3)$. Then

$$NF^z_d(a,b,c) = \int_{\overline{M}} \mathrm{F}_1\ \Omega^a\ \Theta^b\ \Xi^c.$$

When expanding the class of F_1, we can ignore I_1 since it's push-down is zero, so we find

$$= 3z^2\Xi \cdot\ \Omega^a\ \Theta^b\ \Xi^c + 2z\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1^2\ \Omega^a\ \Theta^b\ \Xi^c - z\boldsymbol{\eta}_1\Pi_1\ \Omega^a\ \Theta^b\ \Xi^c.$$

If we let F^z be the generating function for these numbers we see that

$$F^z(u_1,u_2,v_1,v_2) = 3z^2 G_{v_2}(u_1,u_2,v_1,v_2) + 2zQ^1(\mathbf{x},\mathbf{y}) - zP^1(\mathbf{x},\mathbf{y})$$

subject to the usual change of variables, cf. 3.3.7–3.3.8. Now take double derivative with respect to u_1 and use the expressions for P^1 and Q^1 (given in 4.2.7 and 4.2.2); an easy application of the chain rule yields

$$\begin{aligned} F^z_{u_1u_1} &= z\big(3zG_{u_1u_1v_2} - 2G_{u_1u_1v_2} + 2G_{v_1u_2} - 4G_{v_2u_1} \\ &+ 2G_{v_1u_1}G_{u_1u_1u_1} + 4v_1G_{v_1u_1}G_{u_2u_1u_1} + 4v_1G_{v_1u_2}G_{u_1u_1u_1} + (4v_1^2+4v_2)G_{v_1u_2}G_{u_2u_1u_1} \\ &+ G_{u_2}G_{u_1u_1u_1} + G_{u_1}G_{u_2u_1u_1} + 2v_1G_{u_2}G_{u_2u_1u_1} - 2G_{u_1u_1}G_{u_2u_1} - 2v_1G_{u_1u_2}G_{u_1u_2}\big). \end{aligned}$$

While this formula may look ugly on paper, it is in fact a very effective formula, and it's a question of ten minutes typing to implement it in maple, and another minute or so for computing all the numbers up to degree 6.

Perhaps the most interesting case is when $z = 1$ so we are talking about curves with a specified flex line. In the Appendix to this chapter these numbers are listed up to $d = 6$.

4.4.7 Remark. Exploring the finer geometry of $\mathbb{P}^2$, through the space of stable lifts to its second Semple bundle, Susan J. Colley, Lars Ernström and Gary Kennedy [7] have recently given a much more comprehensive formula. Not only does this formula allow any number of triple contact conditions to given curves, but the given curves are also allowed to have double points and cusps! The only drawback of their formula is that the given curves are not allowed to be lines or to have linear components.

4.4.8 Example. THE QUADRIC SURFACE $X = \mathbb{P}^1 \times \mathbb{P}^1$. Continuing the notation from Example 3.3.10, consider the condition of having triple contact with a general curve of bi-degree $(1,1)$. Let $NF_{(m,n)}(a,b,c)$ be the number of rational curves of bi-degree (m,n) which have triple contact with a given curve of bi-degree $(1,1)$, are tangent to b other curves of bi-degree $(1,1)$, tangent to c such curves at specified points, and furthermore passing through a specified points, $(a+b+2c = 2m+2n-3)$.

Let F be the generating function for these characteristic numbers. Then it follows readily from arguments similar to those of the preceding example that

$$F(u_1,u_2,u_3,v_{12},v_3) = 6G_{v_3}(u_1,u_2,u_3,v_{12},v_3) + 2Q^{(12)}(\mathbf{x},\mathbf{y}) - P^{(12)}(\mathbf{x},\mathbf{y})$$

subject to the usual variable changes $x_1 = u_1$, $x_2 = u_2$, $x_3 = u_3 + 2v_{12}$, $y_1 = v_{12}$, $y_2 = v_{12}$, and $y_3 = v_3$. Now take double derivative with respect to $u_{12} = u_1 + u_2$, and use the formulae 4.2.8 and 4.2.3 for $Q^{(12)}_{u_{12}u_{12}}$ and $P^{(12)}_{u_{12}u_{12}}$, getting

$$\begin{aligned}
F_{u_{12}u_{12}} &= 6G_{v_3u_{12}u_{12}} + 2Q^{(12)}_{u_{12}u_{12}} - P^{(12)}_{u_{12}u_{12}} \\
&= 6G_{v_3u_{12}u_{12}} + 4G_{v_{12}u_3} - 8G_{v_3u_{12}} + 2G_{v_{12}u_1}G_{u_2u_{12}u_{12}} + 2G_{v_{12}u_2}G_{u_1u_{12}u_{12}} \\
&\quad +4v_{12}G_{v_{12}u_{12}}G_{u_3u_{12}u_{12}} + 4v_{12}G_{v_{12}u_3}G_{u_{12}u_{12}u_{12}} \\
&\quad +(8v_{12}^2 + 4v_3)G_{v_{12}u_3}G_{u_3u_{12}u_{12}} - 4G_{v_3u_{12}u_{12}} \\
&\quad +G_{u_{12}}G_{u_3u_{12}u_{12}} + G_{u_3}G_{u_{12}u_{12}u_{12}} + 4v_{12}G_{u_3}G_{u_3u_{12}u_{12}} \\
&\quad -2G_{u_3u_{12}}G_{u_{12}u_{12}} - 4v_{12}G_{u_{12}u_3}G_{u_3u_{12}}.
\end{aligned}$$

This equation effectively determines the characteristic numbers of curves with one triple contact from the usual characteristic numbers. Some numbers in low degree are listed in the Appendix to this chapter, starting on page 109.

4.A Appendix: Numbers

Rational curves in $\mathbb{P}^2$ with a triple contact

As described in Example 4.4.6, let $NF_d(a,b,c)$ be the number of rational curves of degree $d \geq 2$ which have triple contact with a given line, are tangent to b other lines, tangent to c further lines at specified points, and furthermore pass through a specified points, $(a+b+2c = 3d-3)$.

There are no curves of degree 2 having a triple contact, so these numbers are not listed.

Curves in $\mathbb{P}^2$ of degree 3.

	$c=0$	$c=1$	$c=2$	$c=3$
$b=0$	21	15	9	3
$b=1$	54	30	12	
$b=2$	108	42	12	
$b=3$	156	42		
$b=4$	162	30		
$b=5$	126			
$b=6$	84			

$a = 6 - b - 2c$

Curves in $\mathbb{P}^2$ of degree 4:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$
$b=0$	1452	876	492	252	108
$b=1$	4446	2406	1188	504	162
$b=2$	12144	5784	2400	792	
$b=3$	29136	11766	3876	990	
$b=4$	59688	19416	4968		
$b=5$	100542	25566	5082		
$b=6$	136416	26976			
$b=7$	149544	23304			
$b=8$	135696				
$b=9$	105888				

$a = 9 - b - 2c$

Curves in $\mathbb{P}^2$ of degree 5:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$	$c=6$
$b=0$	216180	112824	56148	26496	11700	4680	1620
$b=1$	738432	358632	164856	70812	27648	9360	
$b=2$	2340072	1048368	438708	167580	56016	15480	
$b=3$	6824160	2783520	1039488	343152	95184		
$b=4$	18109512	6607872	2147448	595152	135432		
$b=5$	43120440	13764324	3788688	867636			
$b=6$	90589932	24671412	5645796	1065132			
$b=7$	164926152	37573128	7097448				
$b=8$	256801680	48472320	7570128				
$b=9$	340546752	53277552					
$b=10$	386643600	50575824					
$b=11$	380866032						
$b=12$	331319232						

$a = 12 - b - 2c$

Curves in $\mathbb{P}^2$ of degree 6:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$	$c=6$	$c=7$
$b=0$	64150200	29436696	13017240	5533848	2251512	869400	313560	103320
$b=1$	235726344	102342024	42657552	16990272	6412536	2259000	725760	206640
$b=2$	818189568	334431504	130487040	48219408	16647552	5250960	1468800	
$b=3$	2669508288	1020706920	369517680	125130456	38858112	10751400	2565360	
$b=4$	8139313008	2887600464	959185296	293204592	80389296	19155600		
$b=5$	23029216200	7503950376	2256285960	612127368	145633032	29595240		
$b=6$	59965935264	17726144544	4750404192	1125074592	229690656			
$b=7$	142332658848	37631176704	8848316160	1807210368	315333792			
$b=8$	304730764128	71002435680	14462794080	2531545056				
$b=9$	582387730848	118066356864	20674728576	3097966752				
$b=10$	985242672000	172310907456	25876529280					
$b=11$	1468648149984	220837990464	28501139232					
$b=12$	1929083358528	249774611904						
$b=13$	2242895662368	251452729440						
$b=14$	2327182063488							
$b=15$	2177100475200							

$a = 15 - b - 2c$

Rational curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with a triple contact

As described in Example 4.4.8, let $NF_{(m,n)}(a,b,c)$ be the number of rational curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree (m,n) which have triple contact with a given curve Z of bi-degree $(1,1)$, are tangent to b other curves of bi-degree $(1,1)$, tangent to c such curves at specified points, and furthermore pass through a specified points, $(a+b+2c = 2m+2n-3)$.

Since the two generators are interchangeable, we have $NF_{(m,n)}(a,b,c) = NF_{(n,m)}(a,b,c)$, so only numbers with $m \geq n$ are listed.

Curves of bi-degree $(m,0)$ are not treated, since such curves are not immersions, except for $m=1$ (but there are no such curves with triple contact to the

$(1,1)$-curve). As to the curves of degree $(1,1)$, they should satisfy yet another condition, and that's impossible since the only curve having the prescribed contact to Z is Z itself, which cannot satisfy further conditions.

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(2,1)$:

	$c=0$	$c=1$
$b=0$	3	3
$b=1$	12	9
$b=2$	39	
$b=3$	114	

$a = 3 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(3,1)$:

	$c=0$	$c=1$	$c=2$
$b=0$	6	6	6
$b=1$	36	33	30
$b=2$	192	156	
$b=3$	894	642	
$b=4$	3672		
$b=5$	13632		

$a = 5 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(4,1)$:

	$c=0$	$c=1$	$c=2$	$c=3$
$b=0$	9	9	9	9
$b=1$	72	69	66	63
$b=2$	537	483	429	
$b=3$	3654	3042	2466	
$b=4$	22536	16932		
$b=5$	123888	83568		
$b=6$	607824			
$b=7$	2696544			

$a = 7 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(5,1)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$
$b=0$	12	12	12	12	12
$b=1$	120	117	114	111	108
$b=2$	1146	1074	1002	930	
$b=3$	10266	9150	8070	7026	
$b=4$	85488	71496	58656		
$b=5$	653424	506724	380904		
$b=6$	4548360	3215064			
$b=7$	28513992	18251208			
$b=8$	160761792				
$b=9$	821400576				

$a = 9 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(6,1)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$
$b=0$	15	15	15	15	15	15
$b=1$	180	177	174	171	168	165
$b=2$	2091	2001	1911	1821	1731	
$b=3$	23178	21414	19686	17994	16338	
$b=4$	243072	214932	188304	163188		
$b=5$	2389200	2004360	1656240	1343760		
$b=6$	21833160	17200464	13244208			
$b=7$	183786192	134597472	95372832			
$b=8$	1414816896	950873088				
$b=9$	9878202624	6051097344				
$b=10$	62402768640					
$b=11$	358306321920					

$a = 11 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(2,2)$:

	$c=0$	$c=1$	$c=2$
$b=0$	48	36	24
$b=1$	204	132	84
$b=2$	768	456	
$b=3$	2688	1494	
$b=4$	8904		
$b=5$	28176		

$a = 5 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(3,2)$:

	$c=0$	$c=1$	$c=2$	$c=3$
$b=0$	468	318	204	126
$b=1$	2352	1473	882	531
$b=2$	10989	6381	3597	
$b=3$	48006	25962	13890	
$b=4$	196908	99840		
$b=5$	763056	365076		
$b=6$	2811204			
$b=7$	9901848			

$a = 7 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(4,2)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$
$b=0$	3648	2352	1464	888	528
$b=1$	21648	13236	7860	4560	2664
$b=2$	122136	70788	39888	22092	
$b=3$	653928	358044	190848	101172	
$b=4$	3312816	1708608	861936		
$b=5$	15852576	7698624	3685968		
$b=6$	71700096	32848752			
$b=7$	307345152	133251216			
$b=8$	1253319936				
$b=9$	4882278912				

$a = 9 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(5,2)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$
$b=0$	24960	15552	9456	5640	3312	1920
$b=1$	172032	103200	60624	35004	19920	11400
$b=2$	1141824	658104	371772	206412	113304	
$b=3$	7268208	4011408	2168832	1152336	609840	
$b=4$	44196960	23270736	11990160	6081504		
$b=5$	255868800	128057844	62692632	30360540		
$b=6$	1406534124	667209060	310079148			
$b=7$	7329997608	3291324648	1453437144			
$b=8$	36212793456	15397326096				
$b=9$	169850892480	68497985232				
$b=10$	758291752560					
$b=11$	3232531611168					

$a = 11 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(3,3)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$
$b=0$	16728	9804	5472	2916	1512
$b=1$	87984	48282	25284	12798	6552
$b=2$	437028	224772	111012	53892	
$b=3$	2054388	993084	465516	218052	
$b=4$	9166752	4183104	1870944		
$b=5$	38976096	16864968	7227984		
$b=6$	158527248	65301744			
$b=7$	618982416	243577104			
$b=8$	2327680896				
$b=9$	8454918144				

$a = 9 - b - 2c$

Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(4,3)$:

	$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$
$b=0$	427488	229452	118560	59076	28512	13500
$b=1$	2444400	1246209	612258	290943	135108	63405
$b=2$	13361655	6467715	3022131	1372671	616527	
$b=3$	69814362	32084310	14276022	6216714	2711682	
$b=4$	348768324	152281992	64653852	27078336		
$b=5$	1667315616	692655804	281295960	113645772		
$b=6$	7638818820	3025309308	1178213916			
$b=7$	33604175160	12715905480	4760775672			
$b=8$	142250170944	51546944160				
$b=9$	580728436608	201962394624				
$b=10$	2291463338112					
$b=11$	8757848820480					

$a = 11 - b - 2c$

Chapter 5

More enumerative geometry

This final chapter is not conclusive. Rather is it a catalogue of further applications of the techniques presented in the previous chapters, and many of them deserve a deeper treatment than time, space and competence have allowed me to give here. What is done here is mostly the parts which can be done without too much trouble. Refinements that involve correction terms are left to the future.

