Nodal curves on surfaces

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1 Severi varieties

FIRST LECTURE, 09/02/99

Throughout, let S be a surface, and we will always assume it smooth, irreducible and projective. Most of the time we will also assume it to be *regular*, meaning that $q := h^1(S, \mathcal{O}_S) = 0$. Let $\mathcal{L}$ be a complete linear system on S, such that the general curve $C \in \mathcal{L}$ is smooth and irreducible. Mostly we will also assume the system is regular, in the sense that $h^1(S, \mathcal{O}_S(C)) = 0$. Let p_a denote the arithmetic genus of the curves in $\mathcal{L}$.

Let δ be a non-negative integer, $0 \leq \delta \leq p_a$, and set

$$\mathcal{V}_{\mathcal{L},\delta} := \{C \in \mathcal{L} \mid C \text{ is irreducible and has } \delta \text{ nodes and no other singularities}\}^-,$$

the Severi variety of type $(\mathcal{L}, \delta)$ on S. (Here $\{\ \}^-$ indicates the closure.) We have required $\delta \leq p_a$ to ensure that the general curve in $\mathcal{V}_{\mathcal{L},\delta}$ is irreducible. That is not a serious restriction: understanding the Severi varieties of irreducible curves leads also to a complete understanding of reducible curves...

1.1 Example. — Set $S = \mathbb{P}^2$, and take $\mathcal{L}$ to be the complete linear system $\mathcal{L}_n$ of curves of degree n. Then $p_a = \binom{n-1}{2}$.

1.2 Questions. — There are several natural and basic questions:
(i) Existence: Is $\mathcal{V}_{\mathcal{L},\delta}$ non-empty? and when?
(ii) Irreducibility: When is $\mathcal{V}_{\mathcal{L},\delta}$ irreducible?
(iii) Dimension: Compute the dimension of all components of $\mathcal{V}_{\mathcal{L},\delta}$.
(iv) Enumeration: Compute the Hilbert polynomial of $\mathcal{V}_{\mathcal{L},\delta}$ in $\mathcal{L}$

In the case of $S = \mathbb{P}^2$, questions (i), (ii) and (iii) have complete answers. (iv) is known only when $\mathcal{V}_{\mathcal{L},\delta}$ is a finite set of points. The classical result is

Theorem 1 (Severi (1921), Harris (1985)) — *When $S = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{L}_n$, then $\mathcal{V}_{\mathcal{L},\delta}$ is non-empty, and irreducible of codimension δ in $\mathcal{L}$.*

The theorem was stated by Severi, and his proofs for part (i) and (iii) are rigourous, even when translated into modern language. His proof of (ii) was completely wrong. It was only proven by Harris.

Let us sketch Severi's approach to the dimensionality and the non-emptiness. Everything is based on deformation theory.

Take a general curve C in $\mathcal{V}_{\mathcal{L},\delta}$, i.e. a curve with δ nodes. Let Z denote the set of nodes.

We will study the Zariski tangent space $T_{\mathcal{V}_{\mathcal{L},\delta},C}$ of $\mathcal{V}_{\mathcal{L},\delta}$ at C. In particular, the dimension of $T_{\mathcal{V}_{\mathcal{L},\delta},C}$ will give an upper bound for the dimension of $\mathcal{V}_{\mathcal{L},\delta}$.

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_S(C)_{|C} \longrightarrow 0,$$

where the quotient is also the normal bundle $N_{C,S}$ of C in S. The induced long exact cohomology sequence starts like this

$$0 \longrightarrow H^0(\mathcal{O}_S) \longrightarrow H^0(\mathcal{O}_S(C)) \xrightarrow{\tau} H^0(\mathcal{O}_S(C)_{|C}) \longrightarrow 0,$$

where the last zero module is due to the regularity...

Now the tangent space of the whole space $\mathcal{L}$ is

$$T_{\mathcal{L},C} \simeq \frac{H^0(\mathcal{O}_S(C))}{H^0(\mathcal{O}_S)} \simeq \operatorname{Im}\tau = H^0(N_{C,S})$$

(The inclusion $\subset$ always hold; the equality is by chance(?).)

Now note that $\mathcal{L} = \mathbb{P}\big(H^0(\mathcal{O}_S(C))\big)$. Let $f_0 \in H^0(\mathcal{O}_S(C))$ be the section defining C. A tangent $f \in T_{\mathcal{V}_{\mathcal{L},\delta},C}$ is represented by an $f \in H^0(\mathcal{O}_S(C))$ modulo the class of f_0.

$f_0 = t\ f = 0$.

$T_{\mathcal{V}_{\mathcal{L},\delta},C} \subset T_{\mathcal{L},C}$

$T_{\mathcal{V}_{\mathcal{L},\delta},C} = \frac{H^0(\mathcal{I}_Z(C))}{<C>}$.

$$0 \to \mathcal{O}_S \longrightarrow \mathcal{I}_Z(C) \longrightarrow \mathcal{I}_Z(C)_{|C} \to 0,$$

Now consider the first order deformations of the curve C, that is, look aat the family $f(x,y,t) = 0$, where $t^2 = 0$, and x, y are the local parameters of S at the singular point of C, and such that for $t = 0$ we recover f_0, i.e. $f(x,y,0) = f_0$.

We want to deform only in such a way that the singularities are preserved

$x(t), y(t)$

$f(x(t), y(t), t) \equiv 0$

$f_x(x(t), y(t), t) = 0$

$f_y(x(t), y(t), t) = 0$

$$f_x(x(t), y(t), t) \cdot x' + f_y(x(t), y(t), t) \cdot y' + f_t(x(t), y(t), t) = 0$$

Now since both f_x and f_y vanish at the point, this implies that $f_t = 0$. That is, f_t passes through the point.

$f(x,y,t) = f_0(x,y) + f_t(x,y) \cdot t$

f_t is the tangent vector.

Deformation theory provides us with a sheaf on the deformed variety. The cohomology of this sheaf provides us with tangents, and with information about the obstruction to deformation.

Obstructions to deforming with nodes:

$$H^1(\mathcal{I}_Z(C)).$$

If this group is zero then $\mathcal{V}_{\mathcal{L},\delta}$ is smooth and its dimension is the dimension of the tangent space.

Definition. — The Severi variety is said to be *regular* at C when $H^1(\mathcal{I}_Z(C)) = 0$, or equivalently: when $\mathcal{V}_{\mathcal{L},\delta}$ is smooth of dimension $h^0(\mathcal{I}_Z(C)) - 1$.

Let us compute this number: From the short exact sequence

$$0 \longrightarrow \mathcal{I}_Z(C) \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_S(C)_{|Z} \longrightarrow 0$$

we get the long exact sequence

$$0 \longrightarrow H^0(\mathcal{I}_Z(C)) \longrightarrow H^0(\mathcal{O}_S(C)) \xrightarrow{z} \mathbb{C}^\delta \longrightarrow H^1(\mathcal{I}_Z(C) \longrightarrow 0,$$

(where once again the last zero is due to the regularity condition). It follows that

$$h^0(\mathcal{I}_Z(C)) - h^0(\mathcal{O}_S(C)) + \delta - h^1(\mathcal{I}_Z(C) = 0$$

or equivalently,

$$h^0(\mathcal{I}_Z(C)) - 1 = \dim \mathcal{L} - \delta + h^1(\mathcal{I}_Z(C)).$$

Let $c_{\mathcal{L},z}$ denote the dimension of $\operatorname{Im} z$. It is the number of independent conditions imposed on $\mathcal{L}$. Now $\delta = h^1(\mathcal{I}_Z(C)) + c_{\mathcal{L},z}$, so when $h^1(\mathcal{I}_Z(C)) = 0$, the result follows.

