February 1997

Vector Bundles on Compact Riemann Surfaces[1]

Abstract

1. Generalities on vector bundles on CRS.
2. A Theorem of André Weil.
3. Theorem of Narasimhan and Seshadri.
4. Deformations of vector bundles and local moduli.

1 Generalities on Vector Bundles on CRS

Let X be a compact Riemann surface (or, equivalently, a smooth projective curve over $\mathbb{C}$). Our aim is to study vector bundles on X.

Continuous Vector Bundles

Definition. — A *continuous complex vector bundle* of rank n on X is a pair (V, π) where V is a topological space and π is a surjective continuous map $V \to X$ such that

(i) For each $x \in X$, $\pi^{-1}(x)$ is a complex vector space of dimension n, and

(ii) there exists an open cover $\{U_i\}$ of X such that there is a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\phi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{C}^n \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

where ϕ_i is a homeomorphism which is linear on each fibre.

A vector bundle of rank 1 is called a *line bundle*.

Equivalent Definition. — A *continuous complex vector bundle* of rank n on X is an open covering $\{U_i\}$ of X together with continuous maps $\theta_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$ subject to the conditions:

(i) $\theta_{ii} = \mathrm{Id}$

(ii) $\theta_{ij} \circ \theta_{jk} = \theta_{ik}$ on $U_i \cap U_j \cap U_k$.

The maps θ_{ij} are called *transition functions*, and the condition (ii) is referred to as the *cocycle rule.*

[1]Lectures by P. A. Vishwanath, SPIC Math. Inst., Madras, India, partially supported by the FACEPE. Notes taken by Joachim Kock.

A *morphism* of continuous vector bundles on X is a continuous map $\phi : V_1 \to V_2$ such that

$$
\begin{tikzcd}
V_1 \arrow[r, "\phi"] \arrow[d, "\pi_1"'] & V_2 \arrow[d, "\pi_2"] \\
X \arrow[r, equal] & X
\end{tikzcd}
$$

commutes and ϕ is linear on each fibre. An *isomorphism* of vector bundles on X is a morphism which has an inverse.

On vector bundles one can perform almost all the operations that can be performed on vector spaces intrinsically (i.e. without reference to a choice of basis), e.g. direct sums, tensor products, etc. In particular we will need

Definition. — The *determinant* of a vector bundle of rank n is defined as the line bundle $\det V = \bigwedge^n V$. We have $\det(V \oplus W) = \det V \otimes \det W$.

Remark. — Similarly one can define a *smooth* vector bundle: V is then required to be a differentiable manifold and π is required to be a smooth map. There is no difference in the classification of continuous or smooth vector bundles. What is important is that in these two categories we have partitions of unity and hence every short exact sequence of vector bundles is split. The real difference emerges when we pass to the holomorphic category (as we will do from page 4 onwards). Below when we use Sard's Theorem, the category must be assumed to be the category of smooth bundles, but the result of Prop. 1.1 holds in the continuous category as well.

Question. — Classify all continuous complex vector bundles on X up to isomorphism.

The following lemma reduces the question to that of classifying line bundles:

1.1 Proposition. — *Let (V, π) be a continuous complex vector bundle of rank n. Then there is a decomposition*

$$V \simeq X \times \mathbb{C}^{n-1} \oplus L,$$

where L is a line bundle, uniquely determined up to isomorphism. (In fact, this line bundle is isomorphic to $\det V$*.)*

Proof. — Since X is compact, using partitions of unity we can find r $(r \geq n)$ global sections of V which generate V. (A global section of a vector bundle is a continuous map $s : X \to V$ such that the composition $\pi \circ s$ is the identity on X.) The set of global sections $\Gamma(V)$ form a $\mathbb{C}$-vector space. To say that r sections $s_1, \ldots, s_r \in \Gamma(V)$ generate V means that any fibre $V_x = \pi^{-1}(x)$ is generated by the vectors $s_1(x), \ldots, s_r(x)$. Or equivalently: there is a surjective morphism of vector bundles

$$
\begin{array}{ccc}
X \times \mathbb{C}^r & \stackrel{\rho}{\twoheadrightarrow} & V \\
(x, \lambda_1, \ldots, \lambda_r) & \mapsto & \sum \lambda_i s_i(x)
\end{array}
$$

Now look at the kernel K of this map, K has rank $r-n$. So we have the following short exact sequence of vector bundles on X:

$$0 \to K \to X \times \mathbb{C}^r \overset{\rho}{\twoheadrightarrow} V \to 0.$$

To prove the Proposition it is enough to split off a trivial line sub-bundle from V and then use induction. That is, it is enough to show, for $n > 1$, that $V \simeq V' \oplus X \times \mathbb{C}$. Let us see how to get a *trivial* line bundle inside V: Find a line L in $\mathbb{C}^r$ such that the intersection of $\{x\} \times L$ with K_x is trivial for all $x \in X$. To this end, let $\mathbb{P}(K)$ be the projectivised bundle of K, then $\mathbb{P}(K)$ is compact. Now let ϕ be the composition

$$\begin{tikzcd} \mathbb{P}(K) \arrow[r, hook] \arrow[dr, "\phi"'] & X \times \mathbb{P}(\mathbb{C}^r) \arrow[d] \\ & \mathbb{P}(\mathbb{C}^r). \end{tikzcd}$$

Then $\dim \mathbb{P}(K) = r - n$, and since $n > 1$, *Sard's theorem* assures that the map ϕ is not surjective. So there exists a line L in $\mathbb{C}^n$ such that $K \cap (X \times L) = X \times \{0\}$. Then $\rho(X \times L)$ is the required trivial line sub-bundle of V. Hence we get a short exact sequence

$$0 \to X \times \mathbb{C} \to V \to V' \to 0$$

where V' is the quotient. Since we have partitions of unity, this sequence splits. (It is always split locally, and using partitions of unity one can assemble a global splitting from the local ones.) This establishes

$$V \simeq V' \oplus X \times \mathbb{C},$$

and the proof is now completed by induction on the rank. □

Theorem 1 — *There is a bijection*

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{isomorphism classes of complex} \\ \text{continuous vector bundles on } X \end{array} \right\} & \longleftrightarrow & \mathbb{Z} \oplus \mathbb{Z}_+ \\ V & \rightsquigarrow & (\deg V, \operatorname{rk} V), \end{array}$$

that is, the degree and the rank characterise continuous complex vector bundles up to isomorphism.

