1 First lecture: Topology of algebraic varieties, connectedness theorems...

Introduction about topology and algebraic geometry...

Examples of classical results.

1.1 Example. Bézout. Let $Y, Z \subset \mathbb{P}^n$ be subvarieties. If $\dim Y + \dim Z \geq n$ then $Y \cap Z \neq \emptyset$. If $\dim Y + \dim Z > n$ then $Y \cap Z$ is connected.

The non-emptiness can also be expressed as the surjectivity of the map $\pi_0(Y \cap Z) \to \pi_0(Y \times Z)$. Connectedness corresponds to requiring this map to be an isomorphism.

1.2 Example. Bertini. Let Y be a projective variety, and let $f : Y \to \mathbb{P}^n$ be a morphisms such that $\dim f(Y) \geq 2$. Let H be a hyperplane, then $f^{-1}(H)$ is connected.

If furthermore H is general then the intersection is irreducible. (But degeneration of irreducible is connected.) Interpretation: $\pi_0(f^{-1}(H)) \simeq \pi_0(Y)$.

1.3 Example. Lefschetz. Let $Y \subset \mathbb{P}^n$ be irreducible, and let $L \subset \mathbb{P}^n$ be a linear subspace containing $\operatorname{Sing} Y$, and such that $\dim L = n - d$. Then $\pi_i(Y, Y \cap L) = 0$ for $i \leq \dim Y - d$.

Another way to say that is that the map $\pi_i(Y \cap L) \to \pi_i(Y)$ is an isomorphism for $i < \dim Y - d$ and it is still surjective for $i = \dim Y - d$.

1.4 Example. Barth. Let $Y \subset \mathbb{P}^n$ be a smooth subvariety. Then $\pi_i(\mathbb{P}^n, Y) = 0$ for $i \leq 2 \dim Y - n + 1$.

Interpretation as map $\pi_i \to \pi_i$. Surjectivity corresponds to equality in the dimension relation; the map being an isomorphism corresponds to $<$.

If $\dim Y \geq n/2$ then $\pi_1(Y)$ is trivial and $\operatorname{Pic} Y \simeq \mathbb{Z} < \mathcal{O}_Y(1) >$.

Project: to generalise from $\mathbb{P}^n$ to any X with ample tangent bundle. (T ample $\Rightarrow K$ is not nef.)

??some result of Mori? T ample $\Rightarrow X = \mathbb{P}^n$??

$$\begin{tikzcd}
f^{-1}(L) \arrow[r, hook] \arrow[d] & Y \arrow[d, "f \text{ proper}"] \\
L \arrow[r, hook] & P
\end{tikzcd}$$

where $L \to P$ is "sufficiently positive". $\operatorname{codim}(L, P) = d$ the topology of $L \subset P$ must reflect or be the same as the topology of $f^{-1}(L) \subset Y$.

$\pi_i(Y, f^{-1}(L)) \to \pi_i(P, L)$.

$i \leq \dim Y - d -$ deficiency.

For i small, we have $\pi_i(P, L) = 0$.

Now Bézout is a special case of this set-up: take $P = \mathbb{P}^n \times \mathbb{P}^n$ and $L = \mathbb{P}^n$ with the diagonal embedding. The intersection $Y \cap Z$ of Bézout's theorem plays the rôle of Y in the set-up just described.

$$\begin{tikzcd} Y \cap Z \arrow[r, hook] \arrow[d] & Y \times Z \arrow[d, "f"] \\ \Delta \arrow[r, hook] & \mathbb{P}^n \times \mathbb{P}^n \end{tikzcd}$$

Similarly, Bertini comes as

$$\begin{tikzcd} Y \cap H \arrow[r, hook] \arrow[d] & Y \times H \arrow[d, "f"] \\ \Delta \arrow[r, hook] & \mathbb{P}^n \times \mathbb{P}^n \end{tikzcd}$$

and Barth is the case

$$\begin{tikzcd} Y \times Y \arrow[d, "f"] \\ \mathbb{P}^n \times \mathbb{P}^n \end{tikzcd}$$

Theorem 1 *Fulton-Hansen (1980). Let $f : Y \to \mathbb{P}^n \times \mathbb{P}^n$ be a proper morphism. If $\dim f(Y) > n$ then $f^{-1}(\Delta)$ is connected.*

Consequences and applications. We assume subvarieties are irreducible.

1.5 Corollary. (Bézout) *Let $Y, Z \subset \mathbb{P}^n$ be subvarieties such that $\dim Y + \dim Z > n$. Then Y]capZ is connected. More generally, let $u : Y \to \mathbb{P}^n$ be a finite morphism such that $\dim Y > \operatorname{codim}(Z, \mathbb{P}^n)$. Then $u^{-1}(Z)$ is connected.*

Proof. Consider the embeddings $i_Y : Y \hookrightarrow \mathbb{P}^n$ and $i_Z : Z \hookrightarrow \mathbb{P}^n$, and construct the product morphism $f := i_Y \times i_Z : Y \times Z \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n$. Then $Y \cap Z = f^{-1}(\Delta)$ and $\dim f(Y \times Z) = \dim Y + \dim Z > n$. This implies the connectedness.

Next, let f denote the product of u and i_Z and use the same arguments. $\square$

Definition. Let $f : X \to Y$ be a morphism of algebraic schemes over k. We have the diagonal embedding $\Delta \hookrightarrow X \times_Y X$ is closed. Let $I \subset \mathcal{O}_{X \times_Y X}$ denote the ideal sheaf of $\Delta \subset X \times_Y X$. Set $\Omega^1_{X/Y} := I/I^2$.

Now f is called non-ramified at $x \in X$ if $\Omega^1_{X/Y,x} = 0$

This means intuitively injectivity on the tangen spaces. Or equivalently: pull-back of froms is surjective.

Being non-ramified is equivalent to the diagonal embedding being an open embedding, that is, Δ is a connected component of $X \times_Y X$...

1.6 Corollary. *Let X be projective and let $f : X \to \mathbb{P}^n$ be a non-ramified morphism. If $\dim X > n/2$ then f is a closed embedding.*

Proof. We must show f is injective. Let $g : X \times X \to \mathbb{P}^n \times \mathbb{P}^n$ denote the product morphism. Then by Fulton-Hansen, $g^{-1}(\Delta)$ is connected. But since $\Delta_X \subset X \times_{\mathbb{P}^n} X$ is a connected component we must actually have $\Delta_X = X \times_{\mathbb{P}^n} X$ which is to say that f is injective. $\square$

1.7 Corollary. *Suppose $Y \subset \mathbb{P}^n$ with $\dim Y > n/2$. If Y is not normal then the normalisation $Y' \to Y$ is ramified. (Y has a "cusp".)*

1.8 Corollary. *Let $Y \subset \mathbb{P}^n$ with $\dim Y > n/2$, then Y is algebraically simply connected. (I.e., every finite étale morphism $u : Y' \to Y$ with Y' connected is a isomorphism.)*

Recall that étale is non-ramified and flat.