In general, the dimension of $\mathcal{V}_{\mathcal{L},\delta}$ at C is only less or equal than $\dim T_{\mathcal{V}_{\mathcal{L},\delta},C} = \dim \mathcal{L} - c_{\mathcal{L},z}$.

On the other hand, general non-sense of deformation theory tells us that the dimension of $\mathcal{V}_{\mathcal{L},\delta}$ at C is at least $h^0(\mathcal{I}_Z(C)) - 1 - h^1(\mathcal{I}_Z(C)) = h^0(\mathcal{O}_S(C)) - 1 - \delta = \dim \mathcal{L} - \delta$. That is, δ nodes impose at most δ independent conditions.

Let $\widetilde{C}$ be the normalisation of C, i.e., blow up S along Z, and let $\widetilde{C}$ be the strict transform of C. Let E be the union of the exceptional divisors:

$$\begin{tikzcd}
& \widetilde{C} \arrow[dd] \arrow[dr, hook] & \\
E \arrow[rr, hook] \arrow[d] & & \widetilde{S} \arrow[d, "\pi"] \\
Z \arrow[r, hook] & C \arrow[r, hook] & S
\end{tikzcd}$$

Since the nodes of C are ordinary double points, we have $\widetilde{C} \in \pi^*\mathcal{L} - 2E$.

Let us interpret $H^0(\mathcal{I}_Z(C))$ on $\widetilde{S}$: it is the pull-back of $\mathcal{L}$ minus E. Indeed, $\pi^*\mathcal{I}_Z(C) = \pi^*\mathcal{O}_S(C) \otimes \mathcal{O}_{\widetilde{S}}(-E)$, whose corresponding complete linear system is just $\pi^*\mathcal{L} - E$.

Now restrict to $\widetilde{C}$: Consider the short exact sequence

$$0 \to \mathcal{O}_{\widetilde{S}}(-\widetilde{C}) \to \mathcal{O}_{\widetilde{S}} \to \mathcal{O}_{\widetilde{S}}(\widetilde{C})_{|\widetilde{C}} \to 0;$$

tensor with $\mathcal{O}_{\widetilde{S}}(\widetilde{C}+E) = \pi^*\mathcal{O}_S(C) \otimes \mathcal{O}_{\widetilde{S}}(-E)$, and take cohomology:

$$\begin{array}{ccccccccc}
 & & & & H^0(\mathcal{I}_Z(C)) & \longrightarrow & H^0(\mathcal{I}_Z(C)_{|C}) & & \\
 & & & & \| & & \| & & \\
0 & \longrightarrow & H^0(\mathcal{O}_{\widetilde{S}}(E)) & \longrightarrow & \pi^*\mathcal{L} - E & \longrightarrow & (\pi^*\mathcal{L} - E)_{|\widetilde{C}} & \longrightarrow & H^1(\mathcal{O}_{\widetilde{S}}(E)) \\
 & & \| & & & & \| & & \| \\
 & & \mathbb{C} & & & & H^0(\omega_{\widetilde{C}}(-\pi^*K_S) & & 0
\end{array}$$

1.3 Exercise. — To see the three identifications of the last line: From the short exact sequence $0 \to \mathcal{O}_{\widetilde{S}} \to \mathcal{O}_{\widetilde{S}}(E) \to \mathcal{O}_{\widetilde{S}}(E)_{|E} \to 0$ we get

$$\begin{aligned}
&0 \to H^0(\mathcal{O}_{\widetilde{S}}) \to H^0(\mathcal{O}_{\widetilde{S}}(E)) \to H^0(E, \mathcal{O}_E(-1)) \to \\
&\to H^1(\mathcal{O}_{\widetilde{S}}) \to H^1(\mathcal{O}_{\widetilde{S}}(E)) \to H^1(E, \mathcal{O}_E(-1)) \to \dots
\end{aligned}$$

Now as E is just a $\mathbb{P}^1$, we have $H^0(E, \mathcal{O}_E(-1)) = H^1(E, \mathcal{O}_E(-1)) = 0$, whence $H^0(\mathcal{O}_{\widetilde{S}}(E)) = \mathbb{C}$. Also, since we are assuming S (and thus $\widetilde{S}$) regular, it follows that $H^1(\mathcal{O}_{\widetilde{S}}(E)) = 0$.

Concerning the interpretation of $(\pi^*\mathcal{L} - E)_{|\widetilde{C}}$, by adjunction we have $\omega_{\widetilde{C}} = \mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}} + \widetilde{C})_{|\widetilde{C}}$. Recalling that $\pi^*K_S = K_{\widetilde{S}} - E$ and $\pi^*C = \widetilde{C} + 2E$, we find

$$\begin{aligned}
\omega_{\widetilde{C}}(-\pi^*K_S) &= \big(\mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}} + \widetilde{C} - \pi^*K_S)\big)_{|\widetilde{C}} \\
&= \big(\mathcal{O}_{\widetilde{S}}(\widetilde{C} + E\big)_{|\widetilde{C}} \\
&= (\pi^*\mathcal{L} - E)_{|\widetilde{C}}.
\end{aligned}$$

Going back to the long exact sequence, we compute

$$\begin{aligned}
h^0(\omega_{\widetilde{C}}(-\pi^*K_S) &= h^0(\mathcal{I}_Z(C)) - 1 \\
&= \dim T_{\mathcal{V}_{\mathcal{L},\delta},C} \\
&= \dim \mathcal{L} - c_{\mathcal{L},z} \qquad (1.3.1)
\end{aligned}$$

On the other hand, applying Riemann-Roch to the divisor $(\pi^*K_S)_{|\widetilde{C}}$ on $\widetilde{C}$ we find

$$\begin{aligned}
\varepsilon := h^0((\pi^*K_S)_{|\widetilde{C}}) &= h^0(\omega_{\widetilde{C}}(-\pi^*K_S)) + \deg(\pi^*K_S)_{|\widetilde{C}} + 1 - g(\widetilde{C}) \\
&= h^0(\omega_{\widetilde{C}}(-\pi^*K_S)) + K_S \cdot C + 1 - g(\widetilde{C})
\end{aligned}$$

whence, by the genus formula,

$$
\begin{aligned}
h^0(\omega_{\widetilde{C}}(-\pi^* K_S)) &= g(\widetilde{C}) - 1 - K_S \cdot C + \varepsilon \\
&= \tfrac{1}{2}\widetilde{C} \cdot (\widetilde{C} + K_{\widetilde{S}}) - K_S \cdot C + \varepsilon \\
&= \tfrac{1}{2}C \cdot (C - K_S) - \delta + \varepsilon. \qquad (1.3.2)
\end{aligned}
$$

Now we will have to make some assumptions on S to control these cohomology groups. We will treat three cases:
① $K_S \cdot C < 0$
② S is a K3-surface (that is, $K_S \equiv 0$)
③ S is an Enriques surface (that is, $2K_S \equiv 0$)
Note that case ① includes $\mathbb{P}^2$ and also Del Pezzo surfaces. Note that ② and ③ are the only regular surfaces with Kodaira dimension 0.