The degree is defined as

$$\deg V = -\int_X \omega,$$

where ω is any differential 2-form on X which represents the cohomology class of the first Chern class $c_1(V) \in H^2(X, \mathbb{Z})$ under the de Rham isomorphism (We will later give an equivalent definition of degree avoiding the notion of Chern classes). Further, $\deg(V \oplus W) = \deg(V) + \deg(W)$.

Proof. — By the lemma, $V \simeq X \times \mathbb{C}^{n-1} \oplus L$, and since the trivial bundle $X \times \mathbb{C}^{n-1}$ has no Chern class (i.e. it is zero) we have $\deg V = \deg L$. Hence we reduce to proving that there is a bijection

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{isomorphism classes of complex} \\ \text{continuous line bundles on } X \end{array} \right\} & \longleftrightarrow & \mathbb{Z} \\ L & \rightsquigarrow & \deg L. \end{array}$$

To see this, let $\mathcal{C}_X$ denote the sheaf of continuous complex-valued functions on X and let $\mathcal{C}_X^*$ denote the sheaf of nowhere vanishing continuous functions on X. It can then be shown that there is a bijection

$$\left\{ \begin{array}{l} \text{isomorphism classes of complex} \\ \text{continuous line bundles on } X \end{array} \right\} \longleftrightarrow H^1(X, \mathcal{C}_X^*).$$

This amounts to checking that two sets of transition functions define isomorphic line bundles if and only if the two corresponding 1-cocycles define the same cohomology class.

One the other hand, we have an exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{C}_X \overset{\exp}{\to} \mathcal{C}_X^* \to 0$$

where exp denotes the exponential map $f \mapsto e^{2\pi i f}$. In the corresponding long exact sequence the interesting part is

$$H^1(X, \mathcal{C}_X) \longrightarrow H^1(X, \mathcal{C}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{C}_X).$$

The first and the last groups are zero because we have partitions of unity, so $\mathcal{C}_X$ is a *fine sheaf*, and all cocycles are coboundaries. So the connecting map δ, which can be thought of as the Chern class map $\delta(L) = c_1(L)$, is an isomorphism.

Now as X is an oriented compact manifold of real dimension 2, integration defines an isomorphism $H^2(X, \mathbb{Z}) \overset{\sim}{\to} \mathbb{Z}$, and the degree map is the composition of these two isomorphisms:

$$\begin{tikzcd} H^1(X, \mathcal{C}_X^*) \arrow[r, "\delta"] \arrow[dr, "\deg"'] & H^2(X, \mathbb{Z}) \arrow[d] \\ & \mathbb{Z} \end{tikzcd}$$

This completes the proof (and provides an equivalent definition of degree). $\square$

Holomorphic Vector Bundles

Definition. — A *holomorphic* vector bundle is a vector bundle (V, π) where V is a complex manifold, $\pi : V \to X$ is a surjective holomorphic map, and the transition functions are holomorphic.

Question. — Classify all holomorphic vector bundles on X up to isomorphism.

This is a harder question, when compared with the question of classifying continuous bundles. Let us start by looking at vector bundles with fixed rank n and degree d. The first step is

1.2 Lemma. — *The natural map*

$$\left\{\begin{array}{l}\text{isomorphism classes of holo-}\\\text{morphic vector bundles on } X\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{isomorphism classes of}\\\text{continuous complex}\\\text{vector bundles on } X\end{array}\right\}$$

is surjective. That is, any continuous complex vector bundle admits a holomorphic structure.

Since a trivial bundle always has the product holomorphic structure, it is enough, by Proposition 1.1, to show that the natural map

$$\left\{\begin{array}{l}\text{isomorphism classes of holo-}\\\text{morphic line bundles on } X\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{isomorphism classes of}\\\text{continuous complex}\\\text{line bundles on } X\end{array}\right\}$$

is surjective.

Proof. — Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, equal] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathcal{C}_X \arrow[r] & \mathcal{C}_X^* \arrow[r] & 0
\end{tikzcd}$$

and the following part of the corresponding long exact sequence with the induced vertical maps:

$$\begin{tikzcd}
H^1(X, \mathcal{O}_X^*) \arrow[r, "\delta"] \arrow[d] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2(X, \mathcal{O}_X) \\
H^1(X, \mathcal{C}_X^*) \arrow[r, phantom, "\simeq"] & H^2(X, \mathbb{Z}) &
\end{tikzcd}$$

Here $H^1(X, \mathcal{O}_X^*)$ is identified with the group (under the operation of tensor product of line bundles) of isomorphism classes of holomorphic line bundles, and $H^1(X, \mathcal{C}_X^*)$ is identified with the group of isomorphism classes of continuous complex line bundles in an obvious fashion. The map between these two groups is surjective because δ is (the last module in the top row is zero for dimension reasons). This completes the proof. $\square$

Definition. — The *Picard group* of X is

$$\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*) = \left\{\begin{array}{l}\text{isomorphism classes of holo-}\\\text{morphic line bundles on } X\end{array}\right\}$$

The group operation is the tensor product, and inverse is the dual.