For simplicity, and since anyway no attempt is made to be exhaustive, we restrict our attention to the case of $X = \mathbb{P}^r$.

5.1 Enumerative geometry via Porteous' formula

This section takes a quick look into the wide range of possibilities of enumerative applications provided by applying Porteous' formula (cf. [14], Ch. 13) to various jet bundles, set up as follows. Consider a family of *one-pointed* stable maps of degree d to $\mathbb{P}^r$:

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\mu"] \arrow[d, shift left, "\pi"] & \mathbb{P}^r \\ B \arrow[u, shift left, "\sigma"] & \end{tikzcd}$$

In the applications below, for simplicity this family will denote the (stack-theoretic) universal family over $\overline{M}_{0,1}(\mathbb{P}^r, d)$, but most of the arguments work in more general families, satisfying conditions 3.1.3.

For $z \geq 1$, fix a linear system $V^e \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(z))$ of affine dimension e. Consider the pull-back of $\mathcal{O}(z)$ to $\mathfrak{X}$, take the $(f-1)$'th relative jet bundle and pull it back along σ. This vector bundle is denoted L^f. We are interested in the natural vector bundle homomorphism from the trivial vector bundle $\sigma^*\mu^*V^e$ (which by abuse of notation we will also denote V^e) to L^f which locally consists in taking the Taylor expansion of a section of $\mathcal{O}(z)$ and truncate it in f'th order.

The degeneracy loci of such homomorphisms, varying e and f, correspond to various conditions of contacts at p_1 between μ and the hypersurfaces belonging to the linear system V^e. Particularly interesting is the case $z = 1$ so the hypersurfaces are hyperplanes, this will be assumed throughout: then we are talking about various conditions imposed on tangents and osculating planes at the mark p_1. Porteous' formula gives the class of these loci in terms of the Chern classes of V^e and L^f, which in turn are polynomials in eta classes and (modified) psi classes, so the results are enumerative descendants. (The corresponding loci in moduli spaces with more marks then have the class which is the pull-back of the one from $\overline{M}_{0,1}(\mathbb{P}^r, d)$ — that's why the modified psi classes are the correct ones to use.)

Unfortunately, the loci may also contain wrong components. For example, as we saw in the discussion of triple contacts, the locus I of boundary divisors having degree 1 on the marked twig will tend to count as a flex. Similarly, the locus Π_1 will count in many cases. In higher codimension the problems proliferate. Not only are there many new types of boundary corrections, for example the locus of boundary divisors with a twig of degree 2 mapping into H (occurs in codimension 4) and so on, but a more fundamental problem is that the basic ones will almost always appear as excess.

For this reason we shall stick to loci in low codimension.

The projective plane: second order conditions

In the next couple of examples we are concerned with the second order behaviour of maps to $\mathbb{P}^2$ — that measured by the jet bundle $L^2 := \sigma^* J^1_\pi \mu^* \mathcal{O}_{\mathbb{P}^2}(1)$. In these examples, the arguments work for any family satisfying the conditions (A), (B), and (C) of page 61. From the standard jet exact sequence (pulled back along σ_1)

$$0 \longrightarrow \sigma_1^*\big(\nu_2^*\mathcal{O}(1) \otimes \omega_\pi\big) \longrightarrow L^2 \longrightarrow \sigma_1^*\nu_2^*\mathcal{O}(1) \longrightarrow 0$$

it follows that $c_t(L^2) = (1 + \boldsymbol{\eta}_1 t)(1 + \boldsymbol{\eta}_1 t + \overline{\boldsymbol{\psi}}_1 t)$, so $c_1(L^2) = 2\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1$ and $c_2(L^2) = \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)$. There are three possibilities for the choice of linear system V^e. When V^1 is one-dimensional consisting of a single line H, then the vector bundle homomorphism is just a section $\mathcal{O}_{\mathfrak{X}} \to L^2$, and as we have already seen the zero locus is that of maps tangent to H at the first mark,

$$\Theta_1 = c_2(L^2) = \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1).$$

5.1.1 The condition of marked tangent through a given point. Take $V^2 \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ to be the pencil of lines passing through a given point $Q \in \mathbb{P}^2$. The points $b \in B$ where $\phi : V^2 \to L^2$ fails to have rank 2 are exactly the maps whose tangent at p_1 belongs to the pencil, i.e. the *maps whose marked tangent passes through* Q. The class is

$$D_1(\phi) = c_1(L^2) = 2\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1.$$

Push down this cycle from $\overline{M}_{0,1}(\mathbb{P}^2, d)$ to $\overline{M}_{0,0}(\mathbb{P}^2, d)$, forgetting the mark. From (the arguments behind) the divisor and dilaton equations we get $\pi_*(2\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1) = 2d - 2$ times the fundamental class of $\overline{M}_{0,0}(\mathbb{P}^2, d)$. This is the number of ways to put a mark on a curve such that the tangent at that mark passes through Q. In other words, this number is the *class* of a general rational curve of degree d, which of course is well-known from the Plücker formula. More generally, see 5.1.13 below.

5.1.2 Example. Intersect two such cycles, corresponding to distinct points Q and Q'. Then we are talking about the locus of maps whose mark has got a tangent in two distinct pencils. Then it is either a honest tangent to the line $\overline{QQ'}$, or it is a marked cusp, or it is a "marked node" (i.e. a map belonging to Π_1):

$$\begin{aligned}(2\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)^2 &= 4\boldsymbol{\eta}_1^2 + 4\overline{\boldsymbol{\psi}}_1\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1^2 \\ &= \Theta_1 + \mathrm{K}_1 + \Pi_1.\end{aligned}$$

cf. 3.1.1 and 4.3.2. See 5.1.4 for a cleaner point of view.

5.1.3 Marked cusp at given point, bis. The rank zero locus of the vector bundle homomorphism of the previous example consists of those maps μ such that all lines in the pencil are tangent to μ at the mark. Since we are in characteristic zero, this happens if and only if $\mu(p_1) = Q$ and the differential vanishes at p_1. So the condition is equivalent to that of *having a marked cusp mapping to Q*. By Porteous' formula,

$$\begin{aligned} D_0(\phi) &= \begin{vmatrix} c_2(L^2) & 0 \\ c_1(L^2) & c_2(L^2) \end{vmatrix} = c_2(L^2)^2 \\ &= \boldsymbol{\eta}_1^2(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)^2 = \mathrm{Kp}_1 + \boldsymbol{\eta}_1^2\Pi_1. \end{aligned}$$

cf. 4.3.4. Since being tangent to all the lines in a pencil is equivalent to being tangent to any two distinct lines $H \neq H'$ in the pencil, we can also think of this locus as $\Theta_1(H) \cap \Theta_1(H')$, cf. also 5.2.19 below.

5.1.4 The condition of a marked cusp, bis. Take V^3 to be the complete linear system. The locus of points $b \in B$ where $\phi : V^3 \to L^2$ fails to have rank 2 is that of maps μ such that there is an ∞^1 of tangent lines. Since we are in characteristic zero, this happens if and only if the differential $d\mu_{p_1}$ vanishes, and the corresponding pencil of tangent lines is then of course the one of lines through $\mu(p_1)$. In other words, the degeneracy locus is that of *critical mark*, consisting of cusps or "marked node". The class is

$$\begin{aligned} D_1(\phi) = \Delta_1^{(2)}(c_t(L^2)) &= \begin{vmatrix} c_1(L^2) & c_2(L^2) \\ 1 & c_1(L^2) \end{vmatrix} = c_1(L^2)^2 - c_2(L^2) \\ &= (2\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)^2 - \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1) \\ &= 3\boldsymbol{\eta}_1^2 + 3\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 \\ &= \mathrm{K}_1 + \Pi_1. \end{aligned}$$

as we have already seen.

The projective plane: third order conditions

Now look at the second jets: $L^3 := \sigma_1^* J_\pi^2 \mu^* \mathcal{O}_{\mathbb{P}^2}(1)$. The Chern classes are

$$\begin{aligned} c_1(L^3) &= 3\boldsymbol{\eta}_1 + 3\overline{\boldsymbol{\psi}}_1 \\ c_2(L^3) &= 3\boldsymbol{\eta}_1^2 + 6\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + 2\overline{\boldsymbol{\psi}}_1^2 \\ c_3(L^3) &= \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)(\boldsymbol{\eta}_1 + 2\overline{\boldsymbol{\psi}}_1) \end{aligned}$$

Again we will treat the three possibilities of $V^e \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. In 4.4.4 we already studied the case when V^1 is of dimension 1, consisting of a single plane H. The zero scheme of $\phi : V^1 \to L^3$ is of class

$$[D_0(\phi)] = c_3(L^3) = \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)(\boldsymbol{\eta}_1 + 2\overline{\boldsymbol{\psi}}_1)$$

and as cycles it is made up of three parts, $= \mathrm{F}_1 + \mathrm{I} + \boldsymbol{\eta}_1\Pi_1$. The multiplicity 1 at $\boldsymbol{\eta}_1\Pi_1$ is established in the Appendix to this chapter, on page 148.

5.1.5 The condition of marked flex through a given point. Let now V^2 be of dimension 2, and let Q be its base point. The vector bundles $\phi : V^2 \to L^3$ are of rank $e = 2$ and $f = 3$. The rank 1 locus of ϕ is just

$$D_1(\phi) = \Delta_2^{(1)}(c_t(L^3)) = c_2(L^3) = 3\boldsymbol{\eta}_1^2 + 6\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + 2\overline{\boldsymbol{\psi}}_1^2$$

corresponding to the maps μ having a marked flex through Q. However the locus includes the whole of Π_1 as well as the enumeratively irrelevant locus of two-twig maps the marked twig of which is of degree 1 and passes through Q.

5.1.6 The condition of marked flex. Now let $V^3 = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. The locus of points where $\phi : V^3 \to L^3$ fails to have rank 3, is just

$$D_2(\phi) = Z(\bigwedge^3 \phi) = c_1(L^3) = 3\boldsymbol{\eta}_1 + 3\overline{\boldsymbol{\psi}}_1.$$

It is the locus of maps such that the image of p_1 is an inflectional point, but it also includes the enumeratively irrelevant locus made up of boundary divisor whose marked twig is of degree 1.

Pushing down this cycle from $\overline{M}_{0,1}(\mathbb{P}^2, d)$ to $\overline{M}_{0,0}(\mathbb{P}^2, d)$ we recover the formula for the number of flexes of a general plane rational curve of degree d:

$$\pi_*(3\boldsymbol{\eta}_1 + 3\overline{\boldsymbol{\psi}}_1 - \boldsymbol{\xi}_1(1)) = 3d - 6.$$

See also 5.1.13 for more general observations of this kind.

Second order conditions in $\mathbb{P}^3$

Here we are concerned with the second order behaviour of maps to $\mathbb{P}^3$, so let the jet bundle be $L^2 := \sigma^* J^1_\pi \mu^* \mathcal{O}_{\mathbb{P}^3}(1)$. There are four possibilities for the choice of linear system: we can take the constant linear system $\{H\}$ consisting of one fixed plane; we can take a pencil of planes (all containing a given line ℓ); we can take a net of planes (all containing a given point Q); or we can take the complete linear system of all the planes.

When $V^1 \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ is of dimension 1, corresponding to a single plane H, the rank-zero locus of $V^1 \to L^2$ is just the locus of maps tangent to H at p_1.

5.1.7 The condition of marked tangent line incident to a given line. Take $V^2 \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ to be the pencil of planes containing a given line $\ell \in \mathbb{P}^3$. The points $b \in B$ where $\phi : V^2 \to L^2$ fails to have rank 2, corresponds to maps such that one of the planes in the system is a tangent plane to μ at p_1. That is, μ has a tangent plane at p_1 which contains ℓ. Now since the tangent *line* is contained in all tangent planes, this condition is equivalent to that *the tangent line meets* ℓ.