Proof. Let $Z \subset Y'$ be an irreducible component, then $u_{|Z} : Z \to Y$ is finite and non-ramified. Therefore $Z \simeq Y$.

...

then Z is a connected component of Y'... $\square$

1.9 Remark. (Barth) Over $\mathbb{C}$, let $Y \subset \mathbb{P}^n$ be smooth of dimension $\dim Y > n/2$. Then $\pi_1(Y) = 1$. If $\pi_1(Y) \neq 1$, let H be a subgroup pf finite index. $Y' \xrightarrow{u} Y/H = Y \ldots$???

1.10 Projective geometry and hyperplane sections. Recall that a smooth variety $Y \subset \mathbb{P}^n$ of positive dimension is called non-degenerate if it is not contained in any hyperplane.

Let T_yY denote the projective tangent space of Y at $y \in Y$. The tangent variety(?) is $T(Y) = \cup_{y \in Y} T_yY$.

The secant variety of Y is

$$\mathrm{Sec}(Y) = \overline{\cup_{y_1 \neq y_2} < y_1, y_2 >},$$

the closure of the union of all honeste secant lines.

One expect that $\dim T(Y) \leq 2 \dim Y$ and that $\dim \mathrm{Sec}(Y) \leq 2 \dim Y + 1$. If Y is smooth then $T(Y) \subset \mathrm{Sec}(Y)$.

Theorem 2 *Let $Y \subset \mathbb{P}^n$ be smooth. Then either $\dim T(Y) = 2 \dim Y$ and $\dim \mathrm{Sec}(Y) = 2 \dim Y + 1$, or $T(Y) = \mathrm{Sec}(Y)$.*

Proof. Suppose $\dim T(Y) < 2 \dim Y$ but $T(Y) \subset \mathrm{Sec}(Y)$. Let $M \subset \mathbb{P}^n$ be a linear subspace such that $\mathrm{codim}(M, \mathbb{P}^n) \leq 2 \dim Y$ and $M \cap \mathrm{Sec}(Y) \neq \emptyset$. $M \cap T(Y) = \emptyset$.

Consider the projection from M, $\pi_M : \mathbb{P}^n \setminus M \to \mathbb{P}^r$ (where $r \leq 2 \dim Y - 1$). Note that π_M is non-ramified but not injective. Now apply the corollary... ?

Let $g := \pi_{M|Y} : Y \to \mathbb{P}^r$. Recall that $Y \subset T(Y)$ and that $Y \cap M = \emptyset$. Let f denote the product morphism $g \times g : Y \times Y \to \mathbb{P}^r \times \mathbb{P}^r$. Now $\dim f(Y \times Y) = 2 \dim Y \geq r + 1 > r$, so $\dim g(Y) > r/2$ which is a contradiction. $\square$

Definition. A linear subspace $L \subset \mathbb{P}^n$ is said to be tangent to Y at a point $y \in Y$ if $T_yY \subset L$. Equivalently, L is tangent line to Y at y if and only if $Y \cap L$ is singular at y.

Theorem 3 ZAK'S THEOREM OF TANGENTS. *Let $Y \subset \mathbb{P}^n$ be a non-degenerate smooth variety. Let $L \subset \mathbb{P}^n$ be a linear subspace of dimension ℓ such that $\dim Y \leq \ell \leq n-1$. Then*

$$\dim\{y \in Y \mid T_yY \subset L\} \leq \ell - \dim Y.$$

1.11 Corollary. *Let $Y \subset \mathbb{P}^n$ be smooth of dimension d. Consider the Gauss map $g : Y \to \mathbb{G}(d,n)$ which to each point $y \in Y$ associates the projective tangent space. Griffith-Harris: g is generically finite $\Rightarrow$ g is finite.*

Proof. Supose $L \in \mathbb{G}(d,n)$ then $\dim g^{-1}(L) \leq 0 \ldots$??? $\square$

2 second lecture, more Fulton-Hansen

Theorem 4 ZAK'S THEOREM OF TANGENTS. *Let $Y \subset \mathbb{P}^n$ be a non-degenerate smooth variety. Let $L \subset \mathbb{P}^n$ be a linear subspace of dimension ℓ such that $\dim Y \leq \ell \leq n-1$. Then*

$$\dim\{y \in Y \mid T_yY \subset L\} \leq \ell - \dim Y.$$

Proof. Let Z be an irreducible component of the locus above, with $\dim Z > \ell - \dim Y$. Note that $Z \neq Y$ since Y is non-degenerate.

We claim that there exists a linear subspace M of codimension $\ell+1$ such that $M \cap (Y \cup L) = \emptyset$, and such that the projection $\pi_M : \mathbb{P}^n \dashrightarrow \mathbb{P}^\ell$ defined on Y is not one-to-one onto the image $\pi_M(Z)$. To establish this claim, note that since Y is non-degenerate there exists a point $y \in Y \smallsetminus L$ and a point $x \in Z$ such that the line $< y, x >$ is not contained in Y. Now take $z \in < y, x > \smallsetminus Y$, and let M be a generic subspace passing through z, of dimension $n - \ell - 1$.

Now $\pi_M(y) = \pi_M(x) \Rightarrow$ the map is not one-to-one. Set $f := \pi_M \times \pi_{M|Z} : Y \times Z \to \mathbb{P}^n \times \mathbb{P}^n$. Since the dimension of the image of this map is $\dim(Y \times Z) > \dim Y + \ell - \dim Y = \ell$, by Fulton-Hansen we conclude that $f^{-1}(\Delta) = Y \times_{\mathbb{P}^\ell} Z$ is connected. On the other hand, by construction Δ_Z is not the whole of $Y \times_{\mathbb{P}^\ell} Z$ (since π_M is not one-to-one)

Connectedness implies that Δ_Z is not the whole of $Y \times_{\mathbb{P}^\ell} Z \ldots$

This implies that there exists a smooth curve $C \subset Y \times_{\mathbb{P}^\ell} Z$ not entirely contained in Δ_Z. C parametrises a family of secant lines, and its intersection with Δ_Z then corresponds to tangent lines. Now there exists a $t_0 \in C$ such that $\forall t \in C \smallsetminus t_0$ the line $< y_t, x_t >$ is not contained in Δ_Z, while the line $< y_{t_0}, x_{t_0} >$ is contained in Δ_Z (and it's then a tangent line to Y at a point in Z). The other lines are secant lines of Y which meet M. Now pass to the limit: it must be a line contained in L, but then the intersection $L \cap M$ in non-empty, which is a contradiction with the initial assumption. $\square$

2.1 Corollary. *Assume the ground field is $k = \mathbb{C}$. Let $Y \subset \mathbb{P}^n$ be smooth and non-degenerate, and let $Y^* \subset \mathbb{P}^{n*}$ denote the dual variety. Then* $\dim Y^* \geq \dim Y$.