Now you can check that ①, ③ imply $\varepsilon = 0$ and $\chi(\mathcal{O}_S) = 1$, while the case ② of a K3 surface gives $\varepsilon = 1$ and $\chi(\mathcal{O}_S) = 2$. So in any case we have $\varepsilon = \chi(\mathcal{O}_S) - 1$.

Recall that Riemann-Roch for a surface states that

$$\dim \mathcal{L} = \chi(\mathcal{O}_S) - 1 + \tfrac{1}{2}C \cdot (C - K_S) - h^2(\mathcal{L}) + h^1(\mathcal{L})$$

but the last two terms are zero by the assumptions. So under any one of the three assumptions, we see that $\dim \mathcal{L} = \varepsilon + \frac{1}{2}C \cdot (C - K_S)$. Now it follows from (1.3.2) that

$$h^0(\omega_{\widetilde{C}}(-\pi^* K_S)) = \dim \mathcal{L} - \delta,$$

so combining with (1.3.1), we conclude that $\delta = c_{\mathcal{L},z}$, so ①,② or ③ all imply that $\mathcal{V}_{\mathcal{L},\delta}$ is regular.

Non-emptiness is not known for ③.

Sernesi: Let V be a smooth projective variety, and C and irreducible curve on V with δ nodes at Z. Consider the exact sequence

$$0 \to T_C \to T_{V|C} \to N_{C,V} \to T^1 \to 0.$$

The two middle sheaves are locally free, whereas the first one, T_C, is not: it jumps at the nodes. The last sheaf is concentrated at the nodes, so it is isomorphic to $\mathbb{C}^\delta$. Now break the sequence, letting N' be the cokernel of $T_C \to T_{V|C}$. We have then two short exact sequences

$$
\begin{gathered}
0 \to T_C \to T_{V|C} \to N' \to 0 \\
0 \to N' \to N_{C,V} \to T^1 \to 0
\end{gathered}
$$

$H^0(N_{C,V})$ is the tangent space to the Hilbert scheme of C inside V, i.e. the space of infinitesimal deformations of C in V, while the obstruction to such deformations are contained in $H^1(N_{C,V})$.

N' corresponds to deformations preserving the nodes, in the sense that $H^0(N')$ is the space of first order deformations of C in V keeping the nodes, and $H^1(N')$ contains the obstruction to such deformations.

Theorem 2 (Sernesi) — *Suppose that C is unobstructed in V (its Hilbert scheme is smooth), and assume $H^0(N_{C,V}) \to T^1$ is surjective. Consider $\mathcal{H}$, the Hilbert scheme of C, containing $\mathcal{H}_\delta$, the Hilbert subscheme of δ-nodal curves. Then $\mathcal{H}_\delta$ is smooth of codimension δ in $\mathcal{H}$ at C.*

Proof. — Note that we have an injection $H^1(N') \hookrightarrow H^1(N_{C,V})$. This map sends obstructions to obstructions, but on the right hand side they are zero. Now since $H^0(N_{C,V}) \to T^1$ is surjective, we have $h^0(N') = h^0(N_{C,V}) - \delta$. $\square$

Let us see how the dimensionality can apply existence.

Consider first the case of $\mathbb{P}^2$, and the complete linear systems $\mathcal{L}_n$. Since there are rational curves of all degrees we have $\mathcal{V}_{\mathcal{L}_n, \binom{n-1}{2}} \neq \emptyset$, so the nodes impose independent conditions. It follows that the map $H^0(N_{C,V}) \to T^1$ is surjective, and then the theorem applies, for any subset of the nodes ($\delta \leq \binom{n-1}{2}$).

The Del Pezzo case is expected to work similarly, only it's somewhat more complicated since there are many more different complete linear systems. It is expected that $\mathcal{V}_{\mathcal{L},\delta}$ is non-empty of right dimension.

The Enriques case: Let S be an Enriques surface, and let $\mathcal{L}$ be a (regular) complete linear system. Then $\dim \mathcal{L} = g - 1$. If $\mathcal{V}_{\mathcal{L},g-1}$ is non-empty then $\mathcal{V}_{\mathcal{L},g-1}$ is finite. And then it follows that $\mathcal{V}_{\mathcal{L},\delta}$ is non-empty for any $\delta \leq g - 1$. How many points are there then?

2 Severi varieties of K3 surfaces

Second lecture, 11/02/99

Definition. — A K3 surface is a smooth projective surface S such that $h^1(\mathcal{O}_S) = 0$ and K_S is trivial.

It is known that there is a moduli space $\mathcal{S}$ for K3 surfaces, and $\mathcal{S}$ is irreducible and of dimension 20. However, the general element of $\mathcal{S}$ is not algebraic.

For each $g \geq 1$, there is a subspace $\mathcal{F}_g \subset \mathcal{S}$ of dimension 19, such that the general element $S \in \mathcal{F}_g$ is an K3 surface whose Picard group is generated by a single curve C which is smooth, irreducible, and of genus g. In particular, C is indivisible. For $g \geq 2$, this general S is furthermore algebraic.

For $g \geq 3$, the canonical linear system embeds S in $\mathbb{P}^g$:

$$\varphi_{|K|} : S \xrightarrow{\simeq} \Sigma_{2g-2} \subset \mathbb{P}^g$$

— the image Σ_{2g-2} is a surface of degree $2g - 2$. (For $g = 2$, the map also exists, but is not an isomorphism; it is a 2:1 cover of the plane, branched along a smooth sextic.)

Now let S be any K3 surface, and let $C \subset S$ be a smooth curve. For its normal bundle we have $\mathcal{O}_S(C)_{|C} \simeq \omega_C$, by adjunction and since $K_S \equiv 0$. In particular $C^2 = 2g - 2$ (where g is the genus of C). From the short exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_S(C)_{|C} \to 0$$

we find $h^0(\mathcal{O}_S) = 1 + g$, in other terms, $\dim |C| = g$. (This explains why the map $\varphi_{|K|}$ goes to $\mathbb{P}^g$.) There is a surjection $H^0(\mathcal{O}_S(C)) \twoheadrightarrow H^0(\omega_C))$, so the linear system cuts out the canonical system.

If C is not hyper-elliptic then the canonical system is very ample. One moment of reflection will convince that then the map is birational. If C is general enough, not only is it birational but in fact an isomorphism. This is a specially fortunate case for the Severi problem.

Recall that we can impose δ nodes, where $0 \leq \delta \leq p_a$, and we must also bound δ above by the dimension of the system. But in this case $p_a = \dim |C| = g$, so we can go all the way up to g. We have regularity: Let $\mathcal{L} := |C|$, then if $\mathcal{V}_{\mathcal{L},\delta}$ is non-empty, then $\mathcal{V}_{\mathcal{L},\delta}$ is regular. In particular, when $\delta = g$, if $\mathcal{V}_{\mathcal{L},g}$ is non-empty, then $\mathcal{V}_{\mathcal{L},g}$ is a finite set of points, and furthermore, by the argument of the previous lecture, $\mathcal{V}_{\mathcal{L},\delta}$ is non-empty for all $0 \leq \delta \leq g$. Let $\omega_{\mathcal{L},g}$ denote the number of points in the finite $\mathcal{V}_{\mathcal{L},g}$.

