# Tangency quantum cohomology and characteristic numbers

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**Abstract**

This work establishes a connection between gravitational quantum cohomology and enumerative geometry of rational curves (in a projective homogeneous variety) subject to conditions of infinitesimal nature, like for example tangency. The key concept is that of modified psi classes, which are well suited for enumerative purposes and substitute the tautological psi classes of 2D gravity. The main results are two systems of differential equations for the generating function of certain top products of such classes. One is topological recursion while the other is WDVV. In both cases however, the background metric is not the usual Poincaré metric but a certain deformation of it, which surprisingly encodes all the combinatorics of the peculiar way modified psi classes restrict to the boundary. This machinery is applied to various enumerative problems, among which characteristic numbers in any projective homogeneous variety, characteristic numbers for curves with cusp, prescribed triple contact, or double points.

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## 1 Introduction

This article exposes the main ideas and results of the theory of *tangency quantum cohomology*, developed in the author's PhD thesis (Universidade Federal de Pernambuco, 2000), where detailed exposition, complete proofs, and a wealth of examples can be found.

1.1 Brief overview of the scientific context. The most classical problems of enumerative geometry are those of counting curves subject to incidence and tangency conditions; the answers to these questions are called *characteristic numbers.* During the last decade, the discipline has undergone a true revolution instilled by ideas from theoretical physics. The starting point was the discovery of M. Kontsevich (Kontsevich-Manin, 1994) that the numbers of rational curves in $\mathbb{P}^2$ subject to incidence conditions are determined in all degrees by a set of differential equations which amount to the associativity of the *quantum product*, a new structure on the cohomology space of $\mathbb{P}^2$, constructed via *stable maps.* Kontsevich's solution to this century-old problem is unparalleled in elegance — after all, associativity is one of the most basic concepts in all of mathematics.

Since then, stable maps have been used to attack also the more difficult problem of characteristic numbers, and various solutions have been found for rational curves in $\mathbb{P}^2$ (or $\mathbb{P}^r$) (Pandharipande, 1999, Ernström-Kennedy, 1998 & 1999). These solutions are quite complicated though, and the theories underlying them have little connection to physics.

The present work re-establishes that connection, making the powers of *gravitational quantum cohomology* available to enumerative geometry. This not only solves the rational characteristic number problem in a conceptually much simpler way, for *any* projective homogeneous variety, but also allows for a variety of new applications. The most surprising of the results is the construction of a *tangency quantum product.* This is a deformation of the usual quantum product based on *modified psi classes.* The associativity of this product provides a solution to the characteristic number problem which is as simple as Kontsevich's solution to the question involving only incidence conditions.

## 2 Modified psi classes and diagonal classes

2.1 SET-UP. Let $X$ be a projective homogeneous projective variety over the field of complex numbers. Fix a basis $T_0, \ldots, T_r$ for the cohomology space $H = H^*(X, \mathbb{Q})$.

All constructions take place in the stack $\overline{M}_{0,n}(X, \beta)$ of genus-zero $n$-pointed stable maps to $X$ of class $\beta$. Let $\nu_i : \overline{M}_{0,n}(X, \beta) \to X$ denote the $i$'th evaluation morphism. Pull-backs of cohomology classes of $X$ via evaluation morphisms are called evaluation classes, and top products of evaluation classes are called Gromov-Witten invariants. The reader is referred to Fulton-Pandharipande (1997) for basic properties of stable maps, Gromov-Witten invariants, and quantum cohomology.

2.2 PSI CLASSES. The tautological psi class $\boldsymbol{\psi}_i$ is the first Chern class of the line bundle on $\overline{M}_{0,n}(X, \beta)$ whose fibre at a moduli point $[\mu : C \to X]$ is the cotangent line of $C$ at the $i$'th marked point. The inclusion of psi classes in the Gromov-Witten invariants corresponds in theoretical physics to introducing gravity into the quantum field theories (Witten, 1991). The intersection theory of the psi classes goes under the name *gravitational quantum cohomology* (Manin, 1999). The tautological psi class is not compatible with pull-back along forgetful morphisms, and for this reason it is not directly interpretable in enumerative geometry. To this end, a modification of the psi class is introduced.

2.3 MODIFIED PSI CLASSES. The *modified psi class* $\overline{\boldsymbol{\psi}}_i$ is defined for $\beta > 0$ as the pull-back from the one-pointed space $\overline{M}_{0,1}(X, \beta)$ of the usual psi class $\boldsymbol{\psi}_i$. The motivation for this definition is that many conditions from enumerative geometry allow easy expressions in terms of them, cf. §§5 and 6 below.