Equality does not always hold. Examples of varieties whose dual is of strictly greater dimension are $\mathbb{P}^1 \times \mathbb{P}^n$ or $\mathbb{G}(1,4) \subset \mathbb{P}^9 \ldots$

Proof. Consider $C_Y = \{(y, H) \in Y \times \mathbb{P}^{n*} \mid T_yY \subset H\}$ with projections $\pi_1 : C_Y \to Y$ and $\pi_2 : C_Y \to \mathbb{P}^{n*}$. By definition, $Y^* = \pi_{2*}(C_Y) \subset \mathbb{P}^{n*}$. Set $d := \dim Y$. Now the fibre

$\pi_1^{-1}(y)$ is isomorphic to $\mathbb{P}^{n-d-1}$ which implies that C_Y is irreducible of dimension $n-1$. Now $n-1-\dim Y^*$ is generically equal to the dimension of the fibre, and always greater or equal to... $\square$

2.2 Corollary. *Let $Y \subset \mathbb{P}^n$ be smooth and non-degenerate of dimension d, and let $X := Y \cap H$ be any hyperplane section. Then X is non-singular in codimension $2d-n-1$.*

Proof. Set $\ell := n-1$. Now $\dim \operatorname{Sing}(X) = \dim\{y \in Y \mid T_yY \subset H\} \leq n-1-d$, so

$$\begin{aligned} \operatorname{codim}(\operatorname{Sing}(X), X) &= d-1-\dim \operatorname{Sing}(X) \\ &\geq d-1+d-n+1 \\ &= 2d-n \end{aligned}$$

$\square$

2.3 Remark. The hyperplane section $X = Y \cap H$ is Cohen-Macaulay. If $2d-n \geq 2$ then Y is non-singular in codimension ? By Serre's criterion then X is normal. If $d \geq 2$ then since X is connected it is also irreducible.

Definition. Let Y be an (irreducible) algebraic variety, and denote by $\boldsymbol{Coh}(Y)$ (resp. $\boldsymbol{QCoh}(Y)$ the category of coherent (resp. quasi-coherent) sheaves (of $\mathcal{O}_Y$-modules) on Y. The *cohomological dimension* of Y is defined to be

$$\operatorname{cd}(Y) := \max\{n \geq 0 \mid \exists F \in \boldsymbol{Coh}(Y) \mid H^n(Y,F) \neq 0\}$$

2.4 Remark. Observe that $0 \leq \operatorname{cd}(Y) \leq \dim Y$ and that $\operatorname{cd}(Y) = 0$ if and only if Y is affine. On the other extreme, if Y is projective then $\operatorname{cd}(Y) = \dim Y$. (Since for L ample we have $H^n(Y, L^{-m}) \neq 0$ for sufficiently large m.)

2.5 Lemma. *Let Y be quasi-projective. Then the following two conditions are equivalent.*
(i) $H^n(Y,F) = 0 \quad \forall F \in \boldsymbol{Coh}(Y)$
(ii) $H^n(Y,L) = 0 \quad \forall L \in \operatorname{Pic}(Y)$.

Proof. Let L be ample, then $F \otimes L^m$ is generated by global sections for sufficiently large m. Now F can be written as a quotient $0 \to G \to (L^{-m})^{\oplus p} \to F \to 0$, so the claim follows from the long exact sequence

$$H^n(Y, (L^{-m})^{\oplus p}) \to H^n(Y,F) \to H^{n+1}(Y,G) = 0.$$

$\square$

Theorem 5 (Hartshorne-Lichtenbaum) *Let Y be an irreducible algebraic variety and let $Z \subset Y$ be a closed subvariety. Then $\operatorname{cd}(Y \smallsetminus Z) = \dim Y$ if and only if Z is empty.*

2.6 Remark. If $\operatorname{cd}(Y \smallsetminus Z) \leq \dim Y - 1$ then Z is non-empty. If $\dim Y \geq 2$ and $\operatorname{cd}(Y \smallsetminus Z) \leq \dim Y - 2$ then Z is connected.

Theorem 6 *Let Y be an irreducible variety. Let S be a finitely generated graded k-algebra (but not necessarily generated in degree 1) and consider a finite morphism $f : Y \to \operatorname{Proj} S$. Consider a set of homogeneous elements $t_1, \ldots, t_r \in S_+$ (not necessarily of the same degree) and consider the closed subscheme $V_+(t_1, \ldots, t_r)$. Now if $r \leq \dim Y + 1$ then $f^{-1}(t_1, \ldots, t_r)$ is non-empty. If furthermore $r < \dim Y + 1$ then $f^{-1}(t_1, \ldots, t_r)$ is connected.*

Proof. Given a variety $U = \bigcup_{i=0}^{r} U_i$ with U_i affine, then using Čech cohomology it is not difficult to see that $\operatorname{cd}(U) \leq r$.

The complement in $\operatorname{Proj} S$ of $V_+(t_1, \ldots, t_r)$ is the union $D_+(t_1) \cup \cdots \cup D_+(t_r)$ (where $D_+(t_i) \simeq \operatorname{Spec}(S_{(t_i)})$). Now $Y \smallsetminus f^{-1}\big(V_+(t_1, \ldots, t_r)\big) = \bigcup_{i=1}^{r} f^{-1}(D_+(t_i))$ and then it follows that $\operatorname{cd}\big(Y \smallsetminus f^{-1}(V_+(t_1, \ldots, t_r))\big) \leq r - 1$.

Now?? if $r \leq \dim Y + 1$ then $f^{-1}(t_1, \ldots, t_r)$ is non-empty and if $r \leq \dim Y + 2$ then $f^{-1}(t_1, \ldots, t_r)$ is connected. □

2.7 Corollary. *By Stein factorisation, the same statement holds for proper morphisms.*

Theorem 7 *Let Y be an irreducible projective variety of dimension $n \geq 2$. Let $Z \subset Y$ be a non-empty closed subvariety. If $\operatorname{cd}(Y \smallsetminus Z) \leq n - 2$ then Z is connected.*

Proof. Since Z is non-empty, for all line bundles $L \in \operatorname{Pic}(Y \smallsetminus Z)$ we have $H^n(Y \smallsetminus Z, L) = 0$. Now suppose Z were a disjoint union of non-empty subvarieties $Z = Z_1 \coprod Z_2$. Take a line bundle $L \in \operatorname{Pic}(Y)$ such that $h^n(Y, L) \neq 0$ (this is possible since Y is projective). Now for each $i = 1, 2$ we have

$$H^n_{Z_i}(Y, L) \to H^n(Y, L) \to H^n(Y \smallsetminus Z, L) = 0$$

$h^n_{Z_i}(Y, L) \geq h^n(Y, L)$. Now $0 = H^{n-1}(Y \smallsetminus Z, L) \to H^n_Z(Y, L) \to H^n(Y, L) \to 0$ which implies $h^n_Z(Y, L) = h^n(Y, L)$, contradiction. □

???

3 third: cohomological dimension, generalisation of Fulton-Hansen

Theorem of Hartshorne-Lichtenbaum and its corollary.

3.1 Remark. More generally, Lichtenbaum shows that for any scheme Y over a field k, if $\operatorname{cd}(Y) < \dim Y$ then no n-dimensional irreducible component of Y is proper.