Let S be a general K3 surface in $\mathcal{F}_g$, and set $\mathcal{L}_n = |nC|$. Set $g_n := g(nC) = (g-1)n^2+1$. Finally set $\omega_{n,g} := \omega_{\mathcal{L}_n,g_n}$.

Our goal is to understand the Severi varieties of S. The strategy is to produce rational nodal curves on S. The first result is:

Theorem 3 (Mori-Mukai (1983)) — *If $S \in \mathcal{F}_g$ is general, then there are rational curves in S.*

Note however that since these curves a priori can have triple points or cusps, the result does not help much.

Theorem 4 (Xi Chen (1997)) — *If $S \in \mathcal{F}_g$ is general, then for every $g \geq 3$ and $n \geq 1$, the Severi variety $\mathcal{V}_{\mathcal{L}_n,g_n}$ is non-empty.*

In particular, the non-emptiness implies regularity. The theorem does not say anything about irreducibility; it is still an open problem to determine when $\mathcal{V}_{\mathcal{L},\delta}$ is irreducible. The problem of enumeration is also open, except for the case $n = 1$, $\delta = g$.

2.1 Example. — Let $g = 3$. Then $\varphi_{|K_S|}$ maps to $\mathbb{P}^3$ and the image is a quartic surface. Does it contain rational curves?

In 1996, Ziv Ran made the following claim about irreducibility: *$\mathcal{V}_{\mathcal{L},\delta}$ is irreducible whenever it has positive dimension.* In the case of $\dim \mathcal{V}_{\mathcal{L},\delta} = 0$, he argues that the monodromy group is the full symmetric group. . . ?

Counting rational curves on K3 surfaces

Meanwhile, Yau and Zaslow (1996) [6] gave a formula for the number of g-nodal rational curves in $|C|$ on a general K3 surface. Their techniques and language comes from theoretical physics and is almost impossible to read for an algebraic geometer. Beauville [1] has given an algebro-geometric account for their proof, which we now proceed to describe.

Definition. — Let $\mathcal{H}_g$ be the Hilbert scheme of zero-dimensional schemes of length g on S.

2.2 Properties of $\mathcal{H}_g$. — It is a smooth space of dimension $2g$ (it is a desingularisation of the symmetric product of g copies of S). There is a holomorphic 2-form on $\mathcal{H}_g$ which is everywhere non-degenerate (i.e. the canonical system is free... ?) — that is, it's a so-called symplectic variety).

Theorem 5 (Yau-Zaslow-Beauville)

$$\omega_{1,g} = e(\mathcal{H}_g),$$

where e denotes the Euler-Poincaré characteristic.

Note that $\omega_{1,g}$ is the degree of the Severi variety of rational g-nodal curves. The numbers $e(\mathcal{H}_g)$ have been computed by Göttsche.

Sketch of proof. — There is a rational map $\mathcal{H}_g \dashrightarrow \mathcal{L}_1$ sending "a set of g points" Z to the unique curve $C_Z \in \mathcal{L}_1$ passing through those points (since $\dim \mathcal{L}_1 = g$). Of course for many Z, this map is not well-defined; the points could be in special position, for example. But there exists a resolution of this map: $\mathcal{H}_g$ is replaced by something slightly modified, but since anyway we have not defined $\mathcal{H}_g$ formally, we will just pretend the map is a morphism.

Now take $C \in \mathcal{L}_1$. The fibre is the Jacobian $J(C) \in \mathcal{H}_g$, since the Jacobian of a curve consists of all choices of g points on it. This works when C is general, and in particular smooth. For singular C, there is still a generalised Jacobian which does the job.

Now the Euler-Poincaré characteristic of a total space is the product of the Euler-Poincaré characteristic of the base and the Euler-Poincaré characteristic of the fibre. So

$$e(\mathcal{H}_g) = e(\mathcal{L}_1) \cdot e(J(C)) + \text{ correction terms}$$

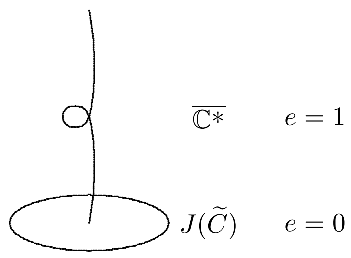
The correction terms come from the locus of degenerate curves and from the fibres over this locus. Now $e(J(C)) = 0$ since it is an Abelian variety. So we are left with the correction terms: Assume C is a nodal curve, and let $\widetilde{C} \to C$ be its normalisation.

?We need a line bundle of degree 0 on $J(C)$. Now there is an exact sequence

$$0 \to \mathbb{C}^* \to J(C) \to J(\widetilde{C}) \to 0$$

where the map $J(C) \to J(\widetilde{C})$ is just pull-back along the normalisation of a line bundle of degree 0. The line bundles on $\widetilde{C}$ are all obtained by gluing together line bundles of the two branches of the normalisation, and since the fibres are just lines, the ways to glue are parametrised by $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$.

Now consider the compactified generalised Jacobian and its map to $J(\widetilde{C})$. The fibres are compactifications of $\mathbb{C}^*$, so they are rational curves with a node. Such curves have Euler-Poincaré characteristic 1, but concerning the base we still have $e(J(\widetilde{C})) = 0$.



So nodal curves give no contribution, unless they are rational...

Now consider a cuspidal curve C. The story is the same, except that we get $\mathbb{C}$ instead of $\mathbb{C}^*$ as kernel of the pull-back map $J(C) \to J(\widetilde{C})$. To see this, take a trivial line bundle on $\widetilde{C}$, and a section s such that the derivative s' is proportional to s, that is, $s' = a \cdot s$, for some $a \in \mathbb{C}$. In coordinates, take s_1, s_2, s_3 on C such that

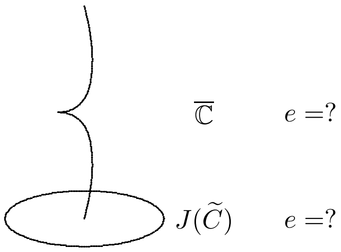
$$\begin{aligned} x_1 &= s_1 \\ x_2 &= s_2 \\ x_3 &= s_3 \end{aligned}$$

defines the cusp. That is, the tangent at $(0, 0, 0)$ is undefined. We want

$$\begin{aligned} x_1' &= s_1' \\ x_2' &= s_2' \\ x_3' &= s_3', \end{aligned}$$

to get it proportional to (s_1, s_2, s_3).

When we pass to the compactifications, we get $\overline{\mathbb{C}}$ which is the cusp.