The class is

$$[D_1(\phi)] = c_1(L^2) = 2\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1.$$

Pushing down in $\overline{M}_{0,0}(\mathbb{P}^3, d)$, we get $\pi_*(2\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1) = 2d - 2$ times the fundamental class. It is the number of tangent lines of a general curve that meet a given line, classically called the rank, cf. more generally 5.1.13 below.

5.1.8 Prescribed tangent line. While the locus of the previous example is described as the set of maps such that there exists at least one plane $H \supset \ell$ with contact 2 with μ at the mark, we might as well ask that *all* planes in the pencil are tangent to μ at the mark, corresponding to the rank-zero locus of $V^2 \to L^2$. This condition is obviously equivalent to the line ℓ being the tangent line of μ at p_1 (which cycle we denote $\Theta_1^{(2)}$), together with the dishonest possibility $\boldsymbol{\eta}_1^2\Pi_1$ of having a "marked node" mapping to ℓ

By Porteous' formula

$$\Theta_1^{(2)} + \boldsymbol{\eta}_1^2\Pi_1 = D_0(\phi) = \Delta_2^{(2)}(c_t(L^2)) = \begin{vmatrix} c_2(L^2) & 0 \\ c_1(L^2) & c_2(L^2) \end{vmatrix} = c_2(L^2)^2 = \boldsymbol{\eta}_1^2(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)^2.$$

Note that this is also $\Theta_1(H)\Theta_1(H')$ for $H \neq H'$ two planes intersecting along ℓ, cf. also 5.2.19 below. (The condition $\Theta_1^{(2)}$ is (the pointed equivalent of) condition T in Schubert's treatment of twisted cubics, [43], p.164.)

5.1.9 The tangent line goes through a specified point. (Pointed equivalent of condition ϱ' of Schubert [43].) Now take V^3 to be the net of planes containing a given point Q. The rank-1 locus of $V^3 \to L^2$ consists of maps μ such that there is an ∞^1 of p_1-tangent planes in the net. In other words there is a line $\ell \ni Q$ such that all the planes in the pencil ℓ^* are tangent to μ at p_1, and obviously that line must be the tangent line at p_1, (or otherwise we are dealing with one of those fake tangents of the maps in Π_1). The class is

$$\begin{aligned} D_1(\phi) = \Delta_1^{(2)}(c_t(L^2)) &= \begin{vmatrix} c_1(L^2) & c_2(L^2) \\ 1 & c_1(L^2) \end{vmatrix} = c_1(L^2)^2 - c_2(L^2) \\ &= (2\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)^2 - \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1) \\ &= 3\boldsymbol{\eta}_1^2 + 3\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}^2. \end{aligned}$$

5.1.10 Example. — Just to show how soon these methods come short, or otherwise need more sophisticated arguments: Take V^4 to be the complete linear system. The locus where $V^4 \to L^2$ fails to have rank 2 is that of critical mark: meaning that all hyperplanes through $\mu(p_1)$ are tangent planes. But this degeneracy locus has got excess along Π_1... There is also a contribution from the locus $\Pi\!\Pi_1$ of "marked triple points", more precisely the locus of four-twig maps, the marked twig of which has degree zero and meets all the other twigs.

Third order conditions in $\mathbb{P}^3$

We are concerned with the second jet bundle $L^3 := \sigma_1^* J^2_\pi \mu^* \mathcal{O}_{\mathbb{P}^3}(1)$. Its Chern classes are

$$\begin{aligned} c_1(L^3) &= 3\boldsymbol{\eta}_1 + 3\overline{\boldsymbol{\psi}}_1 \\ c_2(L^3) &= 3\boldsymbol{\eta}_1^2 + 6\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + 2\overline{\boldsymbol{\psi}}_1^2 \\ c_3(L^3) &= \boldsymbol{\eta}_1(\boldsymbol{\eta}_1 + \overline{\boldsymbol{\psi}}_1)(\boldsymbol{\eta}_1 + 2\overline{\boldsymbol{\psi}}_1) \end{aligned}$$

Again we will treat the four possibilities of $V^e \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. When V^1 is of dimension 1, the rank-zero locus of $V^1 \to L^2$ is just the locus of maps μ having H as osculating plane at p_1. (together with the loci $\boldsymbol{\eta}_1\Pi_1$ and I), as described in 4.4.4.

5.1.11 The osculating plane contains a given line. (Pointed equivalent of condition ν' of Schubert [43].) Let $V^2 \subset H^0(\mathbb{P}^3, \mathcal{O}(1))$ be of dimension 2, and let ℓ be its base line. The rank-1 locus of $\phi : V^2 \to L^3$ is of class

$$[D_1(\phi)] = \Delta_2^{(1)}(c_t(L^3)) = c_2(L^3) = 3\boldsymbol{\eta}_1^2 + 6\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + 2\overline{\boldsymbol{\psi}}_1^2$$

and has three parts: the important locus consists of maps whose marked osculating plane contains ℓ, and the second is Π_1, and finally there is the enumeratively irrelevant locus of two-twig maps, the marked twig of which is of degree 1 and whose image is incident to ℓ.

5.1.12 Example. Now let $V^3 \subset H^0(\mathbb{P}^3, \mathcal{O}(1))$ be the net of planes containing a given point Q. The rank 2 locus of $\phi : V^3 \to L^3$ is just

$$D_2(\phi) = Z(\textstyle\bigwedge^3 \phi) = c_1(L^3) = 3\boldsymbol{\eta}_1 + 3\overline{\boldsymbol{\psi}}_1.$$

It is the locus of maps whose the osculating plane at p_1 contains Q (together with the irrelevant locus L_1, since the plane spanned by Q and the image of the linear twig osculates this twig).

Pushing down in $\overline{M}_{0,0}(\mathbb{P}^3, d)$ we get

$$\pi_*(3\boldsymbol{\eta}_1 + 3\overline{\boldsymbol{\psi}}_1) = 3d - 6$$

times the fundamental class, recovering the formula for the number of osculating planes passing through a given point, of a general rational space curve of degree d, cf. 5.1.13.

Weierstrass marks

5.1.13 Example. (Weierstrass points of a rational curve in $\mathbb{P}^r$) The *Weierstrass points* of a curve in $\mathbb{P}^r$ are the points where the curve possesses a hyperosculating hyperplane, i.e. a hyperplane to which it has contact of order $r+1$.

More generally, given a linear system $V^e \subset H^0(\mathbb{P}^r, \mathcal{O}(1))$ of affine dimension e, (i.e. the set of hyperplanes that contain a given $\mathbb{P}^{r-e} \subset \mathbb{P}^r$) consider the homomorphism of vector bundles on $\overline{M}_{0,1}(\mathbb{P}^r, d)$

$$V^e \to L^e := \sigma^* J_\pi^{e-1} \mu^* \mathcal{O}(1).$$

The rank-$(e-1)$ locus of this homomorphism is the locus of *Weierstrass marks* for the linear system. This is the locus of maps μ one of whose contact-e hyperplanes at p_1 contains the given $\mathbb{P}^{r-e}$. Its class is

$$c_1(L^e) = \sum_{k=0}^{e-1} (\boldsymbol{\eta}_1 + k\overline{\boldsymbol{\psi}}_1) = e(\boldsymbol{\eta}_1 + (e-1)/2 \cdot \overline{\boldsymbol{\psi}}_1).$$

Push-down to $\overline{M}_{0,0}(\mathbb{P}^r, d)$ getting $e(d+e-1)$ (times the fundamental class) which is just the Brill-Segre formula for Weierstrass points on a rational curve.

The easiest interpretable cases are $e = 1$, which is just the degree of a curve; $e = 2$ is the locus of maps having a tangent hyperplane containing a $\mathbb{P}^{r-2}$, or equivalently, those whose tangent line is incident to the $\mathbb{P}^{r-2}$. The case $e = r$

gives the locus of maps whose osculating hyperplane pass through a given point; and $e = r + 1$ yields the locus of maps possessing a hyper-osculating hyperplane.

The numbers resulting from pushing down the corresponding cycles reproduce the classical formulae stating that a rational curve in $\mathbb{P}^r$ has got

$(r+1)(d+r)$ hyper-osculating hyperplanes (Weierstrass points)

$r(d+r-1)$ points whose osculating hyperplane pass a given point (this is called the class of the curve)

$2d-2$ tangent lines incident to a given codimension-2 linear space (this is the rank of the curve).

5.1.14 Project. Many of the constructions of this section are similar to the constructions in the classical paper [10] of Steven Diaz and Joe Harris: Working in a partial compactification of the Severi variety of plane curves, they defined three intrinsic divisors denoted A, B and C. The push-down of $\boldsymbol{\eta}_1^2$ corresponds to their divisor A; the push-down of $\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1$ corresponds to their B; while finally $\pi_*\overline{\boldsymbol{\psi}}_1^2$ corresponds to C.

They then defined a wealth of divisors corresponding to geometrical conditions, and realised them as degeneracy loci of vector bundle homomorphisms. Only the simplest of these constructions have been carried over to the present context. For example they find that the locus of cuspidal curves is of class $3A+3B+C-\Delta$ (where Δ is the boundary), compared to which the expression in 4.3.2 is just the pointed analogue (lift to the universal family). Among the other examples worked out in [10] are the loci of curves having

— a triple point,
— a hyperflex,
— a nodal tangent through a given point,
— a tacnode,
— a prescribed bi-tangent,
— any tri-tangent, etc.

It seems that in each case, the construction of [10] will carry over to the setting of stable maps, and that it should be possible to get the class of the corresponding condition in this case.

In comparison with their work there are two main things to note: most importantly, the space of stable maps is compact, so the determination of the class of a given condition easily allows for actually solving enumerative problems, whereas the Severi variety of [10] is not. Second, the techniques are not limited to $\mathbb{P}^2$, as illustrated by the example for $\mathbb{P}^3$ above. Indeed, the loci in $\overline{M}_{0,n}(\mathbb{P}^3, d)$ corresponding to many of the conditions treated in [10] are the conditions studied by Schubert [43] for twisted cubics, like for example the condition of having a secant line through a given point.

A careful adaptation of the constructions of [10] to the setting of stable maps will certainly lead to beautiful enumerative geometry.

5.2 2-pointed conditions: secant conditions

Double points

In this subsection, we place ourselves in the projective plane.

5.2.1 The locus N of marked double point. In a family of two-pointed maps $\mathfrak{X} \to B \times \mathbb{P}^2$, which satisfy (A), (B), (C) of page 61, let $\mathrm{N}_{12} \subset B$ denote the closure of the locus of irreducible maps such that $\mu(p_1) = \mu(p_2)$. To construct it and compute its class, consider the morphism $B \to \mathbb{P}^2 \times \mathbb{P}^2$ given as the product of the two evaluation morphisms, and take the inverse image of the diagonal $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$. Now $\nu_1 \times \nu_2$ is not flat and the inverse image has excess along D_{12} (codimension 1, whereas codimension 2 is expected). Alongside appear the loci Π_{12} and N_{12}. Now the excess intersection formula (Fulton [14], Cor. 9.2) gives the equivalence $3\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1 - \boldsymbol{\delta}_{12}^2$ of D_{12}, so the formula for N_{12} is

$$\mathrm{N}_{12} = \boldsymbol{\eta}_1^2 + \boldsymbol{\eta}_1\boldsymbol{\eta}_2 + \boldsymbol{\eta}_2^2 - 3\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1 + \boldsymbol{\delta}_{12}^2 - \Pi_{12}.$$

See also 5.2.16 for a more general version of this construction.

5.2.2 Remark. While an immersion has only a finite number of double points, a multiple-cover map has an ∞^1 of double points; therefore such maps can more easily fulfil the requirement than immersions. So in a family where all maps have multiple-cover twigs, the locus described above would have only codimension 1. (This possibility, however, was excluded by assumption (C), since for such maps Σ has codimension only 1.)

In families where multiple-cover maps occur already in codimension 1 (as for example $\overline{M}_{0,n}(\mathbb{P}^2, 2)$ or in $\overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, \beta)$ for any β), the locus N_{12} does not have an immersion as general element, and thus it fails to satisfy (C). In the example of $\overline{M}_{0,n}(\mathbb{P}^2, 2)$, all maps in N_{12} are double covers, since a smooth conic can have no double points.

In order to prevent this behaviour it is necessary and sufficient to impose the following strengthening of condition (C):

(C2) The locus of multiple-cover maps is of codimension at least 2 in B.

(This condition is fulfilled by $\overline{M}_{0,n}(\mathbb{P}^2, d)$ except for $d = 2$.) With this extra restriction on the original family, N_{12} with its induced family will satisfy (A), (B), (C), but not in general (C2).