3.2 Corollary. *If $\operatorname{cd}(Y \smallsetminus Z) \leq \dim Y - 1$ then Z is non-empty. If $\dim Y \geq 2$ and $\operatorname{cd}(Y \smallsetminus Z) \leq \dim Y - 2$ then Z is connected.*

Proof. It is easy to see that if Z is empty then $\operatorname{cd}(Y) = \dim Y$, since Y is projective. Let us show that if Z is non-empty then $\operatorname{cd}(Y \smallsetminus Z) \leq \dim Y - 1$. Set $U := Y \smallsetminus Z$. Then $H^n(U, F) = 0$ for all $F \in \boldsymbol{Coh}(Y)$. It is sufficient to show the vanishing for invertible sheaves. The case $n = 1$ is trivial since then Y is a curve, and since Z is non-empty the complement is affine.

Now induction. . . □

3.3 Lemma. *Let U be as above with $n := \dim U \geq 2$. Then there exists a closed subvariety $W \subset U$ of dimension $n-1$ such that $U \smallsetminus W$ is affine.*

Proof. Let $\pi : Y' \to Y$ denote the blow-up of Y along Z, and let $Z' \subset Y'$ denote the exceptional divisor. (So Y' is an irreducible projective variety.) By Bertini there is an irreducible and very ample Cartier divisor $H' \subset Y'$ such that $H' \cap Z' \neq \emptyset$. For sufficiently large m, the Cartier divisor $Z' + mH'$ is very ample.

$Y' \smallsetminus (Z' \cup H')$ is affine.

Now consider the restriction of π to $Y' \smallsetminus Z'$, which maps isomorphically onto $U \subset Y$. Let $W \subset U$ denote the image of H' under this map. Note that W is irreducible as it is open in an irreducible variety, and that $\dim W = n-1$. We need to show that $U \smallsetminus W$ is affine. Now $\pi(Z' \cup H') \cap U = \pi(H') \cap U$.

$U \smallsetminus W \simeq Y' \smallsetminus (Z' \cup H')$. W is quase-projective but not projective.

... some induction argument... □

Proof of the theorem. Let $i : U \smallsetminus W \hookrightarrow U$ denote the open inclusion. We must study torsion free coherent sheaves F on U. Consider the (Serre?) map $\alpha : F \to i_* i^* F$. This last sheaf is quasi-coherent on U, but not in general coherent. Now the support of the kernel ($\operatorname{Ker} \alpha$) is contained in W????????, but since $\operatorname{Ker} \alpha$ is torsion free, it must be zero. Let now G denote the cokernel of α,

$$0 \to F \to i_* i^* F \to G \to 0$$

then $\operatorname{Supp} G \subset W$.

In the piece of the long exact cohomology sequence

$$H^{n-1}(U, G) \to H^n(U, F) \to H^n(U i_* i^* F)$$

the last module is zero since i is an affine morphism. (Since the inverse image is the intersection of two affines (of a separated scheme)), therefore $H^n(U, i_* i^* F) = H^n(U \smallsetminus W, i^* F) = 0$.

It remains to show that $H^{n-1}(U, G) = 0$. But this follows easily from a general fact: since $\operatorname{Supp} G \subset W$ and $\operatorname{cd}(W) \leq n-2$ by induction hypothesis, we conclude $H^{n-1}(W, G) = 0$.

(This is the general fact: if $W \subset U$ is closed, then for each $k > 0$ the following are equivalent:

(i) $H^k(W, F) = 0 \quad \forall F \in \boldsymbol{Coh}(W)$

(ii) $H^k(U, F) = 0 \quad \forall F \in \boldsymbol{Coh}(U) \mid \operatorname{Supp}(F) \subset W$.)

□

3.4 Corollary. *Let S be a graded finitely generated k-algebra, and let $f : Y \to \operatorname{Proj} S$ be a finite morphism. let $t_1, \ldots, t_r$ be homogeneous elements of S_+. Now if $r \leq \dim Y$ then $f^{-1}(V_+(t_1, \ldots, t_r))$ is non-empty, and if also $r \leq \dim Y - 1$ then $f^{-1}(V_+(t_1, \ldots, t_r))$ is connected.*

We always assume $k = \overline{k}$.

In general we have various irreducible components...

Example. In $\mathbb{A}^4$ consider the intersection of two coordinate planes $\pi_1 : \ x_1 = x_2 = 0$ and $\pi_2 : \ x_3 = x_4 = 0$. Now the intersection minus the origin, $\pi_1 \cap \pi_2 \setminus \{0\}$ is not connected!

Definition. A variety Y (not necessarily irreducible) is said to be d-connected, if all irreducible components of Y are of dimension at least $d+1$ and $Y \setminus W$ is connected for any closed subset W of dimension $\dim W < d$.

The previous example is 0-connected but not 1-connected.

3.5 Example. Any irreducible variety Y of dimension n is $(n-1)$-connected.

3.6 Proposition. *In the notation of preceding results, $f^{-1}(V_+(t_1, \dots, t_r))$ is $(n-r-1)$-connected (where $n = \dim Y$).*

3.7 Example. Let Y and Z be subvarieties in $\mathbb{P}^r$ with $\dim Y + \dim Z > n$
$Y \cap Z$ — the bigger the number $\dim Y + \dim Z$ the more connected is the intersection...

Theorem 8 *Let S be a graded finitely generated k-algebra, $t_1, \dots, t_r \in S_+$ homogeneous. Let $U \subset \operatorname{Proj} S$ be an open set containing $L := V_+(t_1, \dots, t_r)$. If $f : Y \to U$ is a finite morphism, and $r \leq n-1$ then $f^{-1}(L)$ is $(n-r-1)$-connected.*

3.8 Weighted projective space. Let $e = (e_0, \dots, e_n)$ be a vector of positive integers. Consider the polynomial ring $S = k[t_0, \dots, t_r]$ with grading $\deg(t_i) = e_i$. Now by definition, the weighted projective space $\mathbb{P}^n(e)$ is $\operatorname{Proj} S$. It can also be constructed as a quotient, like usual projective space: Let the multiplicative group $\mathbb{G}_m = k^*$ act on $k^{n+1} \setminus \{0\}$ by

$$\lambda.(t_0, \dots, t_n) = (\lambda^{e_0} t_0, \dots, \lambda^{e_n} t_n).$$

The the weighted projective space $\mathbb{P}^n(e)$ is the quotient of this action.