In general, if C is a singular curve, then we get an exact sequence

$$0 \to \mathbb{C}^k \oplus \mathbb{C}^{*h} \to J(C) \to J(\widetilde{C}) \to 0.$$

— this is always true, it is just a matter of decomposing the algebraic group $J(C)$ into its compact part $J(\widetilde{C})$ and its non-compact part $\mathbb{C}^k \oplus \mathbb{C}^{*h}$. Suppose $g(\widetilde{C}) > 0$. Then for the compactified Jacobian we find $e(\overline{J}(C)) = 0$ because $e(J(\widetilde{C})) = 0$, so we conclude that the only contribution come from rational curves.

Suppose that

$$\text{Assumption: } \textit{all rational curves on } S \textit{ are nodal} \qquad (*)$$

Then $J(C) = (\mathbb{C}^*)^g$. (To construct line bundles of degree 0, take those of the normalisation and glue. For each of the g nodes, there are $\mathbb{C}^*$ ways to glue...) It follows that the compactified Jacobian $\overline{J}(C)$ is the product of g cuspidal curves. And $e(J(C)) = 1$.

This shows that $e(\mathcal{H}_g)$ is the number of rational curves on S, provided the assumption $(*)$ holds.

Göttsche and Fantecchi have shown that even if there are rational curves with other singularities, their contribution can be interpreted as a multiplicity of the curves as limits of nodal curves.

Note also that the assumption would follow from the theorem of Xi Chen together with the claim of Ziv Ran. □

Proof of Xi Chen's theorem

Let S be a general surface of $\mathcal{F}_g$. For simplicity, let us say $g = 3$; then S is a quartic surface in $\mathbb{P}^3$. We must show that there is at least one rational curve cut out by $\mathcal{O}(n)$.

The idea is to produce a very special quartic surface F_0 that does contain a rational curve, and then deform it, keeping track of what happens to the singularities of the rational curve.

Let F_0 be a reducible quartic, union of two general quadrics Q^1 and Q^2. The intersection $Q^1 \cap Q^2$ is a smooth quartic curve E of genus 1. On each quadric Q^i, there are two pencils of lines (the rulings) which we will denote H_1^i and H_2^i.

Now construct a curve $C \subset F_0$ which belongs to $|\mathcal{O}(n)|$. Let C have $2n$ irreducible components such that n of them are on Q^1, denoted $C_1^1, \ldots, C_n^1$, and the other n are on Q^2, denoted $C_1^2, \ldots, C_n^2$. More precisely, the first $n-1$ components on each Q^i are lines belonging to the first ruling, whereas the last component C_n^i is in the linear system $|H_1^i + nH_2^i|$:

$$\begin{array}{rcll} C_j^i & \in & \left|H_1^i\right| & i = 1,2,\ j = 1, \ldots, n-1 \\ C_n^i & \in & \left|H_1^i + nH_2^i\right| & i = 1,2 \end{array}$$

We see that each n-component curve C^i belongs to the linear system $|\mathcal{O}(n)|$ on Q^i.

Now we will specify the incidence relations among the C_j^i's: Choose a point $q_0 \in E$. Let C_1^1 be a secant to E joining q_0 with another point $q_1 \in E$. Now on the other quadric Q^2, take C_1^2 to be a secant to E joining q_1 with some $q_2 \in E$. Continue like this until C_{n-1}^2 joins $q_{2n-3} \in E$ with $q_{2n-2} \in E$.

Now the construction of the last two components is crucial: Observe that since E is of type $(2,2)$ on Q^1, the linear system $|H_1^1 + nH_2^1|$ on Q^1 cuts E in $2 + 2n$ points, giving a complete linear series g_{2+2n}^{2n+1}, to which the divisor $q_{2n-2} + (2n+1)r$ belongs. (Similarly for Q^2.)

Now let C_n^1 be such that it cuts this divisor on E, that is, it has contact of order $2n+1$ with E at r. Now it only remains to specify C_n^2 in Q^2, but since the total curve C^2 is cut out by $\mathcal{O}(n)$, this is forced to be the curve joining q_0 with r, and having contact of order $2n+1$ with E at r.

the picture

Now it is an exercise to check that all this is possible. The curve $C = C^1 \cup C^2$ is rational, and it has the following singularities: the points $q_0, \ldots, q_{2n-2}$ are ordinary nodes. Furthermore, on each quadric Q^i, each of the first $n-1$ components intersects the last component in n points. Indeed, $C_1^1 \cdot C_n^1 = H_1^1 \cdot (H_1^1 + nH_2^1) = n$. This gives further $n(n-1)$ singularities on each quadric, and one checks that these are also ordinary nodes. Finally there is the singularity r which is a tacnode of order $2n-1$. This tacnode counts as $2n$ nodes, in the sense that $2n$ blow-ups are needed to resolve the singularity. So all together, there are

$$2n - 2 + 2n(n-1) + 2n = 2n^2 + (2n-2)$$

nodes. Now the $2n-2$ nodes on E are going to be smoothened when we smoothen F_0. So we will end up with $2n$ nodes, which is one less than we want. But we will deform F_0 in a way such that the first node q_0 is preserved. To do this, take a general quartic passing through q_0, and let the family be

$$F_0 + tF.$$

Now q_0 will be a singular point of the total space of the family, and the node will not disappear. However, we want to apply Sernesi's theorem, that requires a smooth ambient variety, so we must make a slight modification of the family. Blow up the base point of the family

$$F_0 + tF \subset \mathbb{A}^3 \times \mathbb{A}^1$$

with coordinates (x, y, z) on $\mathbb{A}^3$ and t on $\mathbb{A}^1$. The family is singular where the double curve E meets F. Now F_0 is analytically given by $xy = 0$, while F is given by $z = 0$, so the singularity is $xy + tz = 0$, which is a quadric with a double point. Blow up. The exceptional divisor is a quadric, it has two rulings and Q^1 cuts it along a line of one ruling whereas Q^2 cuts it along a line of the other ruling. Now contract all the lines of on of the rulings. (Small resolution... ?) The result is the original family with q_0 replaced by a new line (on Q^2). So C_1^1 and C_1^2 do no longer meet; instead there is new rational curve joining them. Now there is the right number of nodes.

The total family is smooth, it remains to verify that the nodes we want to keep impose independent conditions to the surfaces of degree n. Then we can apply a suitable modification of Sernesi's theorem. (Exercise...)

3 Severi varieties of general surfaces

THIRD LECTURE, 18/02/99

Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$. By the general discussion of the first lecture, we know that the dimensionality can be understood whenever the Kodaira dimension is low, $-\infty, 0 or 1$. In these cases, $\mathcal{V}_{\mathcal{L},\delta}$ is regular.

The situation is much more complicated in higher Kodaira dimension; and particularly difficult when the Kodaira dimension is 2, so we are talking about surfaces of general type.

Not much is known in general. Let us list some results for particular surfaces, first without proofs, then maybe later on we can return to the proofs.

Theorem 6 (Chiantini-Sernesi (1997)) — *Let S be a surface of general type, i.e. minimal and K_S nef. Suppose $C \subset S$ is an irreducible curve with δ nodes and no other singularities, numerically equivalent to pK_S for some rational number $p \geq 2$. Assume that $\mathcal{L} := |C|$ contains smooth curves. Suppose either that*

$$\delta < \frac{p(p-2)}{4} K_S^2$$

or that

$$\delta < \frac{(p-1)^2}{4} K_S^2, \quad p \textit{ odd, and } \mathrm{NS}(S) = \mathbb{Z} < K_S >$$

Then $\mathcal{V}_{\mathcal{L},\delta}$ is regular at C (that is, the nodes impose independent conditions, so $\mathcal{V}_{\mathcal{L},\delta}$ is of dimension $\dim \mathcal{L} - \delta$.