2.4 DIAGONAL CLASSES. The $ij$'th *diagonal class* $\boldsymbol{\delta}_{ij}$ is by definition the sum of all boundary divisors having marks $p_i$ and $p_j$ together on a contracting twig. The name is justified by the following basic properties enjoyed by the diagonal classes: First, if $\pi_i$ is the morphism that forgets $p_i$ then $\pi_{i*}\boldsymbol{\delta}_{ij} = 1$, the fundamental class. And second,

$$\begin{aligned} \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{ik} &= \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{jk} \\ -\boldsymbol{\delta}_{ij}^2 = \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_i &= \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_j. \end{aligned}$$

It follows that every top product involving evaluation classes, modified psi classes, and diagonal classes can be reduced to products involving only evaluation classes and modified psi classes.

2.5 KEY FORMULA. The diagonal classes appear as correction terms when restricting a modified psi class to a boundary divisor $D$ both of whose twigs are of positive degree. If $D$ is the image of the gluing morphism

$$\rho_D : \overline{M}_{0,n'+1}(X,\beta') \times_X \overline{M}_{0,n''+1}(X,\beta'') \longrightarrow \overline{M}_{0,n}(X,\beta),$$

then

$$\rho_D^* \overline{\boldsymbol{\psi}}_i = \overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{i\bullet},$$

where $\bullet$ denotes the gluing mark.

## 3 The tangency quantum potential and topological recursion

3.1 THE TANGENCY QUANTUM POTENTIAL. We define an *enumerative descendant* to be a top product of modified psi classes and evaluation classes, and set (for $\gamma_i \in H$)

$$\langle\, \overline{\tau}_{k_1}(\gamma_1) \cdots \overline{\tau}_{k_n}(\gamma_n) \,\rangle_\beta := \int \overline{\boldsymbol{\psi}}_1^{k_1} \cup \nu_1^*(\gamma_1) \cup \cdots \cup \overline{\boldsymbol{\psi}}_n^{k_n} \cup \nu_n^*(\gamma_n) \cap [\overline{M}_{0,n}(X,\beta)].$$

Of particular interest are the *first enumerative descendants*, i.e., those where the exponent of each modified psi class is at most 1:

$$\langle\, \overline{\boldsymbol{\tau}}_0^{\mathbf{a}}\, \overline{\boldsymbol{\tau}}_1^{\mathbf{b}} \,\rangle_\beta := \langle\, \prod_{k=0}^{r} (\overline{\tau}_0(T_k))^{a_k} (\overline{\tau}_1(T_k))^{b_k} \,\rangle_\beta .$$

The generating function of these invariants is called the *tangency quantum potential*:

$$\Gamma(\mathbf{x},\mathbf{y}) = \sum_{\beta>0} \sum_{\mathbf{a},\mathbf{b}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \frac{\mathbf{y}^{\mathbf{b}}}{\mathbf{b}!} \langle\, \overline{\boldsymbol{\tau}}_0^{\mathbf{a}}\, \overline{\boldsymbol{\tau}}_1^{\mathbf{b}} \,\rangle_\beta .$$

Here $\mathbf{x} = (x_0, x_1, \ldots, x_r)$ and $\mathbf{y} = (y_0, y_1, \ldots, y_r)$ are formal parameters, and we employ the usual multi-index notation $\mathbf{a}! = a_0! a_1! \cdots a_r!$, and $\mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_r^{a_r}$. The variables $\mathbf{x}$ are the usual formal variables from quantum cohomology, so when $\mathbf{y}$ is set to zero, then $\Gamma$ reduces to the usual (quantum part of the) genus-zero Gromov-Witten potential.

3.2 DEFORMATION OF THE POINCARÉ METRIC. While the usual quantum potential is based on the trace map $\int_{\mathbf{0}} : H \to \mathbb{Q}$ (integration over the fundamental class), and the Poincaré metric $g_{ij} = \int_{\mathbf{0}} T_i \cup T_j$, the tangency quantum potential relates more naturally to a deformation of these structures, a certain "metric" with values in $\mathbb{Q}[[\mathbf{y}]]$. The deformed trace map is

$$\int_{\mathbf{y}} \mathbf{z} \;:=\; \sum_{\mathbf{s}} \frac{(-2\mathbf{y})^{\mathbf{s}}}{\mathbf{s}!} \int_{\mathbf{0}} \mathbf{T}^{\mathbf{s}} \cup \mathbf{z}, \qquad\qquad \mathbf{z} \in H,$$

and the new metric is given by

$$\gamma_{ij} := \int_{\mathbf{y}} T_i \cup T_j.$$

Let $(\gamma^{ij})$ denote the inverse matrix to $(\gamma_{ij})$.