Example: $\mathbb{P}^n(1, \dots, 1)$ is just the usual projective space. $\mathbb{P}^n(1, s, \dots, s)$ is the cone over $\nu_s(\mathbb{P}^{n-1})$. It can be written as $\operatorname{Proj} k[t_0, \dots, k_n] \simeq \operatorname{Proj}\big(k[t_1, \dots, t_n]^{(s)}[t_0]\big)$.

$$\begin{tikzcd}
 & Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f \text{ finite}"] \\
\mathbb{P}^{2n+1} \supset & U \arrow[r] & \mathbb{P}^n(e) \times \mathbb{P}^n(e) \\
 & H \arrow[u, hook] \arrow[r, "\simeq"'] & \Delta \arrow[u, hook]
\end{tikzcd}$$

Proof of the corollary. We can suppose Y is normal. Let $Z' \subset \operatorname{Proj} S$ be closed and consider its normalisation Z in $K(Y)$... ? Let L denote the closed subset. Then $f'^{-1}(L) = g^{-1}(L)$... $\square$

Theorem 9 (Main theorem (of today?)) *Let $f : Y \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$ be a finite morphism. If $\dim Y > n$ then $f^{-1}(\Delta)$ is $(\dim Y - n - 1)$-connected.*

3.9 Example. $Y \times Z \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$, $Y \cap Z = f^{-1}(\Delta)$ etc…

Let the first factor be $\mathbb{P}^n(e) = \operatorname{Proj} k[t_0, \ldots, t_n]$ with $\deg t_i = e_i$, and let the second be $\mathbb{P}^n(e) = \operatorname{Proj} k[u_0, \ldots, u_n]$ with $\deg u_i = e_i$. Consider the weighted projective space made out of all these variables and their degrees: $\mathbb{P}^{2n+1}(e, e) = \operatorname{Proj} k[t_0, \ldots, t_n, u_0, \ldots, u_n]$. Consider the two disjoint coordinate planes in $\mathbb{P}^{2n+1}(e, e)$ given by $L_1 = V_+(t_0, \ldots, t_n)$ and $L_2 = V_+(u_0, \ldots, u_n)$. Let $U \subset \mathbb{P}^{2n+1}$ be the complement of their union.

Now consider the two projections form each of the planes,

$$g_i : \mathbb{P}^{2n}(e, e) \smallsetminus L_i \to \mathbb{P}^n(e)$$

and the product of these two maps

$$\begin{array}{rcl} g := g_1 \times g_2 : U & \longrightarrow & \mathbb{P}^n(e) \times \mathbb{P}^n(e) \\ [\mathbf{t}, \mathbf{u}] & \longmapsto & ([\mathbf{t}], [\mathbf{u}]) \end{array}$$

This map is a $\mathbb{G}_m$-bundle. Let Δ be a subvariety that meets each of these lines in one point.

Let $H := V_+(t_0 - u_0, \ldots, t_n - u_n) \subset U$. Now the restriction of g to H is an isomorphism onto Δ.

$$\begin{tikzcd} & Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f \text{ finite}"] \\ \mathbb{P}^{2n+1} \supset & U \arrow[r] & \mathbb{P}^n(e) \times \mathbb{P}^n(e) \\ & H \arrow[r, "\simeq"'] \arrow[u, hook] & \Delta \arrow[u, hook] \end{tikzcd}$$

f' is also finite, Now apply the other theorem $r = n + 1$ and $\dim Y' = \dim Y + 1 > n + 1$, then we conclude that $f'^{-1}(H)$ is $(\dim Y - n - 1)$-connected.

3.10 Corollary. (Fulton-Hansen for weighted projective space) *Let $Y \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$ a proper morphism such that $\dim f(Y) > n$. Then $f^{-1}(\Delta)$ is connected.*

Proof. Stein factorisation… □

3.11 Example. Counter example to some possible generalisations. Consider the Fermat quintic surface $S \subset \mathbb{P}^3$ given by $x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$. (assume the characteristic of the ground field is different from 5.) Let the group μ_5 of fifth roots of unity act on $\mathbb{P}^3$ by

$$g.[x_0, x_1, x_2, x_3] = [x_0, gx_1, g^2x_2, g_3x_3]$$

The fixpoints are the four standard unit vectors. μ_5 acts on S, the quotient $Y = S/\mu_5$ is Godeaux's surface, it is smooth with $p_g(Y) = q(Y) = 0$, but $|K_Y| \neq \emptyset$.

Consider the space $P := \mathbb{P}^3/\mu_5$. Now consider $f : X = S \times S \to Y \times Y \to P \times P$. it's a finite map, and $\dim X = 4 > 3$, but one can show that $f^{-1}(\Delta)$ has five connected components.

4 fourth: Complete intersections and Hartshorne's conjectures

Two parts: (1) Hartshorne's conjectures (2) rank-2 vector bundle on $\mathbb{P}^n$.

Let $X \subset \mathbb{P}^n$ be an irreducible variety of codimension e. Let $S = k[x_0, \dots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. Let $I_X \subset \mathcal{O}_{\mathbb{P}^n}$ be the sheaf of ideals of X.

Definition. X is said to be *locally a complete intersection* if I_X is generated locally by e elements, i.e., $I_{X|U_i} = (f_i, \dots, f_e)$.

X is called a *set-theoretical intersection* if there exist $f_1, \dots, f_p \in S$ such that $X = V_+(f_1) \cap \dots \cap V_+(f_p)$.

X is called a *set-theoretical complete intersection* if it is a set-theoretical intersection and $p = e$.

4.1 Remark. Note that any smooth $X \subset \mathbb{P}^n$ is locally a complete intersection.

4.2 Open problem. It is not known whether every smooth curve $C \subset \mathbb{P}^3$ is a set-theoretical complete intersection. It is known only in positive characteristics!

4.3 Result by Eisenbud-Harris. A (smooth?) subvariety $X \subset \mathbb{P}^n$ is always the set-theoretical intersection of n hypersurfaces.

Recall that the saturation of an ideal I is

$$I^{\text{sat}} := \{f \in S \mid \exists N : x_i^N f \in I \forall i \neq 0\}.$$

Other characterisations of saturation are: $I \subset S$ is saturated if and only if $\operatorname{depth}_{S_+}(S/I) \geq 1$, or equivalently, if the natural map $\alpha : S/I \to \oplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$ is injective.

Let $I(X) \subset S$ denote the saturation of the ideal of $X \subset \mathbb{P}^n$.

Definition. X is a scheme-theoretical intersection of $f_1, \dots, f_p$ if

$$\oplus_{i=1}^p \mathcal{O}(-d_i) \to I_X$$

is surjective. (Here d_i is the degree of $f_i \in S$.)

Fulton has shown that any $X \subset \mathbb{P}^n$ is the scheme-theoretical intersection of $n+1$ equations.

Definition. X is a complete intersection if $I(X)$ is generated by e elements.

4.4 Example. Let $C \subset \mathbb{P}^3$ be a twisted cubic, then $I(X) =< q_1, q_2, q_3 >$, generated by three quadrics $q_i \in H^0(\mathcal{O}(2))$. Then C is not a complete intersection. Set-theoretically, C is the intersection of a quadric and a cubic.

4.5 Example. Let $S \subset \mathbb{P}^4$ be the projection of the Verones in $\mathbb{P}^5$. So $\deg S = 4$. Let H be a general hyperplane, then $C = H \cap S$ is a quartic curve in $\mathbb{P}^3$. We claim that $I(S)$ is generated by cubics, so C is scheme-theoretical intersection of cubics. Now $H^0(I_C(2)) \neq 0$ — there is at least one quadric that vanishes along C. C is a scheme-theoretical intersection...