There is a smaller group of interest in $\operatorname{Pic} X$:

$$\operatorname{Pic}^\circ X = \left\{\begin{array}{l}\text{isomorphism classes of}\\\text{degree zero holomorphic}\\\text{line bundles on } X\end{array}\right\} = \operatorname{Ker} \delta,$$

where δ is the connecting map $H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z})$. More generally, if we set

$$\operatorname{Pic}^d X = \left\{ \begin{matrix} \text{isomorphism classes of degree } d \\ \text{holomorphic line bundles on } X \end{matrix} \right\},$$

then $\operatorname{Pic} X$ is the disjoint union $\coprod_{d \in \mathbb{Z}} \operatorname{Pic}^d X$. For any $d \in \mathbb{Z}$ there is a set-theoretic bijection

$$\operatorname{Pic}^\circ X \xrightarrow{\sim} \operatorname{Pic}^d X$$

which is quite non-canonical: for example, choosing $x \in X$, and sending

$$L \mapsto L \otimes \mathcal{O}(d \cdot x)$$

defines one such bijection. Because of such a bijection, it is enough to study the smaller group $\operatorname{Pic}^\circ X$. The following result, in this sense, completes the classification of holomorphic line bundles.

1.3 Proposition. — *The group* $\operatorname{Pic}^\circ X$ *is a complex torus of dimension* g*, that is:* $\operatorname{Pic}^\circ X \simeq \mathbb{C}^g / \mathbb{Z}^{2g}$*, (where* g *denotes the genus of* X*.)*

Proof. — The group $\operatorname{Pic}^\circ X$ sits in the short exact sequence

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \operatorname{Pic}^\circ X \to 0$$

so $\operatorname{Pic}^\circ X \simeq H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$. Here $H^1(X, \mathcal{O}_X)$ is a complex vector space and $H^1(X, \mathbb{Z})$ is a lattice. We shall give a more precise meaning to this:

Consider the natural inclusion $\mathbb{R} \hookrightarrow \mathcal{O}_X$. Hodge theory tell us that the induced map

$$H^1(X, \mathbb{R}) \overset{\alpha_X}{\xrightarrow{\sim}} H^1(X, \mathcal{O}_X), \tag{1}$$

is an isomorphism of real vector spaces. Now use α_X to give a complex structure to the real vector space $H^1(X, \mathbb{R})$. So we find that

$$\operatorname{Pic}^\circ X \simeq H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}) \simeq \mathbb{C}^g / \mathbb{Z}^{2g}.$$

□

Remarks. — Note that the structure of a complex manifold on $\operatorname{Pic}^\circ X$ is related to the complex vector space structure on $H^1(X, \mathbb{R})$ induced via α_X, cf. [C], [N]. A theorem of Torelli in fact roughly asserts that non-isomorphic curves have non-isomorphic Picard groups:

$$X \not\simeq X' \quad \Rightarrow \quad \operatorname{Pic}^\circ X \neq \operatorname{Pic}^\circ X'$$

(this statement is not quite correct, but is true for most curves).

For $g \geq 1$ there is a natural injection (one for each $x_0 \in X$) $i : X \hookrightarrow \operatorname{Pic}^\circ X$ defined by $i(x) = \mathcal{O}(x - x_0)$. This makes it possible to translate questions concerning X into those concerning $\operatorname{Pic}^\circ X$ (see [C]). Also, note that in the case when $g = 1$, $\operatorname{Pic}^\circ X$ can be identified with X.

Constructing Holomorphic Line Bundles on X

There are two ways of constructing holomorphic line bundles on X: one, using divisors, and another, using characters of the fundamental group.

1.4 Divisors and Line Bundles. (cf. [F], [N]) — If D is a divisor on X, that is, a finite formal sum $\sum_{p\in X} n_p p$ ($n_p \in \mathbb{Z}$), then there is a line bundle $\mathcal{O}(D)$ associated with it. Conversely, to any line bundle is associated a divisor D_L: any line bundle has a meromorphic section, then the associated divisor is the divisor of zeros and poles of such a section, counted with multiplicities. These two associations are inverse to each other in the sense that if two line bundles are isomorphic then the corresponding divisors are linearly equivalent, and vice versa. (Actually the correspondence gives a bijection between divisors and sections of line bundles.) So classifying line bundles up to isomorphism is equivalent to classifying divisors up to linear equivalence.

In particular this gives yet another definition of degree:

$$\deg L = \deg D_L = \sum n_p.$$

1.5 Characters of the Fundamental Group and Line Bundles. — Let $\pi = \pi_1(X)$ denote the fundamental group of X. A *character* of this group is a group homomorphism $\rho : \pi \to \mathbb{C}^*$. Now to each such character ρ there can be associated a line bundle L_ρ. There are two ways of describing this association. The first way is more functorial, the second is in terms of transition functions.

Let $\widetilde{X} \xrightarrow{p} X$ be the universal cover of X. It is a holomorphic principal bundle with structure group π. Each character ρ of π gives an action of π on $\widetilde{X} \times \mathbb{C}$, the trivial line bundle on $\widetilde{X}$, defined by

$$\gamma.(\widetilde{x}, \lambda) = (\widetilde{x}\gamma, \rho(\gamma)^{-1}\lambda), \qquad (\gamma \in \pi).$$

Denote by $\widetilde{X} \times_\pi \mathbb{C}$ the orbit space of this action, that is, $\widetilde{X} \times_\pi \mathbb{C}$ is the quotient of $\widetilde{X} \times \mathbb{C}$ by the equivalence relation

$$(\widetilde{x}, \lambda) \sim (\widetilde{x}', \lambda') \Leftrightarrow \exists \gamma \in \pi : \widetilde{x}' = \widetilde{x} \cdot \gamma, \quad \lambda' = \rho(\gamma^{-1}) \cdot \lambda.$$

Now the line bundle L_ρ associated to ρ is by definition $L_\rho = \widetilde{X} \times_\pi \mathbb{C}$ with structure map

$$\begin{array}{rcl} \phi : L_\rho & \longrightarrow & X \\ \text{class of } (\widetilde{x}, \lambda) & \longmapsto & p(\widetilde{x}). \end{array}$$

The second description is in terms of transition functions: Define L_ρ by taking its transition functions to be

$$\theta_{ij} = \rho(\gamma_{ij}),$$

where γ_{ij} are the transition functions of the principal bundle $\widetilde{X} \xrightarrow{p} X$. That is, γ_{ij} in π relates the trivialisation of $\widetilde{X}$ on U_i to the trivialisation on U_j.