5.2.3 Example. Push down N_{12} from $\overline{M}_{0,2}(\mathbb{P}^2, d)$ to $\overline{M}_{0,0}(\mathbb{P}^2, d)$ forgetting the two marks. Note that Π_{12} has zero push-down. Forgetting p_1, the class $\boldsymbol{\eta}_1^2$ pushes down to $\boldsymbol{\iota}$ which in turn has zero push-down in $\overline{M}_{0,0}(\mathbb{P}^2, d)$ for dimension reasons. For the other classes, use the usual push-down properties, getting

$$\pi_* \mathrm{N}_{12} = \pi_*(\boldsymbol{\eta}_1^2 + \boldsymbol{\eta}_1\boldsymbol{\eta}_2 + \boldsymbol{\eta}_2^2 - 3\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1 + \boldsymbol{\delta}_{12}^2) = d^2 - 3d + 2 = (d-1)(d-2)$$

times the fundamental class, which is just twice the number of double points of a general rational plane curve. (For each double point there are two ways to label the pre-images.)

5.2.4 Remark. Since Π_{12} pushes down to zero, for most enumerative applications it is irrelevant. However it is sometimes interesting to intersect with δ_{12}, forcing the marks together (cf. also Examples 5.2.15 and 5.2.19):

5.2.5 Example. The double point becomes a cusp when the marks are forced together, (and then we might as well forget one of them). Precisely, applying $\pi_{2*}\boldsymbol{\delta}_{12}$ to N_{12} yields K_1. Indeed,

$$\begin{aligned}
\boldsymbol{\delta}_{12}\mathrm{N}_{12} &= \boldsymbol{\delta}_{12}\big(\boldsymbol{\eta}_1^2 + \boldsymbol{\eta}_1\boldsymbol{\eta}_2 + \boldsymbol{\eta}_2^2 - 3\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1 + \boldsymbol{\delta}_{12}^2 - \Pi_{12}\big) \\
&= 3\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1^2 - 3\boldsymbol{\delta}_{12}^2\boldsymbol{\eta}_1 + \boldsymbol{\delta}_{12}^3 - \boldsymbol{\delta}_{12}\Pi_{12} \\
&= 3\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1^2 + 3\boldsymbol{\delta}_{12}\overline{\boldsymbol{\psi}}_1\boldsymbol{\eta}_1 + \boldsymbol{\delta}_{12}\overline{\boldsymbol{\psi}}_1^2 - \boldsymbol{\delta}_{12}\Pi_1 \\
&= \boldsymbol{\delta}_{12}\mathrm{K}_1.
\end{aligned}$$

5.2.6 Example. Cutting with the divisor $\boldsymbol{\eta}_1$ we get the (two-pointed) locus of a double point on a specified line:

$$\boldsymbol{\eta}_1\mathrm{N}_{12} = \boldsymbol{\eta}_1^2\boldsymbol{\eta}_2 + \boldsymbol{\eta}_1\boldsymbol{\eta}_2^2 - 3\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1^2 + \boldsymbol{\delta}_{12}^2\boldsymbol{\eta}_1 - \boldsymbol{\eta}_1\Pi_{12}$$

and applying $\boldsymbol{\eta}_1^2$ we get the locus Np_{12} of (two-pointed) double point at a specified point:

$$\boldsymbol{\eta}_1^2\mathrm{N}_{12} = \boldsymbol{\eta}_1^2\boldsymbol{\eta}_2^2 - \boldsymbol{\delta}_{12}\overline{\boldsymbol{\psi}}_1\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_1^2\Pi_{12}$$

5.2.7 Characteristic numbers for curves with specified double point. According to Remark 5.2.2, for $d \geq 3$, the locus $\mathrm{Np}_{12} \subset \overline{M}_{0,n}(\mathbb{P}^2, d)$ of two-pointed specified double point satisfies the three conditions (A), (B), (C) necessary for the tangency conditions to work without further corrections, so we can now compute the characteristic numbers $NP_d(a,b,c)$ (of curves with one specified double point, passing through another a points, tangent to b lines and tangent to c lines at specified points), simply multiplying with the classes Ω, Θ and Ξ. Since these are E-classes, we can disregard the term Π_{12} (cf. 4.1.2). Concerning the first two terms, $\boldsymbol{\eta}_1^2\boldsymbol{\eta}_2^2$ is interpreted as the condition $\Omega \cdot \Omega$ of passing through two distinct points, while $\boldsymbol{\delta}_{12}\overline{\boldsymbol{\psi}}_1\boldsymbol{\eta}_1^2$ is interpreted as the condition Ξ of passing through two infinitely near points. This shows

$$2\ NP_d(a,b,c) = N_d(a+2,b,c) - N_d(a,b,c+1).$$

Let GP denote the potential for the numbers $NP_d(a,b,c)$, then the above equation is equivalent to

$$2GP = G_{u_2u_2} - G_{v_2}.$$

(Another recursion (quite complicated) was previously given by Ravi Vakil [46]).

Let us have a look at the intersection of various double point loci. Intersecting three or more N cycles, say $\mathrm{N}_{12} \cdot \mathrm{N}_{34} \cdot \mathrm{N}_{56}$ gives excess. Indeed, in the initial family $\overline{M}_{0,n}(\mathbb{P}^2, d)$, the locus of maps with a double-cover twig is of codimension 2. For each time we step into a locus N this codimension drops; in N_{12} the locus has codimension 1, and in $\mathrm{N}_{12} \cdot \mathrm{N}_{34}$ there is a whole component of such maps, namely two codimensions to be in the locus, one codimension to force p_2 to the pre-image of $\mu(p_1)$, and a fourth codimension to do the same for p_4 with respect to p_3. Now applying N_{56} gives excess.

However when further conditions are imposed on the same marks, the intersection works better. In $\mathrm{Np}_{12} \cdot \mathrm{Np}_{34}$ there is still a whole component of maps with a double-line twig — the line is $\overline{QQ'}$ — but note that the excessive ∞^1 of double points on this twig have been fixed, so when imposing yet another fixed double point Q'' which by generality is off the line $\overline{QQ'}$, the abundance of double points on the double-line twig cannot contribute, and in fact the intersection $\mathrm{Np}_{12} \cdot \mathrm{Np}_{34} \cdot \mathrm{Np}_{56}$ is proper. This phenomenon works upto $d-1$ imposed specified double points:

5.2.8 Lemma. *The intersection of upto $d-1$ loci* Np *(for general points) is proper.*

Proof. The proof is a dimension count. Consider first the possibility of having irreducible multiple-cover maps contributing. In $\overline{M}_{0,n}(\mathbb{P}^2, k\,e)$ the locus of k-fold covers of a degree-e curve has codimension $(k-1)(3e-2)$ (see [30], p.27–28). Now it is easy to check that for $k \geq 2$ this number is always greater or equal to $ke-1$, indeed $(k-1)(3e-2)-(ke-1) = (2k-3)(e-1)$. Therefore the locus of irreducible multiple-covers is of codimension greater or equal to $d-1$, and since we impose at most $d-1$ conditions, we do not get so far as to get excess. The two numbers are equal exactly for $e = 1$, the d-fold covered lines, which occur in just codimension $d-1$. So as much as this argument is concerned, when imposing $d-1$ double points, there might be contribution from d-fold covered lines. However, mostly this is not so, because the imposed double points are in general position, so the line supporting the image of the map cannot possibly pass through all of them, unless there are at most two of them. So in conclusion: there are only two cases where multiple lines give contribution, namely for conics which are double lines, subject to the condition of one specified double point, and the case of cubics which are triple-lines, subject to the condition of two prescribed double points.

The next case to consider is that when the whole map is made up of multiple-cover twigs. Suppose there are t twigs C_i, and that the map restricted to C_i is a k_i-fold cover of a curve of degree e_i, and we assume that $k_i \geq 2$ for all $i = 0, \ldots, t$. The locus of such maps is of codimension

$$(t-1) + \sum_{i=1}^{t} (k_i - 1)(3e_i - 2)$$

where the first term is the codimension caused just by the number of nodes of the tree. Exactly as in the first case, it is easy to see that this number is always greater or equal to $d-1 = \sum k_i e_i - 1$, in fact this is a twig-by-twig argument. The same argument shows that only in the case where $e_i = 1$ for all i (that is, all the twigs are multiple covers of lines) do we get equality. This already shows there is no excess from these configurations. Now as observed in the first case, each twig can satisfy at most two double point conditions (due to our favourite basic fact $N_1 = 1$), so unless $2t \geq d-1$ there is no contribution. This means that all twigs must be just double-lines, except possibly one (if d is odd) which must then be a triple-line. The case of even $d = 2t \geq 4$ shows that the $(d-1)$-bound is sharp. The locus of maps which consist of t double-lines is of codimension $2t-1 = d-1$, so if we imposed d double point conditions we would get excess (and indeed the locus is not empty because the t supporting lines can pass through the $2t$ specified points).

Finally there is the case where some twigs of the map are birational maps and some are multiple-covers — this is the crucial argument. Since having various twigs which are birational maps would only increase the codimension without causing any more trouble, we can assume there is only one such twig, and since we already considered the case of no such twigs we can assume there is exactly one, C_0. Now think of the remaining twigs as being broken off C_0 one by one. Breaking off a "k_i-fold cover of a degree-e_i curve" increases the codimension by $1 + (k_i - 1)(3e_i - 2)$. Now this number may be very low (and in particular much lower than $d - 1$), for example breaking off a double-line gives codimension $1 + 1 = 2$ as already noted. However, this number is always greater or equal to $3e_1 - 1$, the number of (double) points that the supporting image curve can actually pass through. Indeed, $1+(k_i-1)(3e_i-2)-(3e_i-1) = (k_i-2)(3e_1-2)$, which is clearly positive since $k_i \geq 2$. So after imposing double point number $3e_i-1$, the image curve of the multiple-cover is completely fixed, and even though it appears in codimension-zero in the family at this stage, it will not give excess in the next step, since it is impossible for its ∞^1 of double points to take part in the fulfilment of the next double point condition. This argument goes for each of the multiple-cover twigs of the general map we are considering, and thus for each possible configuration of mixed birational and multiple-cover twigs. This completes the proof. □

5.2.9 Project. Since the dimension is correct, we can impose tangency conditions and incidence conditions to get "characteristic numbers $NP_d^k(a, b, c)$ of rational curves with $k \leq d - 1$ prescribed double points". However, the numbers will count the reducible solutions as well. Now the project is to correct for these contributions. It might not be too difficult since the configurations consist of a fixed part (the multiple cover of some curve, accounting for a lot of double points) and a variable curve which is then some number $NP_{d'}^{k'}(a, b, c)$ for lower values $d' < d$ and $k' < k$. For example the characteristic number $NP_4^2(5, 2, 0)$ of quartics with two prescribed double points, (5 incidence and 2 tangency conditions) get a contribution from curves which are the union of a double-line and a smooth conic; the double line then accounts for the two double point as well as two tangency conditions (namely the two foci mapping to the prescribed tangent lines) while the smooth conic takes the five incidence conditions. So modulo multiplicities (yet to be determined) the correction to the brute answer is something like $N_2(5, 0, 0)$, times 2^2 for the number of ways to chose the attachment point for the two twigs, times 2^2 for the number of ways to label the marked double points...

5.2.10 The generating function for (virtual) characteristic numbers of curves with many fixed double points. Even though it may not always have satisfactory enumerative meaning, let $G(u_1, u_2, v_1, v_2, w)$ denote the potential for characteristic numbers, with an extra formal parameter w corresponding to the condition of having a specified double point. More precisely, since possibly that does not make sense, it is the generating function for the top integrals with k factors of type Np, and w being the corresponding formal parameter. Now since Np admits an easy expression in terms of $\overline{\tau}_0$ and $\overline{\tau}_1$ (after eliminating the diagonal classes) we readily get the differential equation

$$2GP_w = GP_{u_2 u_2} - GP_{v_2}$$

— this is just a repetition of the arguments of 5.2.7 above. The next paragraph shows how to make some enumerative use of this potential:

5.2.11 Incidence conditions only, or at most one tangency condition. Since the multiple-cover contributions have their multiple-cover curve fixed physically, it cannot satisfy any more incidence conditions, or more precisely, it can only satisfy exactly two tangency conditions, as in the example just described. Therefore, if we impose at most one (say zero) tangency condition there are no contributions from multiple-cover maps. So in this case the potential described above is enumerative; precisely, its restriction to $v_1 = 0$ is enumerative.

The numbers with $b_2 = 0$ (but more generally allowing points of multiplicity $m \geq 2$) were previously determined in Göttsche-Pandharipande [24], and in Gathmann [18] using Gromov-Witten invariants of blow-ups of $\mathbb{P}^2$. The method described above reproduce the numbers listed in these two papers, but comes short in providing a water-proof multiplicity-1 argument...

5.2.12 Remark on the intersection of various cusp cycles. The problems of intersecting cusp cycles is very similar to the problems just described. The intersection of two cusp cycles has excess, but the intersection of two "cusp at specified point" cycles is proper. However it has a component consisting of a double line, whose foci are fixed at the two specified points. Again, this configuration is completely fixed and does not cause excess in the next step.

More general secant conditions

Now we describe a slightly more general approach to double points and more generally certain secant conditions.