3.3 TOPOLOGICAL RECURSION. Since $\overline{\boldsymbol{\psi}}_i$ admits an expression in terms of boundary divisors, a sort of topological recursion works. Note however that in contrast to the topological recursion for the usual psi classes (Witten, 1991), the Key Formula 2.5 introduces a lot of diagonal classes which must then be eliminated as explained in 2.4. Miraculously, the deformed metric encodes all this combinatorics of diagonal classes. Put $\Gamma_{x_i} := \frac{\partial}{\partial x_i}\Gamma$ and $\Gamma_{(x_i x_j)} := \sum_{k=0}^{r} g_{ije} g^{ef}\, \Gamma_{x_f}$ (the directional derivative with respect to $T_i \cup T_j$). Then, *The tangency potential satisfies the differential equations*

$$\Gamma_{y_k x_i x_j} = \Gamma_{x_k (x_i x_j)} - \Gamma_{(x_k x_i) x_j} - \Gamma_{(x_k x_j) x_i} + \sum_{e,f} \Gamma_{x_k x_e}\, \gamma^{ef}\, \Gamma_{x_f x_i x_j}.$$

These equations determine all first enumerative descendants from the Gromov-Witten invariants.

## 4 The tangency quantum product and Frobenius structure

4.1 THE TANGENCY QUANTUM PRODUCT. Define a multiplication on $H \otimes \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ by the rule

$$T_i * T_j := T_i \cup T_j + \sum_{e,f} \Gamma_{ije}\, \gamma^{ef}\, T_f.$$

Here and in the sequel, for simplicity, we put $\Gamma_i = \Gamma_{x_i}$.

Now the crucial result is that this *tangency quantum product* is associative. The proof follows the proof of associativity of the usual quantum product (cf. Fulton-Pandharipande, 1997), and again the deformed metric enters in the quadratic term to encode the combinatorics of the diagonal classes. (In the special case $X = \mathbb{P}^2$, this product was previously constructed by ad hoc methods in Ernström-Kennedy (1999).)

4.2 FROBENIUS STRUCTURE. From the observation $T_i \cup T_j = \sum \gamma_{ije} \gamma^{ef} T_f$, it follows that if we extend the tangency quantum potential to $\beta = 0$ setting

$$\Phi \ := \ \Gamma \ + \ \sum_{i,j,k} \frac{x_i x_j x_k}{3!} \int_{\mathbf{y}} T_i \cup T_j \cup T_k,$$

then the tangency quantum product can be written

$$T_i * T_j = \sum_{e,f} \Phi_{ije} \ \gamma^{ef} \ T_f,$$

and the associativity equations become

$$\sum_{e,f} \ \Phi_{ije} \ \gamma^{ef} \ \Phi_{fk\ell} = \sum_{e,f} \ \Phi_{jke} \ \gamma^{ef} \ \Phi_{fi\ell}.$$

It follows readily from these constructions that *The cohomology* $\mathbb{Q}[[\mathbf{y}]]$-*module* $H[[\mathbf{y}]]$ *with bilinear non-degenerate pairing* $\gamma : H[[\mathbf{y}]] \otimes H[[\mathbf{y}]] \to \mathbb{Q}[[\mathbf{y}]]$, *equipped with the tangency quantum potential* $\Phi \in \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ *constitutes a formal Frobenius manifold over* $\mathbb{Q}[[\mathbf{y}]]$. (See Manin (1999) for definitions and theory of Frobenius manifolds.)

## 5 Characteristic numbers

5.1 TANGENCY CONDITIONS. Let $Z \subset X$ be a very ample hypersurface, and denote by $\boldsymbol{\eta}_i$ the pull-back of the class of $Z$ along the $i$'th evaluation morphism. Now one can show that the locus of maps which are tangent to $Z$ at the $i$'th marked point is of class

$$\boldsymbol{\eta}_i(\boldsymbol{\eta}_i + \overline{\boldsymbol{\psi}}_i).$$

The proof of this involves a jet-bundle construction, Porteous' formula, and a transversality argument. Furthermore, the top intersections of such loci are transverse for general hypersurfaces, so that the characteristic numbers are expressed as top products of modified psi classes and evaluation classes.

5.2 CHARACTERISTIC NUMBER POTENTIAL AND COORDINATE CHANGES. Let $G$ be the generating function for the characteristic numbers of rational curves in $X$. Now a combinatorial argument shows that $G$ is a linear change of coordinates of $\Gamma$. In this way, an easy application of the chain rule translates the equations of 3.3 and 4.1 into differential equations for $G$, completely determining the characteristic numbers form the numbers involving only incidence conditions (i.e., the Gromov-Witten invariants).