1.6 Lemma. — *The line bundles arising from this construction have degree zero:*

$$\deg L_\rho = 0.$$

1.7 Observation. — The commutative diagram

$$\begin{tikzcd} \widetilde{X} & \widetilde{X} \times \mathbb{C} \arrow[l] \arrow[d] \\ X \arrow[from=u, "p"'] & L_\rho \arrow[l, "\phi"] \end{tikzcd}$$

can be shown to be a fibre square. That is, L_ρ is characterised (up to isomorphism) by the properties that $p^*L_\rho \simeq \widetilde{X} \times \mathbb{C}$ and that the natural action of π on p^*L_ρ when transported to $\widetilde{X} \times \mathbb{C}$ is given by the character ρ.

Furthermore, let $s \in \Gamma(L_\rho)$ be a global section, and consider $p^*s \in \Gamma(\widetilde{X} \times \mathbb{C})$. As a section to a trivial bundle p^*s is a holomorphic function $p^*s : \widetilde{X} \to \mathbb{C}$, and as such it is π-equivariant (here π acts on $\mathbb{C}$ via ρ). This gives a bijection

$$\Gamma(L_\rho) \longleftrightarrow \left\{ \begin{matrix} \text{space of } \pi\text{-equivariant} \\ \text{holomorphic maps } \widetilde{X} \to \mathbb{C} \end{matrix} \right\}.$$

A reference for these constructions is [N].

Proof of Lemma 1.6 — Choose a meromorphic section s of L_ρ, and consider $F = p^*s$. Think of this as a π-equivariant meromorphic function $F : \widetilde{X} \dashrightarrow \mathbb{C}$. Now take $\widetilde{\omega} = \frac{dF}{F}$. This is a π-equivariant meromorphic 1-form on $\widetilde{X}$, and because it is π-equivariant it goes down to a meromorphic 1-form ω on X.

Now by the residue theorem we have

$$\sum_{x \in X} \mathrm{Res}_x(\omega) = 0,$$

and locally, $\mathrm{Res}_x(\omega) = \mathrm{ord}_x(s)$ (by construction of ω). This implies

$$\deg s = \sum_{x \in X} \mathrm{ord}_x(s) = 0,$$

as promised. □

Next we shall show that all degree zero line bundles come from characters, that is, the map

$$\begin{array}{rcl} \mathrm{Rep}(\pi, \mathbb{C}^*) & \longrightarrow & \mathrm{Pic}^\circ X \\ \rho & \longmapsto & L_\rho \end{array}$$

is surjective. Here $\mathrm{Rep}(\pi, \mathbb{C}^*)$ denotes the group of characters of π. As an abstract group it is isomorphic to $(\mathbb{C}^*)^{2g}$. Indeed, every character factors as

$$\begin{tikzcd} \pi \arrow[rr] \arrow[dr] & & \mathbb{C}^* \\ & H_1(X, \mathbb{Z}) \arrow[ur] & \end{tikzcd}$$

for, $H_1(X,\mathbb{Z}) \simeq \pi/[\pi,\pi]$ and $\mathbb{C}^*$ is abelian. So we get:

$$\mathrm{Hom}(\pi,\mathbb{C}^*) \simeq \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C}^*) \simeq (\mathbb{C}^*)^{2g}.$$

It turns out that we can get nicer functorial properties by restricting our attention to the subgroup $\mathrm{Rep}(\pi,S^1) \subset \mathrm{Rep}(\pi,\mathbb{C}^*)$ of *unitary* representations, that is, the group of homomorphisms $\pi \to \mathrm{U}(1)$. This group is isomorphic to $(S^1)^{2g}$. In fact

1.8 Proposition. — *There is a canonical isomorphism*

$$\mathrm{Rep}(\pi,S^1) \xrightarrow{\sim} \mathrm{Pic}^\circ X$$

which establishes an equivalence of the category of unitary characters of $\pi_1(X)$ with the category of degree zero line bundles on X.

Proof. — From the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, equal] & \mathbb{R} \arrow[r, "\exp"] \arrow[d] & S^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_X^* \arrow[r] & 0
\end{tikzcd}$$

we get the following commutative diagram of long exact sequences with induced vertical maps:

$$\begin{tikzcd}
0 \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(X,\mathbb{R}) \arrow[r] \arrow[d, "\simeq"] & H^1(X,S^1) \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & \mathrm{Pic}^\circ X \arrow[r] & 0
\end{tikzcd}$$

The top row really continues with $\to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$; but this last map is injective since there is no torsion in $H^2(X,\mathbb{Z})$. So the map $H^1(X,\mathbb{R}) \to H^1(X,S^1)$ is surjective. The middle vertical map is once again the isomorphism (1) on page 6. So we conclude that η is an isomorphism. Now the space $H^1(X,S^1)$ is canonically isomorphic to $\mathrm{Hom}(H_1(X,\mathbb{Z}),S^1)$ (by the universal coefficient theorem) which again we have identified with $\mathrm{Rep}(\pi,S^1)$. It can now be checked that we have a commutative diagram:

$$\begin{tikzcd}
 & H^1(X,S^1) \arrow[dr, "\eta"] & \\
 & & \mathrm{Pic}^\circ X \\
\mathrm{Rep}(\pi,S^1) \;=\; & \mathrm{Hom}(H_1(X,\mathbb{Z}),S^1) \arrow[uu, "\simeq"] \arrow[ur] & \\
 & \rho \arrow[r, mapsto] & L_\rho
\end{tikzcd}$$

This proves that $\rho \mapsto L_\rho$ is an isomorphism from $\mathrm{Rep}(\pi,S^1)$ to $\mathrm{Pic}^\circ X$. $\square$

This theme for bundles of higher rank will be taken up in Section 3.