5.2.13 Secant to a hyperplane. Let $H \subset \mathbb{P}^r$ be a hyperplane and consider the intersection of the two divisors $\boldsymbol{\eta}_1(H) \cap \boldsymbol{\eta}_2(H)$. It follows from an easy 1-parameter family argument that this intersection is transversal. It has got two components: one component is just $\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1(H)$ where the two marks have come together already on the source, and the other is the closure of the honest locus of maps whose marks both map to H but without having come together on the source. This second locus we call Λ_{12}. That is,

$$\Lambda_{12} = \boldsymbol{\eta}_1\boldsymbol{\eta}_2 - \boldsymbol{\delta}_{12}\boldsymbol{\eta}_1$$

5.2.14 Remark. Note that a general curve cuts H in d points, so there are $\binom{d}{2}$ ways to choose a pair, and further two ways of labelling. So Λ_{12} should provide a $d(d-1)$-fold cover of $\overline{M}_{0,0}(\mathbb{P}^r, d)$. This agrees with the push down: $\pi_{12*}\Lambda_{12} = \pi_{12*}\big(\boldsymbol{\eta}_1\boldsymbol{\eta}_2 - \boldsymbol{\delta}_{12}\boldsymbol{\eta}_1\big) = d^2 - d$. (This can be seen as another proof of the multiplicities 1 above.)

5.2.15 Example. Among the honest curves are also those in which the two marks have come together, thus leaving the curve tangent to H. To be precise we are talking about the locus

$$\boldsymbol{\delta}_{12}\Lambda_{12} = \boldsymbol{\delta}_{12}\big(\boldsymbol{\eta}_1\boldsymbol{\eta}_2 - \boldsymbol{\delta}_{12}\boldsymbol{\eta}_1\big) = \boldsymbol{\delta}_{12}\big(\boldsymbol{\eta}_1^2 + \overline{\boldsymbol{\psi}}_1\boldsymbol{\eta}_1\big)$$

which pushes down to Θ_1, the locus of tangency to H, giving a new construction of this locus (cf. 3.1.1).

5.2.16 Secant to codimension-2 plane. The locus $\Lambda_{12}^{(2)}$ of honest curves both of whose marks are mapped to a specified codimension-2 plane is

$$\begin{aligned} \Lambda_{12}^{(2)} &= \Lambda_{12}\Lambda_{12} - \boldsymbol{\eta}_1^2\Pi_{12} \\ &= \boldsymbol{\eta}_1^2\boldsymbol{\eta}_2^2 - \boldsymbol{\delta}_{12}\big(2\boldsymbol{\eta}_1^3 + \boldsymbol{\eta}_1^2\overline{\boldsymbol{\psi}}_1\big) - \boldsymbol{\eta}_1^2\Pi_{12}. \end{aligned}$$

Proof. Let the codimension-2 plane be given as the intersection of two hyperplanes H and H'. Consider first the intersection of $\boldsymbol{\eta}_1(H')$ with $\Lambda_{12}(H)$. It is transversal, since the eta divisors can be translated to be transversal to anything. Now apply $\boldsymbol{\eta}_2(H')$ to this locus. There are three components in this intersection: one is $\Lambda_{12}^{(2)}$, another one is $\boldsymbol{\delta}_{12}\boldsymbol{\eta}_1\Lambda_{12}$ and the last one is $\boldsymbol{\eta}_1\boldsymbol{\eta}_2\Pi_{12}$. It is easy to show (cf. the Appendix) that everything has multiplicity 1. Now expand Λ_{12} via 5.2.13. □

5.2.17 Remark. If we apply $\boldsymbol{\delta}_{12}$ to the intermediate locus $\boldsymbol{\eta}_1\Lambda_{12}$ of the proof, we recover the locus of pointed tangency Ξ_1.

5.2.18 Example. In the plane, this is just the locus of a doubly marked specified double point, whose formula we recover, cf. 5.2.6:

$$\begin{aligned}\Lambda_{12}^{(2)} &= \Lambda_{12}\Lambda_{12} - \boldsymbol{\eta}_1^2\Pi_{12}\\ &= \boldsymbol{\eta}_1^2\boldsymbol{\eta}_2^2 - \boldsymbol{\delta}_{12}^2\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_1^2\Pi_{12}\end{aligned}$$

5.2.19 Example. Forcing the marks together we get

$$\begin{aligned}\pi_{2*}\big(\boldsymbol{\delta}_{12}\Lambda_{12}^{(2)}\big) &= \pi_{2*}\big(\boldsymbol{\delta}_{12}\big(\boldsymbol{\eta}_1^2\boldsymbol{\eta}_2^2 - 2\boldsymbol{\delta}_{12}^2\boldsymbol{\eta}_1^2\boldsymbol{\eta}_2 + \boldsymbol{\delta}_{12}\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_1^2\Pi_{12}\big)\big)\\ &= \pi_{2*}\big(\boldsymbol{\delta}_{12}\big(\boldsymbol{\eta}_1^4 - 2\overline{\boldsymbol{\psi}}_1\boldsymbol{\eta}_1^3 + \overline{\boldsymbol{\psi}}_1^2\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_1^2\Pi_{12}\big)\big)\end{aligned}$$

whose push-down is $\Theta_1^{(2)} := \Theta_1\Theta_1 - \boldsymbol{\eta}_1^2\Pi_1$, the locus of maps tangent to a given codimension-2 plane.

5.2.20 Example. In $\mathbb{P}^3$, the two conditions $\Lambda_{12}^{(2)}$ and $\Theta_1^{(2)}$ are (the pointed equivalents of) Schubert's condition B and T.

5.3 Linear stable maps

5.3.1 De Jonquières theory for stable maps? It is an obvious project to use the jet bundle machinery to describe more generally the locus of maps having various multiple contacts to a fixed very ample hypersurface, and thus establish a sort of de Jonquières theory for stable maps. Unfortunately, this is a very hard problem, as there are soon boundary loci that contribute where they shouldn't or are even of excessive dimension.

This theory has recently been developed by Andreas Gathmann [19], who has overcome the obstacles, in a much more general setting: he gives a recursive description of the cycles of maps having prescribed contacts with a very ample hypersurface in any smooth ambient space. The difficult part — that of making the boundary corrections — is handled using the concept of the virtual fundamental class of moduli spaces having components of too high dimension, while the determination of the multiplicities of the correction terms relies on methods developed by Ravi Vakil [45]. However, his recursion is very complicated, and it is not a relation among classes in Kontsevich moduli spaces; rather does it establish relations among classes belonging to a jungle of different auxiliary non-equidimensional moduli spaces, and it is not clear (to me) to what extent the integrals over the virtual classes of these auxiliary spaces yield enumeratively significant numbers.

This section treats only the simplest of all cases, namely the case of $d = 1$, in which there is no hard boundary and everything works as fine as one could hope for. The results compiled in this section are all known, or easily doable by classical techniques — after all $\overline{M}_{0,0}(\mathbb{P}^r, 1)$ is naturally isomorphic to the Grassmannian of lines in $\mathbb{P}^r$, and under this identification, the pointed spaces are $\mathbb{P}^1[n]$ configuration space bundles over $\mathrm{Gr}(1, \mathbb{P}^r)$, whose intersection rings are known (cf. [15]).

The reasons for including these examples are, first, to illustrate the workings of the modified psi classes and diagonal classes in a transparent situation, and second, to constitute an introduction to the recent work of Andreas Gathmann [19].

5.3.2 Lines and hypersurfaces. Throughout, let $Z \subset \mathbb{P}^r$ be a general hypersurface of degree z. We are concerned with linear stable maps having prescribed contacts with Z. The lines in $\mathbb{P}^r$ vary in a family of dimension $2r-2$, and the locus of lines contained in Z is of codimension $z+1$ (since it's the zero locus of a regular section of $\pi_* \mu^* \mathcal{O}(z)$). As long as we do not reach that codimension, the locus of lines having k'th contact with Z is given as the vanishing of a section of a $(k-1)$'th jet bundle.

When imposing contact order $z+1$, the lines of the resulting locus are now completely contained in Z, and there is no meaning in imposing further contacts. Since we are mostly interested in imposing contacts until we reach dimension zero, there are naturally two cases: for hypersurfaces of degree $z \leq 2z - 3$, we impose a contact of order $z+1$ getting the locus of lines in Z. This is the pointed equivalent of the Fano scheme of lines in Z. The first series of examples belongs to this case. There is no need to have more than one mark to force the maps into Z, so we are just in $\overline{M}_{0,1}(\mathbb{P}^r, 1)$ which has no boundary at all. The main point in these cases is that the recursion relation 1.6.3 boils down to a closed formula.

The second case is that of $z > 2z-3$; then Z contains no lines, and we can freely impose various contacts at various marks, solving problems of multisecants of Z (or of complete intersections of various hypersurfaces). In these arguments, the diagonal classes play a prominent rôle.

5.3.1 1-pointed lines in hypersurfaces

In the case $d = 1$, there are no positive degree partitions, so the big sum in the recursion relation 1.6.3 is absent and only the three special terms survive. Suppose furthermore $n = 3$, then the theorem states that

$$\begin{aligned}\overline{\boldsymbol{\psi}}_1 \cdot \boldsymbol{\eta}_1^{a_1}\overline{\boldsymbol{\psi}}_1^{b_1} \cdot \boldsymbol{\eta}_2^{a_2}\overline{\boldsymbol{\psi}}_2^{b_2} \cdot \boldsymbol{\eta}_3^{a_3}\overline{\boldsymbol{\psi}}_3^{b_3} \quad = \quad & + \boldsymbol{\eta}_1^{a_1}\overline{\boldsymbol{\psi}}_1^{b_1} \cdot \boldsymbol{\eta}_2^{a_2+a_3}\overline{\boldsymbol{\psi}}_2^{b_2+b_3} \cdot \boldsymbol{\eta}_3 \\ & - \boldsymbol{\eta}_2^{a_2}\overline{\boldsymbol{\psi}}_2^{b_2} \cdot \boldsymbol{\eta}_3^{a_1+a_3}\overline{\boldsymbol{\psi}}_3^{b_1+b_3} \cdot \boldsymbol{\eta}_1 \\ & - \boldsymbol{\eta}_3^{a_3}\overline{\boldsymbol{\psi}}_3^{b_3} \cdot \boldsymbol{\eta}_1^{a_1+a_2}\overline{\boldsymbol{\psi}}_1^{b_1+b_2} \cdot \boldsymbol{\eta}_2.\end{aligned}$$

For monomials of type $\boldsymbol{\eta}_1^{r-k}\overline{\boldsymbol{\psi}}_1^{k+j}\boldsymbol{\eta}_2^{r-j}\boldsymbol{\eta}_3$ this yields a closed formula:

5.3.3 Proposition. *In the space $\overline{M}_{0,2}(\mathbb{P}^r, 1)$ we have, for all integers k, j less or equal to r,*

$$\int \boldsymbol{\eta}_1^{r-k}\overline{\boldsymbol{\psi}}_1^{k+j}\boldsymbol{\eta}_2^{r-j} = (-1)^k \binom{k+j}{k} + (-1)^{k+j-r+1}\binom{k+j}{k+j-r},$$

(with the convention that binomial coefficients with a negative entry are zero).

Proof. Induction on $k + j$. The formula holds if k or j is negative: then the integral has a factor $\boldsymbol{\eta}^{r+1} = 0$, while both the binomial coefficients have negative entries. Also the formula holds for $k = j = 0$, since $\int \boldsymbol{\eta}_1^r\boldsymbol{\eta}_2^r = 1$, a unique line through two distinct points. Let $f(k, j)$ denote the integral on the left hand side of the formula. The recursion relation above then reads (with $a_3 = 1$ and using the divisor equation on each of the four terms)

$$f(k, j) = f(k, j-1) - f(k-1, j) - f(k+j-r, r-1).$$

Now by induction, the claimed formula holds for each of the three terms on the right hand side, giving

$$\begin{aligned} = \quad & (-1)^k\tbinom{k+j-1}{k} + (-1)^{k+j-r}\tbinom{k+j-1}{k+j-1-r} \\ & - (-1)^{k-1}\tbinom{k+j-1}{k-1} - (-1)^{k+j-r}\tbinom{k+j-1}{k+j-1-r} \\ & - (-1)^{k+j-r}\tbinom{k+j-1}{k+j-r} - (-1)^{k+j-r}\tbinom{k+j-1}{k+j-r-1}.\end{aligned}$$

Here the second and the fourth term cancel each other; the first and the third combine to give $(-1)^k\binom{k+j}{k}$, and finally the two last terms combine to give $(-1)^{k+j-r+1}\binom{k+j}{k+j-r}$, thus proving the formula. $\square$

In case $j = r-1$, the divisor equation removes the $\boldsymbol{\eta}_2$-factor, and we find:

5.3.4 Corollary. *For any $r \geq 1$ and for any $0 \leq k \leq r$, the following formula holds in $\overline{M}_{0,1}(\mathbb{P}^r, 1)$:*

$$\int \boldsymbol{\eta}_1^{r-k}\overline{\boldsymbol{\psi}}_1^{r-1+k} = (-1)^k\binom{r+k}{k}.$$

and in particular,

$$\int \boldsymbol{\eta}_1^{r}\overline{\boldsymbol{\psi}}_1^{r-1} = 1,$$

Since $\overline{M}_{0,1}(\mathbb{P}^r, 1) \simeq \mathbb{P}(T\mathbb{P}^r)$, this formula should be well-known, arising as some Segre classes...