It can alos happen that the ideal has generators of higher degree:

4.6 Example. Let $C \subset \mathbb{P}^5$ be an elliptic curve of degree 7; the projection of the elliptic normal curve in $\mathbb{P}^6$. Then C is the scheme-theoretical intersection of quadric. However there is a cubic generator in $I(C)$. (Eisenbud-Ein-Katz).

There exist examples of projectively normal curves scheme-theoretically defined by quadrics, but whose homogeneous ideal is not generated by quadrics. (Katz).

4.7 Example. Let $C \subset \mathbb{P}^n$ be ??? normal, immersed by L, $\deg L \geq 2g-2$. Then $I(C)$ is generated by quadrics (Saint-Donat).

Being complete intersection is essentially an algebraic property.

Given $f_1, \ldots, f_r \in S = k[x_0, \ldots, x_n]$, spanning an ideal I, how do one determine if they form a regular sequence?

Look at the reslution

$$0 \to \bigwedge^r \oplus S_i \to \ldots \to \bigwedge^2 \oplus S_i \to \oplus S(-d_i) \to I$$

where in the higher terms we ignore the grading. The maps are all called d, and $d(a_1 \wedge \cdots \wedge a_p) = \sum_i (-1)^i f_i (a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_p)$.

Now $(f_1, \ldots, f_p)$ form a regular sequence if and only if this complex is a minimal free resolution of S/I.

The projective dimension is $\operatorname{pd}(S/I) = r$. By Auslander-Buchsbaum then $\operatorname{depth}_{S_+}(S/I) = \dim S - \operatorname{pd}(S/I) = n+1-r$. Which in turn implies that the homogeneous coordinate ring S/I of $X \subset \mathbb{P}^n$ is the complete intersection of $f_1, \ldots, f_r$ is Cohen-Macaulay.

Definition. $X \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay if $S(X) = S/I(X)$ is Cohen-Macaulay.

4.8 Corollary. *If X is a complete intersection then X is arithmetically Cohen-Macaulay.*

...

...

Let X be a complete intersection of $f_1, \ldots, f_r$ (of degrees $d_1, \ldots, d_r$ resp.) Then $I_X/I_X^2 \simeq \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-d_i)_{|X}$.

X is of pure dimension $n-r$.

$$\omega_X \simeq \omega_{\mathbb{P}^n|X} \otimes \det I_X/I_X^2 \simeq \mathcal{O}_X(\sum d_i - n - 1)$$

X is a sub-canonical variety.

4.9 Example. If X is a complete intersection then it is the scheme-theoretical intersection of $\operatorname{codim} X$ hypersurfaces.

$$\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-d_i) \to I_X \to 0$$

$$0 \to N_{X/\mathbb{P}^n} \xrightarrow{\varphi^\vee} \oplus_{i=1}^r \mathcal{O}(d_i) \to K \to 0$$

4.10 Corollary. *X is a complete intersection if and only if $\varphi^\vee : N_{X/\mathbb{P}^n} \to \oplus_{i=1}^r \mathcal{O}(d_i)$ is an isomorphism.*

Now suppose $k = \mathbb{C}$, and let X be smooth.

Theorem 10 (Lefschetz, $\sim$ '30) *Let $j : X \hookrightarrow \mathbb{P}^n$ be an embedding, then*

(i) $j^* : H^i(\mathbb{P}^n, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ *is injective for* $i = \dim X$ *and an isomorphism for* $i < \dim X$.

(ii) $\pi_1(X) = 1$ *for* $\dim X \geq 2$.

(iii) $\mathrm{Pic}(X) \simeq \mathbb{Z} < \mathcal{O}(1) >$ *for* $\dim X \geq 3$.

The hard part is (i). See Milnor: *Morse Theory.* (iii) follows from (i) by GAGA and the exponential sequence: (for $\mathbb{P}^n$ and for X):

$$\begin{array}{ccccc} 0 = H^1(\mathcal{O}_{\mathbb{P}^n}) & \longrightarrow & \mathrm{Pic}\,\mathbb{P}^n & \longrightarrow & H^2(\mathbb{P}^n, \mathbb{Z}) \\ \| & & \downarrow & & \downarrow \\ 0 = H^1(\mathcal{O}_X) & \longrightarrow & \mathrm{Pic}\,X & \longrightarrow & H^2(X, \mathbb{Z}) \end{array}$$

4.11 Remark. (Noether-Lefschetz) $X \subset \mathbb{P}^3$ smooth surface of degree $d \geq 4$ general...

(ii) and (iii) are the Grothendieck-Lefschetz, valid over any ground field...

Theorem 11 (Barth-Larsen) *Let $j : X \hookrightarrow \mathbb{P}^n$ be an embedding of a smooth variety of dimension m. Then*

(i) $j^* : H^i(\mathbb{P}^n, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ *is an isomorphism for* $i < 2m - n$.

(ii) $\pi_1(X) = 1$ *for* $m \geq \frac{1}{2}(n + 1)$.

(iii) $\mathrm{Pic}(X) \simeq \mathbb{Z} < \mathcal{O}(1) >$ *for* $m \geq \frac{1}{2}(n + 2)$.

— the smaller $2m - n$, the more restrictions... ?

5 5th: Second Hartshorne conjecture and the four Severi varieties

Hartshorne's conjectures: Let $X \subset \mathbb{P}^n$ be a smooth non-degenerate variety of dimension m (and set $e = n - m$).

First: if $e < m/2$ then X is a complete intersection.

Second: if $e < m/2 + 1$ then X is linearly normal.

Theorem 12 (Zak) *Second conjecture holds.*

Other ways to state the bound:

$$n - m < m/2 + 1 \quad \Leftrightarrow \quad n < \tfrac{3}{2}m + 1 \quad \Leftrightarrow \quad m > \tfrac{2}{3}(n - 1)\text{check this!}$$

5.1 Example. Consider the Grassmannian of lines in four-space $\mathbb{G}(1, 4) \subset \mathbb{P}^9$. Then $m = 6$, $e = 3 = m/2$; the degree is $\deg \mathbb{G}(1, 4) = 5$, the ideal is generated by quadrics, therefore it cannot be a complete intersection.

5.2 Example. Let $Q_8 \subset \mathbb{P}^9$ be a smooth quadric, and let $L \subset Q_8$ be a linear subspace with $\dim L \leq 4$. The variety of such linear subspaces is the disjoint union of two varieties $X_1 \coprod X_2$. Now X_1 is a smooth variety of dimension 10 and degree 12, embedded in $\mathbb{P}^{15}$ (so $m = 10$, $e = 5 = m/2$) and its ideal is generated by quadrics, so it is not a complete intersection.

5.3 Example. The smoothnes condition is necessary in Hartshorne's conjecture: Let $C \subset \mathbb{P}^3 \subset \mathbb{P}^n$ be a twisted cubic, and let $\mathbb{P}^{n-3}$ be a complementary $(n-3)$-plane. Now consider the cone over C: $C_V(C) \subset \mathbb{P}^n$ is singular with $e < \frac{1}{2}m$ but it is not a complete intersection.