2 A Theorem of André Weil

Holomorphic Vector Bundles on X

Given a representation $\rho : \pi \to \mathrm{GL}(n, \mathbb{C})$, we can copy the construction of 1.5 to define a vector bundle V_ρ of rank n on X associated to ρ. The association $\rho \mapsto V_\rho$ has the following functorial properties:

(i) $V_{\rho_1 \oplus \rho_2} = V_{\rho_1} \oplus V_{\rho_2}$
(ii) $V_{\rho_1 \otimes \rho_2} = V_{\rho_1} \otimes V_{\rho_2}$
(iii) $V_{\wedge^r \rho} = \bigwedge^r V_\rho$, (for $1 \leq r \leq n$)
(iv) $V_\rho^* = V_{{}^t\rho^{-1}}$

(In (iv), V^* denotes the dual bundle of V, and ${}^t\rho$ denotes the transpose of ρ.)

Remark. — From rule (iv) it follows that $\deg V_\rho = 0$. Indeed, $\deg V_\rho := \deg \bigwedge^n V_\rho = \deg V_{\wedge^n \rho}$, but $V_{\wedge^n \rho}$ is the line bundle induced by the character $\bigwedge^n \rho$, so it has degree zero, by Lemma 1.6.

Question. — Characterise those degree zero vector bundles that come from representations.

The answer was given by André Weil in 1938: The degree zero vector bundles induced by representations are exactly those whose indecomposable components all have degree zero (cf. Theorem 3 on page 11). This answers the question about the image of the map $\rho \mapsto V_\rho$. A related question is: which representations ρ give isomorphic vector bundles V_ρ? This is very hard to decide, as we shall see from examples later.

Vector Bundles and Locally Free Sheaves

The following digression applies to any complex variety X.

A locally free (coherent) sheaf of rank n on X is a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that there exists an open cover $\{U_i\}$ of X with $\mathcal{F}_{|U_i} \simeq \mathcal{O}_{U_i}^n$, for some $n \geq 1$.

Now there is an equivalence of categories between the category of locally free coherent $\mathcal{O}_X$-modules, and the category of holomorphic vector bundles on X: Given a holomorphic vector bundle V on X, the corresponding sheaf $\mathcal{O}(V)$ is simply the sheaf of its local sections. And given a locally free sheaf $\mathcal{F}$ the corresponding (geometric) vector bundle is gotten by copying the transition functions.

This equivalence does *not* preserve sub-objects: If $\mathcal{F} \hookrightarrow \mathcal{G}$ is a monomorphism in the category of locally free $\mathcal{O}_X$-modules, then the corresponding morphism $F \to G$ need not be a monomorphism in the category of holomorphic vector bundles on X.

Example. — Let x_0 be a point of X and consider the injection of sheaves $\mathcal{O}(-x_0) \hookrightarrow \mathcal{O}_X$ then the corresponding morphism of line bundles $L \to X \times \mathbb{C}$ is not injective. Indeed, the line bundle L has degree -1, whereas the second has degree zero. The map is an injection at all points $x \neq x_0$, but falls in rank at x_0.

2.1 Observations. — When X is a compact Riemann surface, if $\mathcal{F} \hookrightarrow \mathcal{G}$, one can talk of the the subbundle of G generated by $\mathcal{F}$: for, $\mathcal{F} \hookrightarrow \mathcal{G}$ induces an injection $F \hookrightarrow G$ if and only if the quotient sheaf $\mathcal{G}/\mathcal{F}$ is *torsion free* (and in particular, locally free); if $(\mathcal{G}/\mathcal{F})_{\text{tor}}$ denotes the torsion part of the quotient, then the kernel of the map $\mathcal{G} \to (\mathcal{G}/\mathcal{F})_{\text{tor}}$ will be a sheaf $\mathcal{F}'$ containing $\mathcal{F}$ which will agree with it away from the support of the quotient sheaf $\mathcal{G}/\mathcal{F}$. Now, because of the construction of $\mathcal{F}'$, $\mathcal{F}' \hookrightarrow \mathcal{G}$ induces a bundle injection $F' \hookrightarrow G$ and F' is called the sub-bundle of G generated by the subsheaf $\mathcal{F}$.

If $\varphi : F \to G$ is a morphism of vector bundles such that the rank of φ_x is independent of $x \in X$, then $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are again holomorphic vector bundles, and we have an exact sequence of vector bundles:

$$0 \longrightarrow \operatorname{Ker}\varphi \longrightarrow F \xrightarrow{\varphi} G \longrightarrow \operatorname{Coker}\varphi \longrightarrow 0.$$

We will finally sketch how to get the sheaf of sections of V_ρ, for a representation $\rho : \pi \to \mathrm{GL}(n, \mathbb{C})$. (This could also be taken as the definition of the locally free sheaf associated to a representation.)