3.1 Corollary. — *Let $S \subset \mathbb{P}^5$ be a smooth surface of degree $d \geq 5$, and set $\mathcal{L}_n := |\mathcal{O}(n)|$, with $n \geq 2d - \delta$. (Note that this linear system enjoys the properties of the theorem.) Suppose $C \in \mathcal{L}_n$ is irreducible and has δ nodes. Now if*

$$\delta < \frac{nd(n - 2d + \delta)}{4}$$

then $\mathcal{V}_{\mathcal{L}_n,\delta}$ is regular at C.

Proof. — Observe that $K_S = (d-4)H$ where $H \in \mathcal{L}_1$. Now if C is in $\mathcal{L}_n$ then $C \equiv \frac{n}{d-4} K_S$. Now apply Chiantini-Sernesi (with $p = d - 4$). □

3.2 Remark. — If we instead take $d = 5$, a general quintic surface S will have $\mathrm{Pic}\, S = \mathbb{Z} < K_S >$. Now similar properties give a better bound…

In this case $C \equiv nK_S$, where $K_S = H$, (n plays the rôle of p). Now for

$$\delta < \frac{(n-1)^2}{4} \cdot 5$$

$\mathcal{V}_{\mathcal{L},\delta}$ is regular at C.

These bounds are sharp in the sense that there exists a surface such that if δ attains the limit then $\mathcal{V}_{\mathcal{L},\delta}$ is not regular.

Definition. — A curve $C \subset \mathbb{P}^3$ which is irreducible and non-degenerate is said to be *geometrically linearly normal* (GLN) if there is no curve $\widetilde{C} \subset \mathbb{P}^r$, $r \geq 4$ and a projection $\pi : \mathbb{P}^r \dashrightarrow \mathbb{P}^3$ such that $\widetilde{C} \to C$ is birational, and such that the centre of projection is disjoint from $\widetilde{C}$.

3.3 Remark. — If we furthermore require the map $\widetilde{C} \to C$ to be an isomorphism then we recover the usual definition of being linearly normal (without the attribute "geometrically").

Theorem 7 — *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d; let $C \in \mathcal{L}_n$ be irreducible with δ nodes whose union we denote by Z. Then C is geometrically linearly normal if and only if Z imposes independent conditions to the linear system $\mathcal{L}_{n+d-5}$.*

In particular, if $d = 5$, then C is geometrically linearly normal if and only if $\mathcal{V}_{\mathcal{L}_n,\delta}$ is regular at C.

The following example with general S and a curve C on it such that δ attains the limit and C is not geometrically linearly normal.

3.4 Example. — Let n be even, say $n = 2m$. Let $X \subset \mathbb{P}^4$ be a complete intersection surface of type $(2, m)$, i.e., the intersection of a quadric and a hypersurface of degree m. Let $X' \subset \mathbb{P}^3$ be the image of a general projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$. Then $\deg X' = 2m = n$ and X' has a double curve of degree $m^2 - m$.

Let $S \subset \mathbb{P}3$ be a general quintic surface and consider the curve $C := S \cap X'$ which belongs to $\mathcal{L}_n$. Let $\delta = 5(m^2 - m) = 5(n^2/4 - n/2) = 5\frac{n(n-2)}{4}$. Since n is even, the second bound of the theorem does not apply, but let us show that the first bound is sharp.

Claim. *C is not geometrically linearly normal.* So in particular, the Severi variety $\mathcal{V}_{\mathcal{L},\delta}$ is not regular. Proof of the claim. Let $P \in \mathbb{P}^4$ be the centre of the projection, and let $F \subset \mathbb{P}^4$ be the cone from P over S. Now let $\widetilde{C}$ be the intersection $F \cap X \subset \mathbb{P}^4$. Now the map $\widetilde{C} \to C$ is birational (since we are not inside the double points locus…)

3.5 Remark. — It is an interesting fact that even though $\mathcal{V}_{\mathcal{L},\delta}$ is not regular at C, it is true that C is a singular point of a regular component of $\mathcal{V}_{\mathcal{L},\delta}$. (Is this component unique? — see Question below…)

Moral of the theorem: when the nodes are few they impose independent conditions. There can be special curves in $\mathcal{V}_{\mathcal{L},\delta}$ such that the nodes do not impose independent conditions, but generically on the component of $\mathcal{V}_{\mathcal{L},\delta}$ it holds.

3.6 Question 1. — Try to find $f(d, n)$ such that for a general surface $S \subset \mathbb{P}^3$ of degree d, and an irreducible curve $C \subset S$ with δ nodes, $C \in \mathcal{L}_n$, then for $\delta < f(n, d)$ we have C on a regular component of $\mathcal{V}_{\mathcal{L}_n,\delta}$.

3.7 Question 2. — Is the regular component of C unique?

3.8 Question 3. — Similar questions, with *regularity* replaced by *irreducibility.*

Proof of the interesting fact. — Suppose not. Take V to be any component of $\mathcal{V}_{\mathcal{L},\delta}$ containing C, and let C' be a general member of V. So C is not geometrically linearly normal, so it comes from $\mathbb{P}^4$. SO for each C' there is a curve $\widetilde{C}' \subset \mathbb{P}^4$ and a projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ such that the image of $\widetilde{C}'$ is $C' \subset S \subset \mathbb{P}^3$. All these curves $\widetilde{C}'$ have the same degree, and since the special one is a complete intersection all of them are.

Now let P be the centre of projection, and let F be the cone from P over $C \subset S$. Let us deform $\widetilde{C}$ inside F, and next move the cone. The maximal number of parameters for deforming $\widetilde{C}$ inside F is

$$\begin{aligned} h^0(N_{\widetilde{C},F}) &= h^0\big(\mathcal{O}_{\widetilde{C}}(2) \oplus \mathcal{O}_{\widetilde{C}}(m)\big) \\ &= 14 + (5m^2 - 10m + 14) \end{aligned}$$

and adding the four parameters corresponding to moving the cone, we see that $\widetilde{C}$ varies in a family of dimension at most

$$5m^2 - 10m + 32.$$

On the other hand, we know that

$$\dim_C \mathcal{V}_{\mathcal{L}_n,\delta} \geq \dim \mathcal{L}_n - \delta = 5m^2 + 4$$

which is bigger than what we just found. So the curves of the bad type can never fill the component.

$\square$

Let us ask ourselves the following question.

3.9 Problem. — For which δ's may we expect that $\mathcal{V}_{\mathcal{L}_n,\delta}$ is generically regular? — or has a component which is?

First of all, $\delta \leq \dim \mathcal{L}_n$. Secondly, remark that by Riemann-Roch, $\dim LL_n < g(C)$, the geometric genus of $C \in \mathcal{L}_n$, except in the cases ①,② and ③. So in particular, imposing g nodes (i.e. look for rational curves) is very hard

3.10 Question 3 — What is the minimal geometric genus of curves in $\mathcal{L}_n$? I.e. what is the maximal number of nodes you can impose and still have $\mathcal{V}_{\mathcal{L}_n,\delta} \neq \emptyset$?