5.3.5 Higher contacts of linear stable maps. Let $\mathbf{J}_{(z)}(m) \subset \overline{M}_{0,1}(\mathbb{P}^r, 1)$ denote the cycle of maps having m-order contact with $Z = Z(f)$ at p_1. For $m \leq z+1$ it is the zero locus of the section $\sigma_1^*\partial_\pi^{m-1}\mu^* f$ of the jet bundle $\sigma_1^* J_\pi^{m-1}\mu^*\mathcal{O}(z)$. No correction terms are needed, since there is no boundary, and no multiple-cover maps in $\overline{M}_{0,1}(\mathbb{P}^r, 1)$.

This jet bundle has a filtration of line bundles, coming from the standard jet short exact sequences

$$0 \longrightarrow \sigma_1^*\big(\nu_2^*\mathcal{O}(1)\otimes\omega_\pi^{\otimes m}\big) \longrightarrow \sigma_1^* J_\pi^m\mu^*\mathcal{O}(z) \longrightarrow \sigma_1^* J_\pi^{m-1}\mu^*\mathcal{O}(z) \longrightarrow 0.$$

Each locus $\mathbf{J}_{(z)}(m)$ is a Cartier divisor inside the previous locus, given as the zero scheme of a section of the line bundle $\sigma_1^*\big(\nu_2^*\mathcal{O}(1)\otimes\omega_\pi^{\otimes(m-1)}\big)$, whose first Chern class, $z\boldsymbol{\eta}_1 + (m-1)\overline{\boldsymbol{\psi}}_1$ we denote by $\boldsymbol{\varphi}_1(m)$. So the class of the m-order contact cycle is

$$\mathbf{J}_{(z)}(m) = \boldsymbol{\varphi}_1(1)\boldsymbol{\varphi}_1(2)\cdots\boldsymbol{\varphi}_1(m)$$

(As usual, in spaces with more marks this expression for the class still holds, since we are talking about E-classes, but the correct jet bundles to use are the pull-backs of the bundles on one-pointed space.)

When imposing contact $z+1$, we are forcing the image lines into Z, so for hypersurfaces of degree $z \leq 2r-3$, the locus $\mathbf{J}_{(z)}(z+1)$ is the locus of pointed lines inside Z. We fix the mark by imposing furthermore the condition $\boldsymbol{\eta}_1(H)$ of the mark mapping to a specified general hyperplane. The locus $\mathbf{J}_{(z)}(z+1)\boldsymbol{\eta}_1$ will abusively be called the Fano cycle or the Fano scheme of lines in Z.

5.3.6 Lemma. *Let $Z \subset \mathbb{P}^r$ be a general hypersurface of degree $z = 2r-3$. Then the number of lines on Z is*

$$\sum_{k=0}^{r-2} (-1)^k \binom{r+k}{r} \; z^{r-1-k} \; \sigma_{r-1+k}(z).$$

(Here $\sigma_{r-1+k}(z)$ is the $(r-1+k)$'th elementary symmetric function on the first z integers.)

Other formulae have been given by van der Waerden [48], Libgober [37], and Debarre-Manivel [8], see below, for generalisations.

Proof. By the discussion, we are talking about the degree of the cycle $\mathbf{J}_{(z)}(z+1)\boldsymbol{\eta}_1 = z\boldsymbol{\eta}_1(z\boldsymbol{\eta}_1+\overline{\boldsymbol{\psi}}_1)\cdots(z\boldsymbol{\eta}_1+z\overline{\boldsymbol{\psi}}_1)\cdot\boldsymbol{\eta}_1$. The rest is just combinatorics: In the expansion, all terms have at least two factors of $\boldsymbol{\eta}_1$, and the highest exponent we need to consider is r, since $\boldsymbol{\eta}_1^{r+1} = 0$ for dimension reasons. Now for k from 0 to $r-2$, let us compute the coefficient of $\boldsymbol{\eta}_1^{r-k}\overline{\boldsymbol{\psi}}_1^{r+k-1} = (-1)^k\binom{r+k}{k}$. From the eta factors we get z^{r-k-1} (the minus one because the last eta appears without the z). Then we need to take all the combinations of $r+k-1$ possible $\overline{\boldsymbol{\psi}}_1$-factors, so the contribution from them is the $(r+k-1)$'th symmetric function on their coefficients. These coefficients are exactly the first z integers. $\square$

More generally, given a cycle in $\overline{M}_{0,1}(\mathbb{P}^r, 1)$ (whose mark has been fixed), let its degree be defined as the integral up against a product of classes $\boldsymbol{\iota} = \pi_*(\boldsymbol{\eta}_2^2)$ of complementary dimension. Under the identification of $\overline{M}_{0,1}(\mathbb{P}^r, 1) \simeq \mathrm{UGr}(1, \mathbb{P}^r)$, the incidence class is just the Schubert divisor class (first Chern class of the tautological bundle), so this definition agrees with the usual definition of degree in a Grassmannian.

5.3.7 Proposition. *The degree of the Fano scheme $\mathbf{J}_{(z)}(z+1)\boldsymbol{\eta}_1$ of lines contained in a degree z hypersurface $Z \subset \mathbb{P}^r$ is equal to*

$$\sum_{k=0}^{r-2} (-1)^k \binom{r+k}{r} \sum_{i=0}^{z} z^{i+1} \; \sigma_{z-i}(z) \; 2^{r-2-k-i} \binom{2r-z-3}{r-2-k-i}.$$

Proof. By the above definition of degree, we are talking about the integral

$$\mathbf{J}_{(z)}(z+1)\boldsymbol{\eta}_1 \cdot \boldsymbol{\iota}^{2r-z-3}$$

Recall from Pandharipande [42], (Lemma 2.2.2) that on $\overline{M}_{0,1}(\mathbb{P}^r, 1)$ we have $\boldsymbol{\iota} = 2\boldsymbol{\eta}_1 + \boldsymbol{\psi}_1$. Now expand and use the arguments from the preceding proof. □

Setting $z = 2r-3$ recovers the formula of the preceding lemma. The case $z = 3$ is that originally studied by Fano. In this case there is a closed formula due to A.B. Altman and S.L. Kleiman [1] (see also Fulton [14], p.275). The degree 320 of the Fano scheme of lines in a quartic threefold was computed by B.R. Tennison [44].

More generally, Libgober [37] gave a formula for the degree of the Fano scheme of lines in a general complete intersection, and Debarre-Manivel [8] have given a formula for the degree of the Fano scheme of any linear subvarieties in a general complete intersection...

The formula produced the following numbers — in agreement with those listed in [8].

	$\mathbb{P}^3$	$\mathbb{P}^4$	$\mathbb{P}^5$	$\mathbb{P}^6$	$\mathbb{P}^7$	$\mathbb{P}^8$
$z=1$	1	2	5	14	42	132
$z=2$	4	8	20	56	168	528
$z=3$	27	45	108	297	882	2754
$z=4$		320	736	1984	5824	18048
$z=5$		2875	6125	16100	46625	143250
$z=6$			60480	154224	439776	1338336
$z=7$			698005	1707797	4783478	14403893
$z=8$				21518336	59021312	175640576
$z=9$				305093061	815556357	2395066806
$z=10$					12493096000	36140368000
$z=11$					210480374951	598308653657
$z=12$						10790608637952
$z=13$						210776836330775

5.3.8 Proposition. *Let $Z, Z' \subset \mathbb{P}^r$ be two general hypersurfaces of common degree $z = r-2$. Then the number of lines on $Z \cap Z'$ is*

$$\sum_{k=0}^{r-3}(-1)^k \binom{r+k}{r} z^{r-1-k} \; \sigma_{r-1+k}(z \cup z).$$

Here $\sigma_{r-1+k}(z \cup z)$ denotes the $(r-1+k)$'th elementary symmetric function on the $2z$-element set consisting of two copies of each of the first z integers.)

Proof. It is the integral of $\mathbf{J}_{(r-2)}(r-1)\mathbf{J}'_{(r-2)}(r-1)\boldsymbol{\eta}_1$, whose expansion is treated as in the proof of 5.3.6. □

The first few numbers are (starting from $r=3$):

$$1,\ 16,\ 1053,\ 111616,\ 19188125,\ 4782986496,\ 1636365422153\ldots$$

The number 16 of lines common to two quadrics is in Hodge and Pedoe [26], III, p.366. The number 1053 of lines in the intersection of two cubics in $\mathbb{P}^5$ was previously computed in Libgober [37].

While for more general complete intersection it is increasingly difficult to handle these expansions and get a closed formula like those of the three preceding results, it is easy to write down the cycle in $\overline{M}_{0,1}(\mathbb{P}^r, 1)$ of the Fano schemes, and then invoke the general recursion to handle the combinatorics to produce a number.

5.3.9 Example. LINES ON COMPLETE INTERSECTION CALABI-YAU THREEFOLDS. The previous examples include the 2785 lines on the quintic in $\mathbb{P}^4$, and the 1053 lines on the intersection of two cubics in $\mathbb{P}^5$. Similarly we recover 1280 lines on the intersection of a quadric and a quartic in $\mathbb{P}^5$; 720 lines on the intersection of two quadrics and a cubic in $\mathbb{P}^6$, and 512 lines on the intersection of four quadrics in $\mathbb{P}^7$.

5.3.10 Proposition. *Through a general point on a general hypersurface $Z \subset \mathbb{P}^{z+1}$ of degree z, there are exactly $z!$ lines.*

Proof. The cycle of lines in Z is $\mathbf{J}_{(r)}(r)$. Now force the mark into a general line ℓ, thus forcing the map to pass one of the $r-1$ points in $\ell \cap Z$. In conclusion, the number is

$$\mathbf{J}_{(z)}(r) \cdot \tfrac{1}{r-1}\boldsymbol{\eta}_1^{r-1} = \left(\prod_{i=1}^{r}((r-1)\boldsymbol{\eta}_1 + (i-1)\overline{\boldsymbol{\psi}}_1\right)\tfrac{1}{r-1}\boldsymbol{\eta}_1^{r-1}.$$

Now since $\boldsymbol{\eta}_1^{r+1} = 0$ for dimension reasons, this big product expands to

$$\boldsymbol{\eta}_1^r \prod_{i=1}^{r-1} i\overline{\boldsymbol{\psi}}_1 = (r-1)!\,\boldsymbol{\eta}_1^r\overline{\boldsymbol{\psi}}_1^{r-1} = (r-1)!$$

by 5.3.4. □

5.3.11 Remark. This can also be proven by elementary analytic geometry, via Bézout's theorem in the embedded tangent hyperplane...

More generally,

5.3.12 Proposition. *Let $z_1, \ldots, z_k$ be positive integers and set $r := 1 + \sum z_i$; let $Z_1, \ldots, Z_k \subset \mathbb{P}^r$ be general hypersurfaces of degree $z_1, \ldots, z_k$. Then the number of lines through a general point on the complete intersection is*

$$\prod_{i=1}^{k} z_i!$$

Proof. Similar. □

Multiple contacts to hypersurfaces of high degree

This final example is the one that clearest shows the workings of the diagonal classes. We are going to impose various contacts on the map, at various marks. We are not really interested here in the case where the contacts are so high that the map is forced into Z — that is accomplished in a much simpler way using only one high contact, as in the above examples — so for simplicity we assume $z > 2r - 3$ from now on.

Let $\mathbf{J}_{(z)}(m_1 p_1 + \cdots + m_n p_n)$ denote the closure of the locus of irreducible maps such that the pull-back divisor $\mu^* Z$ dominates the divisor $\sum m_i p_i$. For example, $\mathbf{J}_{(z)}(2p_1 + 2p_2) \subset \overline{M}_{0,2}(\mathbb{P}^2, 1)$ is the locus of (labelled) bitangents of the plane curve $Z \subset \mathbb{P}^2$.

As in the case of only one contact, each Lambda cycle is a Cartier divisor inside the previous locus, ("previous" meaning a cycle where one of the imposed contacts is one less).

However in this case it is not just given by the vanishing of the next jet, that is applying $\boldsymbol{\varphi}_n(m_n + 1)$.

The simples case, $\mathbf{J}_{(z)}(p_1 + p_2)$, was already treated in 5.2.13 (under the name $\mathbf{J}_{12}$): We saw that it is not just the application of the divisor $\boldsymbol{\varphi}_2(1) = z\boldsymbol{\eta}_2$ to the locus $\mathbf{J}_{(z)}(p_1) = z\boldsymbol{\eta}_1$: a correction $\boldsymbol{\delta}_{12}$ for the case where the marks run together is needed:

$$\begin{aligned} \mathbf{J}_{(z)}(p_1 + p_2) &= \boldsymbol{\varphi}_2(1) \cdot \mathbf{J}_{(z)}(p_1) - \boldsymbol{\delta}_{12} \cdot \mathbf{J}_{(z)}(p_1) \\ &= \boldsymbol{\varphi}_1(1)(\boldsymbol{\varphi}_2(1) - \boldsymbol{\delta}_{12}). \end{aligned}$$

This is the general idea:

5.3.13 Lemma. *The class of the cycle* $\mathbf{J}_{(z)}(\sum m_i p_i)$ *is*

$$\mathbf{J}_{(z)}(m_1,\dots,m_n) = \prod_{i=1}^{n}\prod_{k=1}^{m_i}\big(\boldsymbol{\varphi}_i(k) - \sum_{j=1}^{i-1} m_j \boldsymbol{\delta}_{ij}\big).$$

Additionally, the formula works for Z of low degree $z < 2r-3$ but only up to $\sum m_i \leq z+1$.