Theorem 13 (Zak) *If $e < \frac{1}{2}m + 1$ then X is linearly normal in $\mathbb{P}^n$.*

In other words, there does not/(does?) exist a subvariety $X' \subset \mathbb{P}^{n+1}$ and a projection π_p from a point $p \in \mathbb{P}^{n+1}$ such that $X = \pi_p(X')$ and $\mathrm{Sec}(X') = \mathbb{P}^{n+1}$.

(Note that $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \simeq H^0(\mathcal{O}_X(1))$, let $V \subset H^0(\mathcal{O}_X(1))$ then $\varphi_{|V|} : X \hookrightarrow \mathbb{P}(V)\ldots$

This bound is sharp. For example, let $S \subset \mathbb{P}^4$ be the projection of the Veronese surface in $\mathbb{P}^5$, then $e = 2 = \frac{1}{2}m + 1$.

Theorem 14 *Let $X \subset \mathbb{P}^r$ be a variety of dimension $m > \frac{2}{3}(r-2)$. Then* $\mathrm{Sec}(X) = \mathbb{P}^r$.

the theorem is a consequence of the following proposition.

5.4 Proposition. *Let $X \subset \mathbb{P}^r$ be a smooth variety of dimension m. Suppose* $\dim \mathrm{Sec}(X) = \ell < r$. *Then there exists a hyperplane tangent to X along a subvariety of dimension $2n + 1 - \ell \geq 2n + 2 - r$.*

This in turn follows from Teracini's lemma.

Proof of the theorem. (Zak) Suppose (ad absurdum) that $\mathrm{Sec}(X)$ is not the whole of $\mathbb{P}^r$. By the proposition there exists a hyperplane $H \subset \mathbb{P}^r$ such that

$$2m + 2 - r \leq \dim \mathrm{Sing}(H \cap X) = \dim\{x \in X \mid T_x X \subset H\} \leq r - 1 - \dim X = r - 1 - m$$

by Zak tangent theorem. This is equivalent to $2r - 3 \geq 3m$ in contradiction to the assumption of the theorem. □

Set-up for Teracini's lemma. Let $X, Y \subset \mathbb{P}^n$ be irreducible non-degenerate subvarieties. Their join is

$$S(X, Y) = \bigcup_{\substack{x \in X \\ y \in Y}} < x, y >$$

(so $\dim S(X, Y) = \dim X + \dim Y + 1$.) We have $S(X, X) = \mathrm{Sec}(X)$.

Now let $x \in X$ and $y \in Y$ and let p be a point on the line joining x with y. (But assume that $p \notin X \cup Y$.) Now Teracini's lemma describes a relation between $T_x X$ and $T_y Y$ on one side, and $T_p S(X, Y)$ on the other side:

5.5 Lemma. (Teracini) *If $p \in < x, y > \smallsetminus (X \cup Y)$ then*

$$< T_x X, T_y Y > \subset T_p S(X, Y).$$

(And in characteristic zero, and if p, x, y are generic, then equality holds.)

Proof. Consider the projection $\mathbb{A}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^n$, and the affine cones $C(X) = \pi^{-1}(X)$, $C(Y) = \pi^{-1}(Y)$ and $C(S(X,Y))$. Now $\pi(U+V) =< \pi(U), \pi(V) >$ which implies $C(X)+C(Y) = C(S(X,Y))$. Now consider the differential, — it's a linear map...

The equality statement follows from generic smoothness... □

For the proof of the proposition, recall the construction of the secant variety.

$$S_X := \overline{\{(x,y,z) \mid x \neq y, z \in < x,y >\}} \subset X \times X \times \mathbb{P}^n$$

with projections, notably $\pi_3 : S_X \to \mathbb{P}^n$. By construction, the secant variety is the image of π_3. let $p \in \mathrm{Sec}(X) \setminus X$ be general , then $\dim \pi_3^{-1}(p) = 2m + 1 - \dim \mathrm{Sec}(X)$

Proof of the proposition. Let p be a general point of $\mathrm{Sec}(X) \setminus X$. Then $\dim \pi_3^{-1}(p) = 2m+1-\ell \geq 2m+2-r$. There exists a subvariety $Z_p \subset X$ constituted by the intersections of X with the secants passing through p.

Now since X is not the whole of $\mathbb{P}^r$, there exists a hyperplane $H \subset \mathbb{P}^r$ such that ...

$$\forall x \in Z_p \mid T_x X \subset T_p \mathrm{Sec}(X) \subset H.$$

□

Problem: classify smooth non linearly normal subvarieties $X^m \subset \mathbb{P}^r$ such that $m = \frac{2}{3}(r-2)$. ($\mathrm{Sec}(X) \subsetneq \mathbb{P}^r$).

Theorem 15 (Zak) *It turns out there are only four such varieties, which are now called Severi varieties since the result generalises Severi's result which is the first in the list: They are the isomorphic projections of the following:*

(i) $\nu_2(\mathbb{P}^2) \hookrightarrow \mathbb{P}^5$*, the Veronese surface;*
(ii) $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$*, (Segre embedding);*
(iii) $\mathbb{G}(1,5) \hookrightarrow \mathbb{P}^{14}$*, (Plücker embedding);*
(iv) $E_{16} \hookrightarrow \mathbb{P}^{26}$...

Take a point $p \in \mathrm{Sec}(X)$ which is not a point of X. Set

$$Q_p = \{x \in X \mid < p, x > \text{ is a secant to } X\}.$$

and let Z_p be as before.

Let Σ_p be the cone over Q_p. i.e. the set of secant lines to X which pass through p.

The property which is common to the four cases is:

Q_p is a quadric of dimension $m/2$.

Σ_p is a linear space of dimension $m/2 + 1$.

5.6 Example. Let $S \subset \mathbb{P}^5$ be the Verones surface, image of $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$. Let $p \in \mathrm{Sec}(S) \setminus S$, with $p \in < x,y >$. Now the line $\ell =< \nu_2^{-1}(x), \nu_2^{-1}(y) >$. The image of this line is the conic which pass through the two points. It is Q_p and the plane determined by this conic is Σ_p. All the secant lines through p are in one and the same plane.

Theorem 16 (Severi) *Let* $S \subset \mathbb{P}^5$ *be a smooth non-degenerate surface whose secant variety is of dimension 4. Then* $S \simeq \nu_2(\mathbb{P}^2)$.

Observe that $\dim \mathrm{Sec}(S) \leq 3$ cannot occur since S in non-degenerate.

Proof. We have $\dim \mathrm{Sec}(S) = 4$ and $\dim Q_p = 1$. Let Σ_p be the cone over Q_p. We claim that Σ_p is a plane (and thus Q_p is a conic) (Since a general line in it is a secant line it cuts S in two points, therefore Q_p is a conic) We will show that any plane H that contains a generic directrix of Σ_p cannot contain yet another directrix. Now $H \cap S$ is a non-degenerate curve $C \subset \mathbb{P}^4$.