There are unexpected results on this question:

3.11 Remark. — Set $N(d,n) := \dim \mathcal{L}_n$ on a surface of degree d.

$$N(d,n) = \begin{cases} \binom{n+3}{3} - 1 & n < d \\ \frac{n^2 d}{2} + o(n) & n >> d. \end{cases}$$

The maximal number of nodes you can impose to have a regular component is

$$\delta \sim \frac{nd(n-2d+\delta)}{4} \sim \frac{n^2 d}{4}$$

— this is the Chiantini-Sernesi bound.

"until half way we expect regularity... "

3.12 Example. (1) — Set $d >> n$. Let F be a surface of degree n with a double line and otherwise smooth (for example a cone over a curve with only one node). Then $C := F \cap S$ belongs to $\mathcal{L}_n$ and has $\delta = d$ nodes. Note that $d >> N(d,n)$. Now C belongs to a non-regular component of $\mathcal{V}_{\mathcal{L}_n,\delta}$, in fact to a *unexpected* component: this means that the number of nodes is higher than the dimension. This happens on a general surface, and there are as many curves presenting the phenomenon as you want.

3.13 Example. (2) — In this example we exhibit an expected component which is not regular. Take $n >> d \geq 7$, and n even. Consider in $\mathbb{P}^2$ the Severi variety $\mathcal{V}$ of curves of degree n with

$$\alpha = \binom{n-1}{2} - \frac{n(d-7)}{2} \quad \text{nodes.}$$

What is the dimension of the family?

$$\dim \mathcal{V} = \frac{n(n+3)}{2} - \alpha = \frac{nd-n-2}{2},$$

because $\alpha \leq \binom{n-1}{2}$. Take a general curve in $\mathcal{V}$, take it to be irreducible. Fix a point $O \in \mathbb{P}^3$, and let $S \subset \mathbb{P}^3$ be a general surface of degree d. Cut S with the cone from O over a general curve in $\mathcal{V}$. In this way we get $C \in \mathcal{V}_{\mathcal{L}_n,\delta}$ with

$$\delta := \alpha d = \frac{n^2+4n+2}{2} \cdot d - \frac{nd^2}{2} \quad \text{nodes.}$$

(As the cone has α nodal double lines.) Now

$$\dim \mathcal{L} - \delta = \frac{d^3 - 12d^2 + 11d - 6}{6} \geq 0, \quad \text{for big } d$$

When n is big with respect to d, this number is smaller than $\dim \mathcal{V}$ so we are in the expected range. So it is a lower (upper??) bound, and we conclude that $\dim \mathcal{V}$ is smaller than the dimension of the component containing C.???

Note that the number does not depend on n (assymptotically?), so we come very close to the bound. . . ?

Theorem 8 (Chiantini-Ciliberto (1997)) — *For all δ such that $0 \leq \delta \leq N(d,n)$ on a general S, there exists a regular component of $\mathcal{V}_{\mathcal{L}_n,\delta}$.*

In particular, $\mathcal{V}_{\mathcal{L}_n,\delta}$ is reducible in this range (with $\delta = \alpha d$ as in the example). This is a very negative result.

3.14 Question 4 — (Mild form of question 2): Is such a component unique?

3.15 Question 5. — Is it true that if V is an irregular component of $\mathcal{V}_{\mathcal{L}_n,\delta}$ (and S general), then a general $C \in V$ is cut out on S by a surface which is singular in codimension 1?

4 Lower bounds for the genus of curves on general S

Fourth lecture, 19/02/99

A couple of words about the proofs of the theorems of yesterdays lecture.

For the first theorem of Chiantini and Sernesi, the main tool is Reider's method.

There could be other statements of the same sort, for example: what happens to a general surface S containing a line and what happens to curves such that a multiple of them belongs to the linear system plus or minus the line???

There may be some general statement that could include such variations over the theme.

It should also be possible to take projective Cohen-Macaulay curves (e.g. complete intersections with nodes). Probably the same techniques would apply, but with different analysis. Instead of Reider's method one should use Bombieri's method.

There could be better bounds for general C. (Of course not for every curve: we have already seen that the bounds are sharp.)

Concerning the second theorem of Chiantini and Sernesi, that with geometrically linearly normallity, the proof is quite straighforward and would probably appear as an "exercise" in Arbarello-Cornalba-Griffith-Harris...

It would be interesting to extend the result to other types of curves, like projective Cohen-Macaulay curves.

Concerning the theorem of Chiantini and Ciliberto, the main tool of the proof is a Xi Chen type of analysis, but substantially more complicated (not conceptually though).

In degree 4 we have seen that each component of $\mathcal{V}_{\mathcal{L}_n,\delta}$ has the right dimension. For $d \geq 7$, we have seen that $\mathcal{V}_{\mathcal{L}_n,\delta}$ is reducible and has components of too high dimension.

However, for $d = 5$ and $d = 6$, it is expected that everything is nice. It is an interesting project to investigate these two cases. For example, it might be true that for $d = 5$, $\mathcal{V}_{\mathcal{L}_n,\delta}$ is irreducible... At least, the investigators have been unable to find an easy counter example.

But let us study the case of a general surface $S \subset \mathbb{P}^3$ of degree $d \geq 5$. There can be unexpected components, and there could be curves such that ... ?

4.1 Question. — What is the minimal geometric genus of an irreducible, reduced curve $C \subset S$?

This question is closely related to the question of determining when a variety is *hyperbolic*: A variety V is said to be hyperbolic if there are no maps $C \to V$, in particular no curves on V of genus 0 or 1. Question: Is a general surface of degree ≥ 5 hyperbolic? These questions originate in differential projective geometry... Kobayashi... See notes by Demailly, where various different definitions of hyperbolicity are given and studied...

Theorem 9 (Xu) — *Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 5$. Let $C \subset S$ be an irreducible reduced curve of geometric genus g and degree nd (that is, it is cut out by a surface of degree n). Then*

$$g \geq 2 + \frac{d-5}{2} nd.$$

There are other results (Chiantini and Lopez [4]) of same type:

(1) Similar results for surfaces that are general with respect to the condition of containing a given curve D. The curve C is then assumed to belong to the linear system $|nH - D|$.

(2) Similar results for general projective Cohen-Macaulay surfaces in $\mathbb{P}^4$.

Recent extensions of these results have been given by Chiantini, Lopez and Ran. They give bounds for the geometric genus p_g of subvarieties of given projective varieties under certain conditions.

Let us present the ideas of a nice proof of Xu's theorem, given by Chiantini and Lopez [4].

The central idea is the concept of *foci* of a family of varieties. This is the main tool. The ideas go back to Corrado Segre

Let Y be a smooth irreducible variety, and let B be a smooth variety. Consider a smooth family of subschemes of Y, parametrised by B. Let X_b denote the fibre over a geometric point $b \in B$. We have the following diagram:

$$\begin{tikzcd}
X_b \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d, "p"'] \arrow[drr, "f"'] & \subset & B \times Y \arrow[d, "q"] \\
b & \in \; B & & Y
\end{tikzcd}$$

This is a slightly simplyfied version: the full version operates with maps into $B \times Y$ instead of just looking at inclusions.