Proof. The proof is an inductive description of the cycles, realising each as a divisor inside the previous. There is no doubt about the first case: $\mathbf{J}_{(z)}(p_1) = \boldsymbol{\varphi}_1(1) = z\boldsymbol{\eta}_1$. Now for the recursion step. Suppose we already know the class of $\mathbf{J}_{(z)}(\mathbf{m}) = \mathbf{J}_{(z)}(m_1,\dots,m_n)$, and intersect with the divisor $\boldsymbol{\varphi}_n(m_n+1)$, which is well-defined here inside. Set-theoretically, *either* the condition is fulfilled in the open locus of irreducible maps. In that case we land in $\mathbf{J}_{(z)}(m_1,\dots,m_n+1)$, which has got codimension 1 by the previous remark. *Or* we run into the boundary at a place where p_n coincides with one or more other marks. (In that case p_n is on a contracting twig and so the higher-order vanishing condition $\boldsymbol{\varphi}_n(m_n+1)$ is fulfilled trivially.) These components of the intersection are $\boldsymbol{\delta}_{ni}\mathbf{J}_{(z)}(\mathbf{m})$, for $i = 1,\dots,n-1$. Now we claim the multiplicity of each of these correction terms is m_i (this fact is proved in the Appendix, on page 145) getting the formula

$$\boldsymbol{\varphi}_n(m_n+1)\cdot\mathbf{J}_{(z)}(\mathbf{m}) = \mathbf{J}_{(z)}(\mathbf{m}+\mathbf{e}_n) \;+\; \sum_{i=1}^{n-1} m_i\;\boldsymbol{\delta}_{ni}\;\mathbf{J}_{(z)}(\mathbf{m}).$$

Here $\mathbf{e}_i$ denotes the vector whose only non-zero entry is the i'th which is 1. The relation gives an expression of $\mathbf{J}_{(z)}(m_1,\dots,m_n+1)$ in terms of Lambda classes of lower codimension. Now it is not difficult to expand the recursion to get the stated closed formula. □

5.3.14 Example. *Lines in* $\mathbb{P}^2$. Let $Z \subset \mathbb{P}^2$ be a general plane curve of degree $z \geq 2$. Then the number of flexes, and twice the number of bitangents are, respectively

$$\begin{aligned}\mathbf{J}_{(z)}(3) &= (z-2)\,3z\\ \mathbf{J}_{(z)}(2,2) &= (z-2)(z-3)\,(z^2+3z)\end{aligned}$$

5.3.15 Example. *Lines in* $\mathbb{P}^3$. Let $Z \subset \mathbb{P}^3$ be a general surface of degree $z \geq 4$. Then

$$\begin{aligned}
\mathbf{J}_{(z)}(5) &= (z-4)\,(35z^2-60z)\\
\mathbf{J}_{(z)}(4,2) &= (z-4)(z-5)\,(6z^3+26z^2-60z)\\
\mathbf{J}_{(z)}(3,3) &= (z-4)(z-5)\,(z^4+3z^3+29z^2-60z)\\
\mathbf{J}_{(z)}(3,2,2) &= (z-4)(z-5)(z-6)\,(z^4+9z^3+20z^2-60z)\\
\mathbf{J}_{(z)}(2,2,2,2) &= (z-4)(z-5)(z-6)(z-7)\,(2z^4+12z^3+14z^2-60z)
\end{aligned}$$

5.3.16 Example. *Lines in* $\mathbb{P}^4$. Let $Z \subset \mathbb{P}^4$ be a general hypersurface of degree $z \geq 6$. Then

$$\begin{aligned}
\mathbf{J}_{(z)}(7) &= (z-6)(735z^3-3710z^2+4200z)\\
\mathbf{J}_{(z)}(6,2) &= (z-6)(z-7)(85z^4+345z^3-3310z^2+4200z)\\
\mathbf{J}_{(z)}(5,3) &= (z-6)(z-7)(10z^5+30z^4+455z^3-3390z^2+4200z)\\
\mathbf{J}_{(z)}(4,4) &= (z-6)(z-7)(z^6+z^5+65z^4+395z^3-3350z^2+4200z)\\
\mathbf{J}_{(z)}(5,2,2) &= (z-6)(z-7)(z-8)(10z^5+115z^4+65z^3-2990z^2+4200z)\\
\mathbf{J}_{(z)}(4,3,2) &= (z-6)(z-7)(z-8)(z^6+11z^5+95z^4+115z^3-3030z^2+4200z)\\
\mathbf{J}_{(z)}(3,3,3) &= (z-6)(z-7)(z-8)(z^6+21z^5+40z^4+225z^3-3110z^2+4200z)\\
\mathbf{J}_{(z)}(4,2,2,2) &= (z-6)(z-7)(z-8)(z-9)(z^6+21z^5+125z^4-165z^3-2710z^2+4200z)\\
\mathbf{J}_{(z)}(3,3,2,2) &= (z-6)(z-7)(z-8)(z-9)(2z^6+22z^5+105z^4-115z^3-2750z^2+4200z)\\
\mathbf{J}_{(z)}(3,2,2,2,2) &= (z-6)(z-7)(z-8)(z-9)(z-10)(3z^6+33z^5+115z^4-345z^3-2470z^2+4200z)\\
\mathbf{J}_{(z)}(2,2,2,2,2,2) &= (z-6)(z-7)(z-8)(z-9)(z-10)(z-11)(5z^6+45z^5+105z^4-525z^3-2230z^2+4200z)
\end{aligned}$$

5.3.17 Remark. The polynomials have been factorised to show two things: first of all notice that unless the hypersurface has degree at least equal to the sum of the weights, of course there are no lines with the prescribed contacts. This fact is reflected by the polynomials having roots along these low degrees.

The second observation is just a curiosity: for each r, the linear term of the rest factor (i.e. the factor whose roots are distinct from the roots just described) is always the same. For $r = 2$, this coefficient is 3; for $r = 3$ it is -60 and for $r = 4$ we see 4200. This phenomenon continues for all the dimensions I have bothered to check: here is a list of these mysterious numbers, from $r = 2$ through $r = 8$:

$$3\ ,\quad -60,\quad 4\,200,\quad 535\,040,\quad 167\,650\,560,\quad 68\,497\,228\,800,\quad 40\,070\,878\,848\,000.$$

Let $Z, \widetilde{Z} \subset \mathbb{P}^3$ be two general hypersurfaces of degree z and $\tilde{z}$ respectively. (We assume that z and $\tilde{z}$ are not too small.) Then the intersection is a smooth curve X of

$$\begin{array}{rcl} \text{genus} \quad g &=& \frac{1}{2}z\tilde{z}(z+\tilde{z}-4)+1 \qquad \text{and} \\ \text{degree} \quad n &=& z\tilde{z} \end{array}$$

(cf. eg. Hartshorne [25], Ex.I.7.2).

5.3.18 Apparent nodes. In $\overline{M}_{0,3}(\mathbb{P}^3,1)$, the cycle $\mathbf{J}_{(z)}(p_1+p_2)\cdot\mathbf{J}_{(\tilde{z})}(p_1+p_2)$ is the locus of lines such that both marks map to the intersection $X = Z \cap \widetilde{Z}$. In other words, it is the (pointed) locus of secant lines to X. The cycle

$$\mathbf{J}_{(z)}(p_1+p_2)\cdot\mathbf{J}_{(\tilde{z})}(p_1+p_2)\cdot\boldsymbol{\eta}_3^3$$

is the locus of secants passing through a general point. So the lines appearing here correspond to the apparent nodes of X, but each line appears twice, since we distinguish the marks. Applying the recursion relation 1.6.3 yields the polynomial

$$= z\tilde{z}(z-1)(\tilde{z}-1),$$

which is just twice the classical formula of Schubert [43], p.322, for the numbers of apparent nodes (and of course also in agreement with the usual formula $(n-1)(n-2)/2-g$).

5.3.19 Tri-secants. Similarly, $\mathbf{J}_{(z)}(p_1+p_2+p_3)\cdot\mathbf{J}_{(\tilde{z})}(p_1+p_2+p_3)$ is the locus of tri-secant lines to X. Now take a general line in $\mathbb{P}^3$ and force the tri-secants to intersect that line. We find (in $\overline{M}_{0,4}(\mathbb{P}^3,1)$)

$$\mathbf{J}_{(z)}(p_1+p_2+p_3)\cdot\mathbf{J}_{(\tilde{z})}(p_1+p_2+p_3)\cdot\boldsymbol{\eta}_4^2 = 2z^3\tilde{z}^3-3(z^3\tilde{z}^2+z^2\tilde{z}^3)+6(z^2\tilde{z}+z\tilde{z}^2)-8z\tilde{z}.$$

Dividing by 3! this agrees with the classical formula of Schubert [43], p.323, and in terms of g and n: (Cayley, 1863)

$$\frac{(n-1)(n-2)(n-3)}{3} - g(n-2).$$

5.3.20 Quadri-secants. The cycle $\mathbf{J}_{(z)}(p_1+p_2+p_3+p_4)\cdot\mathbf{J}_{(\tilde{z})}(p_1+p_2+p_3+p_4) \subset \overline{M}_{0,4}(\mathbb{P}^3,1)$ is that of quadri-secants — again counted 4! times due to relabelling of the marks. The class is not very nice to look at:

$$\begin{aligned} 2z^4\tilde{z}^4-6(z^4\tilde{z}^3+z^3\tilde{z}^4)+6z^3\tilde{z}^3+3(z^4\tilde{z}^2+z^2\tilde{z}^4)+18(z^3\tilde{z}^2+z^2\tilde{z}^3)-26z^2\tilde{z}^2 \\ -66(z^2\tilde{z}+z\tilde{z}^2)+144z\tilde{z}, \end{aligned}$$

but when divided by 4! it agrees with the classical formula (Schubert [43], p.323) and also (Cayley, 1863):

$$\frac{(n-2)\,(n-3)^2\,(n-4)}{12} - \frac{g\,(n^2-7\,n+13-g)}{2}$$

5.3.21 Tangents cutting X once more. Similarly, $\mathbf{J}_{(z)}(2p_1+p_2)\cdot\mathbf{J}_{(\tilde{z})}(2p_1+p_2)$ is the number of tangents that cut X in yet another point. We compute

$$z^3\tilde{z}^2 + z^2\tilde{z}^3 - 2z^2\tilde{z}^2 - 6(z^2\tilde{z} + z\tilde{z}^2) + 16z\tilde{z}$$

which is in accordance with the classical formula (Salmon, 1868) $2(n-2)(n-3) + 2g(n-6)$.

5.3.22 Tri-secants to a complete intersection curve in $\mathbb{P}^4$. This time, we are considering a curve $X \subset \mathbb{P}^4$, given as the complete intersection of three hypersurfaces Z_1, Z_2, Z_3, of degree z_1, z_2, z_3 resp. The locus in $\overline{M}_{0,3}(\mathbb{P}^4, 1)$ of tri-secant lines is $\mathbf{J}_{(z_1)}(1,1,1)\cdot\mathbf{J}_{(z_2)}(1,1,1)\cdot\mathbf{J}_{(z_3)}(1,1,1)$, and its class is

$$z_1z_2z_3\big(z_1z_2z_3 - 3(z_1+z_2+z_3) + 10\big)\big(z_1z_2z_3 - 4\big).$$

Of course, X has degree $n = z_1z_2z_3$, and also one can readily compute that $2g-2 = z_1z_2z_3(z_1+z_2+z_3-5)$. Now, recalling that the formula above counts marked configurations, we should divide by 6, and then the formula agrees with the classical one of Berzolari (1895),

$$\frac{(n-2)(n-3)(n-4)}{6} - g(n-4).$$

5.3.23 Remark. See Le Barz [36] or Colley [6] for modern treatments and generalisations of these classical formulae. The formulae of Cayley and Berzolari are cited from [36].

5.A Appendix: 1-parameter families

General set-up

To determine the multiplicity we draw a small arc in the moduli space, i.e. construct a 1-parameter family running inside the locus, and then read off the divisor we are intersecting with. So we will give ourselves a 1-parameter family of n-pointed stable maps

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\mu"] \arrow[d, "\pi"'] & \mathbb{P}^r \\ B \arrow[u, bend left, "\sigma_i"] \arrow[ur, "\nu_i"'] & \end{tikzcd}$$

The base B is a smooth curve, and $\mathfrak{X}$ is a smooth surface. The morphism π is flat and proper, and the sections $\sigma_1, \ldots, \sigma_n)$ are disjoint. The restriction of μ to each fibre is a stable map $\mu_b : \mathfrak{X}_b \to \mathbb{P}^r$ of degree d. We will identify the elements of B with the maps they parametrise. For the present purposes, it is enough to consider families where B is an open set of $\mathbb{A}^1 \subset \mathbb{P}^1$ and the family is a blow-up of the trivial family $B \times \mathbb{P}^1$. We think of this surface as being an open set of $\mathbb{P}^1 \times \mathbb{P}^1$, and μ is then given by three global sections of a line bundle of degree d on each fibre.