Lemma (Generalised Three-secant lemma of Severi) Let $C \subset \mathbb{P}^r$ $(r \geq 3)$ be a curve. Then for $0 \leq s \leq r-2$, any $s+1$ general points on C generate a $\mathbb{P}^s$ such that $\mathbb{P}^s \cap C$ consists in $s+1$ isolated points.

Now Σ_p is a plane π_p, and $\pi_p \cap S$ is the conic Q_p. $\mathrm{Sec}(S)$ is the union of ∞^2 planes, say parametrised by $V \subset \mathbb{G}(2,5)$. So there is a family $\mathfrak{X} \to V$ with $\mathfrak{X} \subset \mathbb{G}(2,5) \times \mathbb{P}^5$ with projection $\pi_2 : \mathfrak{X} \to \mathrm{Sec}(S)$.

$\mathrm{Sec}(S)$ is the union of the planes that cut in conics...

Let $H \to T$ denote this 2-dimensional family of conics. Through a given point of S there are infinitely many members of the family. Now any two of the conics meet in a point, so $C_t \cdot C_{t'} \geq 1$. Here $C_t \subset \pi_t$ and $C_{t'} \subset \pi_{t'}$. $H \supset < \pi_t, \pi_{t'} >$ and $H \cap S = C_t + C_{t'} + D$ for some effective divisor D. We will show that $D = 0$. Now we know that $C_t + C_{t'} + D$ is connected! It follows that $(D \cdot C_t) \geq 1$. Now $2C_tH = C_t^2 + C_tC_{t'} + C_tD$, but the first two terms are equal to 1, so we cnclude that $C_tD = 0$. This imples $D = 0$.

Now look at the linear system $|H - C_t| = |C_{t'}|$ where H is the hyperplane that cuts out the cone. Note that $\dim |H - C_t| = 2$ (we are projecting our surface). The corresponding map $\varphi_{|C_{t'}|} : S \to \mathbb{P}^2$ is a birational morphism (it is indeed a morphism...). Now $C_{t'}^2 = 1$. φ sends the conics $C_{t'}$ to lines in $\mathbb{P}^2$. Let $\psi : \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^5$ denote the inverse map. It's a birational map of $\mathbb{P}^2$ into $\mathbb{P}^5$ whose image is S. Therefore ψ must be defined by a linear system $V \subset H^0(\mathcal{O}_{\mathbb{P}^2}(k))$. Now the image of each line is a conic, therefore $k = 2$, but since the map goes to $\mathbb{P}^5$ we have $\dim V \geq 6$, so in conclusion $V = H^0(\mathcal{O}(2))$, and ψ is an isomorphism, the Veronese. □

The idea of the proof of Zak is an adaptation of the argument of Severi.

The fundamental observation of Zak is that the projection from Σ_p of X is a birational map φ. In the first case (severi's case) the centre of projection is a $\mathbb{P}^2$ and $\varphi : X \dashrightarrow \mathbb{P}^2$. In the next case the centre of projection is a $\mathbb{P}^3$ and the birational map is $\varphi : X \dashrightarrow \mathbb{P}^4$. In the third case the centre of projection is a $\mathbb{P}^5$ and the birational map is $\varphi : X \dashrightarrow \mathbb{P}^8$. Finally in the fourth case the centre of projection is $\mathbb{P}^9$ and the birational map is $\varphi : X \dashrightarrow \mathbb{P}^{16}$. Another thing that generalises is the fact that the inverse map φ^{-1} is always given by a linear system of quadrics vanishing along ?

In the first case, $Y = \ell_1 \cup \ell_2 \subset \mathbb{P}^3 \subset \mathbb{P}^4$?

$H^0(I(2))$

In the second case, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7 \subset \mathbb{P}^8$

In the third case $Y = S_{10} \subset \mathbb{P}^{15} \subset \mathbb{P}^{16}$. all the quadrics which vanish along these loci...

Now the outline of the proof of Zak is

— $\mathrm{Sec}(X)$ is a hypersurface (a cubic)

— Q_p is a quadric of dimension $m/2$ inside $\Sigma_p \simeq \mathbb{P}^{m/2+1}$. The maximal linear subspace of it(?) (of dim $m/4$ verifies the intersection condition... ?

— analysis of the linear subspaces: for $m \geq 4$ we get $m \equiv 0 \mod 4$ and for $m \geq 8$ we get $m \equiv 0 \mod 8$, and always $m \leq 16$. Therefore $m \in \{2, 4, 8, 16\}$.

Another property common to the four cases: there is a ∞^m of quadrics of dimension $m/2\dots$

Over $\mathbb{R}$ there are four division algebras: the real numbers $\mathbb{R}$; the complex numbers $\mathbb{C}$; the Hamiltonians or *quaternions* $\mathbb{H}$; and Cayley numbers or *octonians* $\mathbb{O}$. Let $\mathbb{A}_{\mathbb{R}}$ denote anyone of these four algebras, and let $\mathbb{A} = \mathbb{A} \otimes_{\mathbb{R}} \mathbb{C}$ denote its complexification.

Now let M denote the vector space of symmetric 3×3 matrices, and consider $\mathbb{P}M$. If the matrices in question have entries in $\mathbb{C}$, then the Veronese is the locus in $\mathbb{P}M$ of matrices of rank 1. (And $\mathrm{Sec}(S)$ is the locus of rank at most 2.) Now "symmetric" is the real version of a more general property of being hermitian: there is a notion of being hermitian for each of the four division algebras. Let $M_{\mathbb{R}}$ denote the set of $\mathbb{A}_{\mathbb{R}}$-hermitian (i.e. symmetric) 3×3 matrices. Such a matrix is then of type

$$\begin{bmatrix} \alpha_1 & \overline{\beta}_1 & \overline{\beta}_2 \\ \beta_1 & \alpha_2 & \overline{\beta}_3 \\ \beta_2 & \beta_3 & \alpha_3 \end{bmatrix}$$

where $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{A}_{\mathbb{R}}$, and the bar over the betas denote conjugation.

Let $M = M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification. Now the Severi variety $X \subset \mathbb{P}M \simeq \mathbb{P}^{2m/3+2}$ is the locus of matrices of rank 1 And $\mathrm{Sec}(X)$ is the locus of rank at most 2.

The variety is defiend by $3m/2 + 3$ quadrics. The quadrics which vanish along X: $V = H^0(I_X(2))$. $\dim V = 3m/2 + 3$.

The corresponding rational map

$$\varphi_{|V|} : \mathbb{P}^{3m/2+2} \dashrightarrow \mathbb{P}^{3m/2+2}$$

is in fact birational (a Cremona transformation), whose envelope

Its scheme-theoretical base locus is X

$$\varphi(A) = \omega A^*$$

where ω is a projectivity.

special Cremona transformation of type $(2, 2)$

base locus X is smooth and irreducible. with $\varphi = \varphi^{-1}$ defined by quadrics.??