Consider the universal covering $p : \widetilde{X} \to X$. The fundamental group π of X acts on $\mathbb{C}^n$ by ρ. The invariant direct image $p_*^\pi(\mathbb{C}^n)$ (which as a pre-sheaf is defined by: given $U \subset X$, an open set; $p_*^\pi(\mathbb{C}^n)(U) :=$ vector space of all locally constant π-equivariant continuous maps from $p^{-1}(U)$ into $\mathbb{C}^n$) is a locally constant sheaf of complex vector spaces, a so-called *local system.* Tensoring with $\mathcal{O}_X$ gives the locally free sheaf we want:

$$\mathcal{O}(V_\rho) = p_*^\pi(\mathbb{C}^n) \otimes_{\mathbb{C}} \mathcal{O}_X.$$

The bundle V_ρ is also called a *flat bundle.* That is, a bundle whose transition functions are locally constant. (Cf. [Sp].)

Indecomposable Vector Bundles

Definition. — A vector bundle V on X is said to be *indecomposable* if it is not a direct sum of its proper subbundles. That is,

$$V \not\simeq V_1 \oplus V_2 \qquad \operatorname{rk} V_i < \operatorname{rk} V.$$

Theorem 2 *(Krull–Remak–Schmidt, cf. [N]) — Given a holomorphic vector bundle V on X, then there is a decomposition*

$$V \simeq \bigoplus V_i$$

where the V_i's are indecomposable. These indecomposable components are uniquely *determined up to permutation of indices, and isomorphism (that is, if also $V \simeq \oplus W_j$, with W_j indecomposable, then the components pair up: $V_i \simeq W_{j(i)}$).*

Theorem 3 *(A. Weil, cf. [W], [A]) — A holomorphic vector bundle V on X arises from a representation if and only if its indecomposable components have degree zero.*

Example. — For $x_0 \in X$, the degree zero bundle $\mathcal{O}(x_0) \oplus \mathcal{O}(-x_0)$ does *not* come from a representation since its indecomposable components do not have degree zero.

We shall only prove the following theorem from which Theorem 3 can easily be deduced.

Theorem 3' — *Let V be an indecomposable vector bundle on X. Then, $V \simeq V_\rho$ for some representation $\rho : \pi \to \mathrm{GL}(n, \mathbb{C})$ if and only if* $\deg V = 0$.

Holomorphic Connections in Holomorphic Vector Bundles

Let V be a holomorphic vector bundle on X. Given any open set $U \subset X$ and a holomorphic vector field Y on U,

Definition. — A *holomorphic connection* is a $\mathbb{C}$-linear map

$$\nabla_Y : \Gamma(V_{|U}) \longrightarrow \Gamma(V_{|U})$$

of sheaves of complex vector spaces on U, satisfying:

(i) Leibnitz' rule: $\nabla_Y(f.s) = (Yf).s + f.\nabla_Y s$, for $f \in \mathcal{O}_X(U)$ and $s \in \Gamma(V_{|U})$.
(ii) $\mathcal{O}_X$-linearity in Y: i.e. $\nabla_{f_1Y_1+f_2Y_2} = f_1\nabla_{Y_1} + f_2\nabla_{Y_2}$ as operators, $f_1, f_2 \in \mathcal{O}_X(U)$.

Equivalently, a holomorphic connection can be defined to be a map

$$\nabla : V \longrightarrow V \otimes_{\mathcal{O}_X} \Omega^1_X$$

which is $\mathbb{C}$-linear as a map of sheaves of complex vector spaces, satisfying Leibnitz' rule:

$$\nabla(s.f) = s \otimes df + f.\nabla(s).$$

Here Ω^1_X denotes the holomorphic cotangent sheaf (= sheaf of holomorphic 1-forms).

Definition. — Let Y_1 and Y_2 be local vector fields on X. The *curvature* of a connection ∇ is the map of sheaves of complex vector spaces

$$\begin{array}{rcl} \Omega_\nabla(Y_1, Y_2) : V & \longrightarrow & V \\ s & \longmapsto & \nabla_{Y_1}\nabla_{Y_2}s - \nabla_{Y_2}\nabla_{Y_1}s - \nabla_{[Y_1,Y_2]}s. \end{array}$$

This map is $\mathcal{O}_X$-linear, and it is skew symmetric in Y_1 and Y_2, that is: $\Omega_\nabla(Y_1, Y_2) = -\Omega_\nabla(Y_2, Y_1)$.

This makes it possible to think of the curvature Ω_∇ as an element in $\Gamma(\mathrm{End}\, V \otimes \Omega^2_X)$. Locally, this is just a matrix of holomorphic 2-forms. In particular, if X is a compact Riemann surface, then there are no holomorphic 2-forms, so $\Omega_\nabla \equiv 0$. Such connections are called *flat*.

The term "connection" comes from the fact that it connects fibres along curves. This we are going to make precise in:

2.2 Parallel transport. — Given a smooth complex vector bundle V equipped with a smooth connection ∇ and a smooth curve $\tau : [0,1] \to X$ in X with $\tau(0) = a$ and $\tau(1) = b$, then there is induced a *parallel transport operator* $P_\tau : V_a \to V_b$ which is a $\mathbb{C}$-linear transformation. Furthermore, composition of paths corresponds to composition of the induced parallel transport operators.

If the connection ∇ is *flat* ($\Omega_\nabla \equiv 0$), then parallel transport operator P_τ is invariant under smooth homotopic deformations of the curve τ. Hence, in such a case, we get a representation — the *holonomy representation* —

$$\rho : \pi(X, x_0) \to \operatorname{Aut}_{\mathbb{C}} V_{x_0}.$$

Let V be a holomorphic vector bundle on a compact Riemann surface X, equipped with a holomorphic connection ∇. Viewing this as a smooth connection in the smooth bundle underlying V, we shall soon see that such a connection is flat. It's then a fact that

$$V \simeq V_\rho$$

as holomorphic bundles, where ρ is the holonomy representation given by (V, ∇). Also, conversely, if V is a holomorphic vector bundle on X associated to a representation $\rho : \pi \to \mathrm{GL}(n, \mathbb{C})$, it admits a holomorphic connection. See [A] for details. Because of this fact, we shall only prove the following version of Theorem 3':

Theorem 3" — *Let V be an indecomposable vector bundle on X. Then, V admits a holomorphic connection if and only if* $\deg V = 0$.