Now there is a diagram of exact sequences, called the *focal diagram*:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & T(q)_{|\mathfrak{X}} \arrow[r, "\varphi"] \arrow[d] & \mathcal{N} \arrow[d, equal] & \\
0 \arrow[r] & T_{\mathfrak{X}} \arrow[r] \arrow[d, "df"'] & T_{B \times Y|\mathfrak{X}} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] & 0 \\
 & q^* T_{Y|\mathfrak{X}} \arrow[r, equal] & q^* T_{Y|\mathfrak{X}} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

The map $\varphi : T(q)_{|\mathfrak{X}} \to \mathcal{N}$ is called the *focal map* or the characteristic map. Let us look at it fibrewise, in the fibre X_b

$$T_{B,b} \otimes \mathcal{O}_{X_b} \xrightarrow{\varphi_b} N_{X_b|Y}$$

which we can also think of as a map

$$T_{B,b} \longrightarrow H^0(N_{X_b|Y}).$$

In this interpretation it is the differential of the map to the Hilbert scheme.

The main remark is that

$$\operatorname{Ker} \varphi = \operatorname{Ker} df.$$

In particular, the dimension of the image $f(\mathfrak{X})$ is on one hand equal to $\dim \mathfrak{X} - \operatorname{rk} \varphi$ and on the other hand equal to $\dim \mathfrak{X} - \operatorname{rk} df$. So the problem of computing the rank of the differential df has been expressed in terms of infinitesimal deformations of the variety!

There are two important subschemes of each fibre. The first is the *focal subscheme* of X_b, denoted F_b. It is the set of points $x \in X_b$ such that φ_b is not of maximal rank. Inside F_b we have the *fundamental set* D_b defined as the set of points $x \in X_b$ where the dimension of the fibre $f^{-1}f(x)$ is higher than expected, i.e. $\dim f^{-1}f(x) > \min(0, \dim \mathfrak{X} - \dim Y)$. The points of D_b are called fundamental points.

4.2 Stupid remark. — Suppose F_b is a divisor on X_b. Then F_b is bounded above by $c_1(N_{X_b,Y})$. Here we are thinking of the case where X_b is a curve, but a similar remark works in higher dimension.

The following example is a classical application of these concepts:

4.3 Example. — Let $C \subset \mathbb{P}^3$ be an irreducible non-degenerate curve. *Not every secant is a 3-secant.* PROOF. Consider the family of secant lines

$$\begin{tikzcd} X_b \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d, "p"'] \arrow[drr, "f"] & \subset & B \times \mathbb{P}^3 \arrow[d, "q"] \\ b & \in \; B & & \mathbb{P}^3 \end{tikzcd}$$

where $\dim B = 2$. Now $X_b \cap C \subset D_b \subset F_b$, and F_b is a divisor in X_b (or it could be the whole of X_b, but in a moment we will exclude that possibility). So by the last remark we have

$$F_b < c_1(N_{X_b,\mathbb{P}^3}) = 2.$$

Suppose $X_b \cap C$ contains at least three points. Then F_b would be the whole of X_b, and consequently df would drop rank everywhere. In that case $f(X_b)$ would have dimension 2, and in fact it would be a plane. But this is a contradiction to the assumption that C is non-degenerate.

Proof of the theorem of Xu. — Let $\mathcal{F}$ be the set of pairs (C, S) such that $S \subset \mathbb{P}^3$ is a smooth surfaces of degree d, and $C \subset S$ is a smooth irreducible curve of geometric genus g, belonging to $\mathcal{L}_n$. (More precisely, we will consider a map from a smooth irreducible curve to S such that the image is in $\mathcal{L}_n$.) There is natural map from $\mathcal{F}$ to $\mathbf{P}_d := |\mathcal{O}(d)|$, and we will suppose there is a component of $\mathcal{F}$ that dominates $\mathbf{P}_d$. There is a universal family of curves $\mathcal{C} \to \mathcal{F}$ and there is also a dominant map $\mathcal{C} \twoheadrightarrow \mathbb{P}^3$.

Inside $\mathbf{P}_d$ we take a plane Σ generated by three surfaces S_1, S_2, S_3 (our reference surfaces). Given a surface S_{d-1} of degree $d-1$ we will take S_1 and S_2 such that $S_1 \cdot S_2$ contains $\Gamma := S \cdot S_{d-1}$. Moreover, this intersection contains two general points A and B in S. Note that $S \supset \Gamma$.

Now let $\mathcal{L} \subset \mathbf{P}_d$ denote the subset consisting of all surfaces containing Γ. Then in particular $S \in \mathcal{L}$. We claim that $\dim \mathcal{L} = 4$. Indeed, suppose $\mathcal{L} = \mathbb{P}(V)$ for some vector space of polynomials V. Then from the exact sequence $0 \to < S > \to V \to V_{|S} \to 0$, and the observation that $\mathbb{P}(V_{|S}) = \Gamma + |\mathcal{O}_S(1)|$, we see that $\dim \mathbb{P}(V_{|S}) = 3$ and it follows that $\dim V = 5$.

Mow pull back $\Sigma \subset \mathbf{P}_d$ to $\mathcal{F}$ to get the set $\mathcal{F}_\Sigma$ of all such surfaces plus the possible curves on them. Take a new family parametrised by Σ itself:

$$\begin{tikzcd} C_t \arrow[r, hook] \arrow[d] & \Omega \arrow[r, two heads, "f"] \arrow[d] & \mathbb{P}^3 \\ S_t & \in \; \Sigma & \end{tikzcd}$$

It may be necessary with a base change, but we will assume it is still the same Σ. Note that if Σ is general enough, the map $\Omega \to \mathbb{P}^3$ is still dominant.

Now the first thing to note is that $F_f \subsetneq C_t$; it cannot fill up a whole component, otherwise the image would be a surface, not $\mathbb{P}^3$! Now look at $\Gamma \subset \mathbb{P}^3$ and consider the inverse image $f^{-1}(\Gamma)$. Since $f^{-1}(\Gamma) \cap p^{-1}(\bullet)$ is a finite set of points, we find that $\dim f^{-1}(\Gamma) = 2$.

Secondly, D_t contains the (inproper) intersection $C_t \cdot \Gamma$. Consider the point $A \in S'$, and look at its inverse image. Well, the dimension of $f^{-1}(A)$ is 1; we have a curve such that A is also fundamental point. For these curves C_t we have $F_t \neq C_t$. (This may not be true for every point A, but since A is general, we can move it a little if necessary.)

Eventually, we apply the argument

$$c_1(N_{C_t}) > \deg F_t \geq \deg D_t \geq \deg C = nd(d-1) + 1$$

On the other hand, from the exact sequence

$$0 \to T_C \to f^{-1}(T_{\mathbb{P}^3}) \to N_{C_t} \to 0$$

we get $c_1(N_{C_t}) = 4nd + 2g - 2$. Now it follows that $2g \geq nd(d-5) + 3$, and thus

$$g \geq \frac{nd(d-5)}{2} + \frac{3}{2},$$

and since this first fraction is always an integer, the enounced lower bound follows. □

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