Given a hypersurface $H \subset \mathbb{P}^r$, we consider the pull-back divisor $D := \mu^* H$ which is a curve in $\mathfrak{X}$. Assuming that the general element of the family is not in the locus $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_1(H)$, then we can read off the divisor $\boldsymbol{\eta}_1$ as $\sigma_1^* \mu^* H$, or, what is more convenient, as the intersection of D with the image of the section σ_1 (abusively denoted by the same symbol). To say that the family runs entirely inside $\boldsymbol{\eta}_1$ means that the σ_1 is a component of $\mu^* H$. Inside such a family, the locus Θ_1 is a well-defined divisor: it is given as a section of the line bundle $\sigma^*\big(\omega_\pi \otimes \mu^* \mathcal{O}_{\mathbb{P}^r}(1)\big)$. In other words, it's the intersection of the curve σ_1 with the divisor $D - \sigma_1 \equiv D + K_\pi$, which is effective since D contains σ_1 by assumption.

Similarly, if the family is such that every member has got k-order contact with H, this means that D contains σ_1 as component with multiplicity k. Then $D - k\sigma_1$ is effective and its intersection with σ_1 is the locus where the $(k-1)$'th jet bundle vanishes.

As noted, these loci often include boundary cycles that should not be counted, and we then need to know with what multiplicity to correct for them.

Now recall that these arguments work in the one-pointed case. If more sections are in play, the correct thing to do is to pull back the above argument from the family with only one section. In other words, we can just blow down all (-1)-curves of the fibres before we proceed to perform the argument. Now in practice this is even easier since the families will be constructed starting from these "minimal" families.

Families transversal to $\boldsymbol{\delta}_{12}$

The goal here is simply to construct a family whose two sections intersect transversely. For simplicity we just work with affine coordinates so consider the family of maps given by

$$\begin{array}{rcl} B \times \mathbb{A}^1 & \longrightarrow & \mathbb{P}^2 \\ (b,t) & \longmapsto & \begin{bmatrix} \varphi_1(t) \\ (t^2 - bt)\,\varphi_2(t) \\ \varphi_3(t) \end{bmatrix} \end{array}$$

Assume that $\varphi_3(0) \neq 0$ and that φ_1, φ_2 and φ_3 have no common factor. (Take $B \subset \mathbb{A}^1$ so small that the map has no base points.) Now put two sections on the family setting

$$\begin{array}{rcl} \sigma_1(b) & := & (b,0) \\ \sigma_2(b) & := & (b,b) \end{array}$$

Since the two section intersect at $(0,0)$, this is no longer a stable family, but since the intersect is transverse, a single blow-up yields a stabilisation.

Proof of the multiplicities of Lemma 5.2.16. Let $H \subset \mathbb{P}^2$ be the line with equation $x = 0$ and let $H' \subset \mathbb{P}^2$ be given by $y = 0$. Consider the family where we furthermore require $\varphi_1(t)$ to have a factor t. Then for all members in the family we have p_1 mapping to $(0,0)$, $(= H \cap H')$, while p_2 maps to H. In other words, the whole family is inside the locus $\boldsymbol{\eta}_1(H') \cap \Lambda_{12}(H)$.

We now impose the condition that p_2 also maps to H'. That is, we are looking at the intersection σ_2 with μ^*H'. Now $\mu^*H' = \sigma_1 + R$ where R is some residual curve not passing through $(0,0)$. So the intersection is the transversal one of σ_2 with σ_1. This establishes that $\boldsymbol{\delta}_{12}$ appears with multiplicity 1 in the intersection $\boldsymbol{\eta}_2 \cap \big(\boldsymbol{\eta}_1(H') \cap \Lambda_{12}(H)\big)$. As claimed in 5.2.16. Note that descarting the information about H, the same argument establishes the multiplicity 1 of 5.2.13.

To be correct, the argument runs up in the blow-up, and here the two sections are disjoint, so here we are actually asking for the intersection of $\tilde{\sigma}_2$ with $\tilde{\mu}^*H' = E + \tilde{\sigma}_1 + \widetilde{R}$: the exceptional divisor appears with multiplicity 1 since $(0,0)$ is a simple point of μ^*H' downstairs. $\square$

Now it is easy to prove the multiplicities claimed in Lemma 5.3.13.

Proof of the multiplicities of Lemma 5.3.13. Let $L \subset \mathbb{P}^2$ be the line with equation $x = 0$. We want a family of maps having contact of order m_i with L at the i'th mark. And want to see what happens when two of the marks, say p_1 and p_2 come together. Consider the parametrisation

$$\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t)\prod(t-\lambda_i)^{m_i} \\ \varphi_3(t) \end{bmatrix}.$$

Putting mark p_i at $t = \lambda_i$, this map is in the locus. Now put the map in a family

$$\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t)(t-\lambda_1)(t-b\lambda_2)\prod(t-\lambda_i)^{m_i} \\ \varphi_3(t) \end{bmatrix}$$

and let b approach 0. So all sections are constant, except $\sigma_2(b) := (b, b\lambda_2)$. Sections σ_1 and σ_2 intersect transversely at $(0,0)$. (We leave it at that, since anyway the argument is going to be carried out without the blow-up...)

Now impose one contact higher on p_1. That is, look in $\mathfrak{X}$ for the intersection of σ_1 with $D - m_1\sigma_1$. Since $D = \sum m_i\sigma_i$, and the only one of the sections that intersects σ_1 is σ_2 we see that the intersection is $m_2\sigma_1 \cap \sigma_2$. The multiplicity is m_2 as claimed in 5.3.13 $\square$

Remark. Note that there is a subtlety we have avoided: when there are more than two marks, the divisor $\boldsymbol{\delta}_{12}$ is really made up of various components, and those with more than just p_1 and p_2 together on the contracting twig (say also p_3) should have higher multiplicity (in this case $m_2 + m_3$). However, this is automatic since that component is counted also in $\boldsymbol{\delta}_{13}$ where we get the missing contribution m_3. The important thing to note is that since $\boldsymbol{\delta}_{12}$ is defined to be invariant under pull-back we can just carry out the argument in the space or family with just two marks p_1 and p_2 and then pull back the result.

Finally we need a variant of these arguments to establish the multiplicity 1 of 4.3.5.1

Multiplicity 1 in Formula 4.3.5.1. Consider the family of maps

$$\begin{array}{rcl} B \times \mathbb{A}^1 & \longrightarrow & \mathbb{P}^2 \\ (b,t) & \longmapsto & \begin{bmatrix} (t^2-b^2)\varphi_1(t) \\ t^3\,\varphi_2(t) \\ \varphi_3(t) \end{bmatrix} \end{array}$$

Assume there are no base points and require furthermore that φ_1 has a constant term but no linear term. Then for every value of b the corresponding map is a cusp. Let $\sigma_1(b) := (b,0)$ be the section, it's the one that points to the cusp. So this family runs inside K_1.

Put another section on the family setting $\sigma_2(b) := (b,b)$. Then the whole family runs inside the locus of maps with $\mu(p_2) \in L$ (equation $x=0$). (The two sections are not disjoint: they meet transversely at $(0,0)$.)

Now impose the condition of second contact. That is, compute the intersection of σ_2 with $\mu^*L - \sigma_2$. Now μ^*L consists of two curves: σ_2 and another "diagonal line" (equation $t+b$), and some residual curve not passing through the point $(0,0)$. So the sought for intersection is just the transversal one at $(0,0)$ which shows that in the divisor of second contact, $\boldsymbol{\delta}_{12}$ counts once. □

Families transversal to Π

It is convenient now to use the whole projective fibres of the family, so we start out with the trivial family $B \times \mathbb{P}^1$ with projection π to $B \subset \mathbb{A}^1$ and a rational map μ to $\mathbb{P}^2$ whose restriction to a general fibre is of degree d. We are interested in the situation where μ has got two simple base points in the central fibre. Simple means that blowing up the points resolves the map. Suppose the two base points are of degree $n \geq 1$ and $m \geq 1$ respectively. This means that the resolved map, when restricted to the two exceptional divisors has degree n and m, respectively. If $n+m=d$ then the resolved map contracts the strict fibre F to a point in $\mathbb{P}^2$. In other words, the central fibre is of type Π.

Construction of a family. Here is a rational map that satisfies these requirements:

$$\begin{array}{rcl} B \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ \big(b,[s:t]\big) & \longmapsto & \begin{bmatrix} b^d\,\varphi_1(s,t) \\ b^d\,\varphi_2(s,t) + b\,s^{m-1}t^{n-1}q(s,t) \\ s^m t^n + b^m t^d + b^n s^d \end{bmatrix}. \end{array}$$

Here φ_1 and φ_2 are homogeneous polynomials of degree d, and q is homogeneous of degree 2 and with roots not equal to $[0:1]$ or $[1:0]$.

Restricted to the central fibre ($b = 0$), the map is

$$\begin{bmatrix} 0 \\ 0 \\ s^m t^n \end{bmatrix}$$

which is undefined at $[0:1]$ and $[1:0]$.

The purpose of the second term in the second coordinate (the one linear in b) is to provide the "multiplicity 1", as we shall see in a moment.

But let us first check that the base points are simple: By the symmetry of the situation it is sufficient to check the base point at $[s:t] = [1:0]$, so set $s = 1$. Then the map reads

$$\begin{bmatrix} b^d\ \varphi_1(t) \\ b^d\ \varphi_2(t) + b\ t^{n-1} q(t) \\ t^n + b^m t^d + b^n \end{bmatrix}.$$

Now blow up $(b,t) = (0,0)$, setting $\tilde{b}\ t = \tilde{t}\ b$. Let us first look in the chart where $\tilde{t} = 1$, so substitute $b = \tilde{b}\ t$. We get

$$t^n \begin{bmatrix} \tilde{b}^d\ t^m\ \varphi_1(t) \\ \tilde{b}^d\ t^m\ \varphi_2(t) + \tilde{b}\ q(t) \\ 1 + \tilde{b}^m\ t^{2m} + \tilde{b}^n \end{bmatrix}.$$

Here the factor t^n corresponds to the exceptional divisor, and the vector is the resolved map in this chart. There are no more base points in the strict fibre F (equation $\tilde{b} = 0$).

In the chart $\tilde{b}$, we get, substituting $t = \tilde{t}\ b$:

$$b^n \begin{bmatrix} b^m\ \varphi_1(\tilde{t}b) \\ b^m\ \varphi_2(\tilde{t}b) + \tilde{t}^{n-1} q(\tilde{t}b) \\ \tilde{t}^n + b^{2m}\ \tilde{t}^d + 1 \end{bmatrix}$$

where this time b is the equation of the exceptional divisor. Here the map reads

$$\begin{bmatrix} 0 \\ \tilde{t}^{n-1} q(0) \\ \tilde{t}^n + 1 \end{bmatrix}$$

and since $q(0) \neq 0$ the map is well-defined.

Using this construction it is easy to establish the "multiplicity one" of Lemma 5.2.16 and Lemma 4.4.4:

Proof of multiplicity 1 at Π *in Lemma 5.2.16.* We need a family running entirely inside the locus $\boldsymbol{\eta}_1\Lambda_{12}$, that is: the second mark maps to a given line say $(x = 0)$, while the first mark maps simultaneously to two given lines where the other line L is given by $y = 0$.

We make our family satisfy this condition by letting $\varphi_1(t)$ have a factor $(t-1)(t-2)$, and letting $\varphi_2(t)$ and $q(t)$ have a factor $(t-1)$. Suppose furthermore the other factor of $q(t)$ is $(t-\lambda)$ with $\lambda \neq 2$. Now put the following two constant sections on the family:

$$\begin{aligned} \sigma_1(b) &:= (b,1) \\ \sigma_2(b) &:= (b,2). \end{aligned}$$

Then σ_1 maps constantly to $(0,0) \in \mathbb{P}^2$ while σ_2 maps to the vertical line $x = 0$.

We need the intersection of σ_2 with μ^*L. Now, μ^*L has equation $b \cdot (t-1) \cdot r(b,t)$ where the rest $r(b,t)$ is a curve that does not pass the point $(0,1)$. That is, it contains F (equation $b = 0$) as component with multiplicity 1. Now clearly σ_2 intersects μ^*L transversely at $(0,1)$, and nowhere else in F. This proves that Π_{12} appears with multiplicity 1 in $\boldsymbol{\eta}_2 \cdot \boldsymbol{\eta}_1\Lambda_{12}$. □

Proof of multiplicity 1 in Lemma 4.4.4. This time we are running in a family all of whose members are tangent to the line L (with equation $y = 0$). This is achieved in the family constructed above by letting $\varphi_2(t)$ and $q(t)$ have a common factor $(t-1)^2$ and defining the first section to be

$$\sigma_1(b) := (b,1).$$

Now the pull-back of L is $\mu^*L = 2\sigma_1 + F + R$, where R is a curve not passing through the point $(0,1)$. The locus of third order contact (before the correction we are determining) is read off in the family as the intersection of σ_1 with $\mu^*L - 2\sigma_1 = F + R$. This is just the transverse intersection at $(1,0)$. Thus, Π_1 has multiplicity 1 in the intersection $(\boldsymbol{\eta}_1 + 2\overline{\boldsymbol{\psi}}_1) \cap \Theta_1$. □

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