Proof. (M.F. Atiyah) — The proof goes as follows: *Step 1.* Isolate the *obstruction* to equipping V with a holomorphic connection, interpreting it as a cohomology class $a(V)$. *Step 2.* By Serre duality we can think of this class as a linear form on $H^0(X, \operatorname{End} V)$. *Step 3* establishes the fact that if V is indecomposable then $H^0(X, \operatorname{End} V) \simeq \mathbb{C} \oplus N$, where N denotes the subspace of nilpotent endomorphisms. And finally, in *Step 4* it is shown that $a(V)$ applied to $\varphi \in H^0(X, \operatorname{End} V)$ is zero if and only if $\deg V = 0$.

Step 1. — Starting with a holomorphic connection $\nabla : V \longrightarrow V \otimes_{\mathcal{O}_X} \Omega^1_X$ we will construct a bigger sheaf $D(V) \supset V$ and an $\mathcal{O}_X$-linear map $\overline{\nabla} : D(V) \longrightarrow V \otimes_{\mathcal{O}_X} \Omega^1_X$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\nabla"] \arrow[d, hook] & V \otimes_{\mathcal{O}_X} \Omega^1_X \\ D(V) \arrow[ur, "\overline{\nabla}"'] & \end{tikzcd}$$

As a sheaf of complex vector spaces we set $D(V) = V \oplus (V \otimes_{\mathcal{O}_X} \Omega^1_X)$. Then we define an $\mathcal{O}_X$-module structure on it by

$$f.(s, \alpha) = (f.s, f.\alpha + s \otimes df)$$

for $f \in \Gamma(\mathcal{O}_X), s \in \Gamma(V)$ and $\alpha \in \Gamma(V \otimes \Omega^1_X)$.

This gives us a short exact sequence

$$0 \longrightarrow V \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{i} D(V) \xrightarrow{p} V \longrightarrow 0$$

which we will denote by $\mathcal{R}(V)$. The maps i and p are $\mathcal{O}_X$-homomorphisms. Indeed, $i(\alpha) = (0, \alpha)$ by definition and then $i(f.\alpha) = (0, f.\alpha) = f.(0, \alpha)$ (and similarly one checks that p is also $\mathcal{O}_X$-linear). Now define

$$\begin{array}{rcl} \overline{\nabla} : D(V) & \longrightarrow & V \otimes \Omega^1_X \\ (s, \alpha) & \longmapsto & \alpha - \nabla s, \end{array}$$

then $\overline{\nabla}$ is an $\mathcal{O}_X$-linear splitting to the sequence $\mathcal{R}(V)$. Indeed, let us see that it is $\mathcal{O}_X$-linear: By the definitions of the maps

$$\begin{array}{rcl} \overline{\nabla}(f.(s, \alpha)) & = & \overline{\nabla}(f.s, f.\alpha + s \otimes df) \\ & = & f.\alpha + s \otimes df - \nabla(f.s) \\ & = & f.\alpha + s \otimes df - s \otimes df - f.\nabla s \\ & = & f.(\alpha - \nabla s) \\ & = & f.\overline{\nabla}(s, \alpha) \end{array}$$

Conversely, given such an $\mathcal{O}_X$-linear splitting $\overline{\nabla}$ we can recover the holomorphic connection as $\nabla s = -\overline{\nabla}(s, 0)$. So we can reformulate the notion of a connection as an $\mathcal{O}_X$-linear splitting of the short exact sequence $\mathcal{R}(V)$.

Let us denote by $a(V)$ the class of the extension $\mathcal{R}(V)$ in $\operatorname{Ext}^1(V, V \otimes_{\mathcal{O}_X} \Omega^1_X)$. It is called the *Atiyah class* of V. So we have shown that V admits a holomorphic connection if and only if $a(V) = 0$.

Remark. — Observing that

$$\operatorname{Ext}^1(V, V \otimes_{\mathcal{O}_X} \Omega^1_X) \simeq H^1(X, V^* \otimes V \otimes \Omega^1_X) \simeq H^1(X, \operatorname{End} V \otimes \Omega^1_X) \simeq \operatorname{Ext}^1(T_X, \operatorname{End} V),$$

we can also think of $a(V)$ as an extension

$$0 \to \operatorname{End} V \to \Sigma \to T_X \to 0.$$

This is the natural exact sequence on X (see [A]) associated to the frame bundle of V.

Step 2. — By Serre duality we can think of $a(V)$ as a linear form

$$\begin{array}{rcl} \langle\, a(V) \mid - \,\rangle : H^0(X, \operatorname{End} V) & \longrightarrow & \mathbb{C} \\ \varphi & \longmapsto & \operatorname{Res}(\operatorname{Tr}(a(V) \circ \varphi)) \end{array}$$

Indeed, Serre duality states that the bilinear pairing

$$H^1(X, \operatorname{End} V \otimes \Omega^1_X) \times H^0(X, \operatorname{End} V) \longrightarrow H^1(X, \Omega^1_X) \xrightarrow{\operatorname{Res}} \mathbb{C}$$

is non-degenerate. Here the pairing is given by $(\lambda_{ij}, \varphi) \mapsto \operatorname{Tr}(\lambda_{ij} \circ \varphi)$, followed by the residue map which we proceed to describe: the exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \mathcal{E}^{1,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{1,1} \longrightarrow 0$$

gives the Dolbeault isomorphism $H^1(X,\Omega^1_X) \simeq \Gamma(\mathcal{E}^{1,1})/\overline{\partial}(\Gamma(\mathcal{E}^{1,0}))$, where $\mathcal{E}^{i,j}$ denotes the sheaf of smooth complex valued differential forms of type (i,j) on X. Now given a class $\eta \in H^1(X,\Omega^1_X)$, find a representative $\omega \in \Gamma(\mathcal{E}^{1,1})$ and define $\operatorname{Res}(\eta) = \int_X \omega$. One finally checks that this definition is independent of the choice of ω (see [F] for details).