5.3 EXAMPLE: PLANE CURVES. To give the idea, let $N_d(a,b,c)$ be the number of rational plane curves of degree $d$, passing through $a$ points, tangent to $b$ lines, and tangent to $c$ lines at given points (with $a+b+2c=3d-1$). The characteristic number potential

$$G(s,u,v,w)=\sum_{d>0}\exp(ds)\sum_{a,b,c}\frac{u^a}{a!}\frac{v^b}{b!}\frac{w^c}{c!}\,N_d(a,b,c)$$

is related to the tangency quantum potential $\Gamma$ by

$$G(s,u,v,w)=\Gamma(x_1,x_2,y_1,y_2),$$

subject to the change of variables:

$$\begin{array}{lll} x_0=0 & x_1=s & x_2=u+v \\ y_0=0 & y_1=v & y_2=w \end{array}$$

The deformed metric reads

$$(\gamma^{ef})=\begin{pmatrix}0&0&1\\0&1&2y_1\\1&2y_1&2y_1^2+2y_2\end{pmatrix}=\begin{pmatrix}0&0&1\\0&1&2v\\1&2v&2v^2+2w\end{pmatrix},$$

and 3.3 translates into the following the differential equation satisfied by $G$.

$$G_{vs}=G_{us}-G_u+\tfrac{1}{2}G_{ss}G_{ss}+2vG_{ss}G_{us}+(v^2+w)G_{us}G_{us}.$$

## 6 Further applications

6.1 CODIMENSION-2 CORRECTIONS. In many further applications, curves with a "marked node" contribute where they shouldn't. Let $\Pi_1$ denote the sum of all codimension-2 boundary cycles whose middle twig has degree 0 and carries the mark $p_1$, while the two other twigs have positive degree. The potential for "first enumerative descendants integrated over $\Pi_1$" is related to $\Gamma$ through a quadratic differential equation, since an integral over $\Pi_1$ can be expressed as a product of integrals over the moduli spaces corresponding to its twigs. (A similar observation applies to the potential of "first enumerative descendants integrated against a single factor $\overline{\boldsymbol{\psi}}_1^2$": this follows from one step of topological recursion.)

6.2 Cuspidal curves. Taking the above boundary classes into account, the techniques of modified psi classes also yield the characteristic numbers of cuspidal rational curves (in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$). For example, for $\mathbb{P}^2$, the locus of curves having cusp at the first mark is shown to be of class

$$3\boldsymbol{\eta}_1^2 + 3\boldsymbol{\eta}_1\overline{\boldsymbol{\psi}}_1 + \overline{\boldsymbol{\psi}}_1^2 - \Pi_1.$$

In cuspidal environment, the tangency formula of 5.1 fails, because the cusp mapping to the given line will count as tangency, although it is not a limit of a honest tangency. The correct formula in this case is

$$\boldsymbol{\eta}_i(\boldsymbol{\eta}_i + \overline{\boldsymbol{\psi}}_i - \boldsymbol{\delta}_{1i}).$$

Now the characteristic number problem is solved like this. First, the diagonal classes are expanded using the formulae of 2.4, and next the differential equations for the enriched potentials of 6.1 are used to eliminate $\Pi_1$ and $\overline{\boldsymbol{\psi}}_1^2$. The final result of these manipulations is three differential equations which effectively determine the cuspidal characteristic numbers from the usual ones.

6.3 Other examples. Many other conditions allow an easy description in terms of the classes introduced above. For example, the locus of maps which have triple contact to a given hypersurface $V \subset \mathbb{P}^r$ is of class $\boldsymbol{\eta}_i(\boldsymbol{\eta}_i + \overline{\boldsymbol{\psi}}_i)(\boldsymbol{\eta}_i + 2\overline{\boldsymbol{\psi}}_i) - [\mathrm{I}_i]_3 - \boldsymbol{\eta}_i\Pi_i$. (Here $[\mathrm{I}_i]_3$ denotes a certain correction consisting of two-twig maps with a linear twig mapping into $V$, but it turns out never to give any contribution.) The locus of maps which are secant to a codimension-2 plane (at $p_1$ and $p_2$) is of class $\boldsymbol{\eta}_1^2\boldsymbol{\eta}_2^2 - \boldsymbol{\delta}_{12}\big(2\boldsymbol{\eta}_1^3 + \boldsymbol{\eta}_1^2\overline{\boldsymbol{\psi}}_1\big) - \boldsymbol{\eta}_1^2\Pi_{12}$. (For the plane, this is the locus of curves with specified double point.) Cutting this locus with $\boldsymbol{\delta}_{12}$ forces the two secant points together, and we find the class of being tangent to the codimension-2 plane.

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