Step 3. **Lemma.** — *If V is indecomposable then $H^0(X, \operatorname{End} V) \simeq \mathbb{C} \oplus N$ where N denotes the nilpotent endomorphisms.*

Remark. — Actually the converse is true as well: if $H^0(X, \operatorname{End} V) \simeq \mathbb{C} \oplus N$, where N denotes the subspace of nilpotent endomorphisms, then V is indecomposable. But we shall not use this result.

Proof of the lemma. — Let V be indecomposable of rank n, and let $\varphi \in H^0(X, \operatorname{End} V)$. For each $x \in X$, the map on the fibre $\varphi_x : V_x \to V_x$ is a $\mathbb{C}$-linear transformation. Look at the characteristic polynomial

$$\operatorname{char} \varphi_x = \sum_{j=0}^{n} a_j(x) T^j.$$

It is easy to check that the functions a_j are holomorphic, hence *constant*, so

$$\operatorname{char} \varphi = \sum_{j=0}^{n} a_j T^j = \prod_{i=1}^{k} (T - \lambda_i)^{n_i}$$

is independent of x. (Here the λ_i are distinct, and $\sum_{i=1}^{k} n_i = n$.)

Set $V_{i,x} = \operatorname{Ker}(\varphi_x - \lambda_i)^{n_i}$ (the generalised eigenspace of φ_x). Then $\dim V_{i,x}$ is independent of x, so $V_i = Ker(\varphi - \lambda_i)^{n_i}$ are really subbundles of V, cf. Observation 2.1 on page 10. Furthermore, $V = \oplus_{i=1}^{k} V_i$ as can be checked easily. But this is a contradiction to the assumption that V is indecomposable, unless $k = 1$. Then it follows that $\operatorname{char} \varphi = (T - \lambda)^n$ which means that $\varphi - \lambda$ is nilpotent, whence the result. □

Step 4. — Let V be indecomposable. Then for $\varphi \in H^0(X, \operatorname{End} V)$ we have

$$\langle\, a(V) \,|\, \varphi \,\rangle = \begin{cases} -2\pi i \lambda \cdot \deg V & \text{for } \varphi = \lambda \cdot \mathrm{Id} \\ 0 & \text{for } \varphi \in N, \end{cases}$$

so that $a(V) = 0$ if and only if $\deg V = 0$.

Let φ be nilpotent. For simplicity we assume $\varphi^2 = 0$. Set $V_1 = \operatorname{Im} \varphi$, and $V_2 = \operatorname{Coker} \varphi$, then we have an exact sequence of sheaves:

$$0 \to V_1 \to V \to V_2 \to 0$$

The only problem is that V_1 may not be a subbundle, only a subsheaf. But as mentioned in Observation 2.1 on page 10, there is a natural notion of a subbundle associated to a subsheaf, with which we can work. So we shall assume V_1 to be a subbundle. Set $\varphi_1 = \varphi_{|V_1}$

and φ_2 is the endomorphism of V_2 induced by φ. These are the zero maps and we have the diagram:

$$\begin{tikzcd}
0 \arrow[r] & V_1 \arrow[r] \arrow[d, "\varphi_1"] & V \arrow[r] \arrow[d, "\varphi"] & V_2 \arrow[r] \arrow[d, "\varphi_2"] & 0 \\
0 \arrow[r] & V_1 \arrow[r] & V \arrow[r] & V_2 \arrow[r] & 0
\end{tikzcd}$$

But note that φ need not be zero.

Let us now calculate $\langle\, a(V) \,|\, \varphi \,\rangle = \mathrm{Res}(\mathrm{Tr}(a(V) \circ \varphi))$. We need to show that $\langle\, a(V) \,|\, \varphi \,\rangle = 0$ for $\varphi^2 = 0$.

At this point we remark that the association $V \mapsto \mathcal{R}(V)$ is functorial and exact in the sense that a short exact sequence $0 \to V' \to V \to V'' \to 0$ induces a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & V' \arrow[r] \arrow[u] & V \arrow[r] \arrow[u] & V'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & D(V') \arrow[r] \arrow[u] & D(V) \arrow[r] \arrow[u] & D(V'') \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & V' \otimes \Omega \arrow[r] \arrow[u] & V \otimes \Omega \arrow[r] \arrow[u] & V'' \otimes \Omega \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

This exactness property implies

$$\mathrm{Tr}(a(V) \circ \varphi) = \mathrm{Tr}(a(V_1) \circ \varphi_1) + \mathrm{Tr}(a(V_2) \circ \varphi_2).$$

Since $\varphi_1 = \varphi_2 = 0$, this completes the proof in the case where φ is nilpotent.

Now for $\varphi = \lambda\,\mathrm{Id}$: It suffices to prove this in the case when V is a line bundle. For the general case follows from standard principle of splitting for Chern classes. Denote V by L, a line bundle in what follows.

Let us check $\langle\, a(L) \,|\, \varphi \,\rangle = -2\pi i \lambda {\cdot} \deg L$. First note that $a(L) \in H^1(X, L^* \otimes L \otimes \Omega^1_X) \simeq H^1(X, \Omega^1_X)$. We claim that the map

$$\begin{array}{ccc} H^1(X, \mathcal{O}_X^*) & \longrightarrow & H^1(X, \Omega^1_X) \\ L & \longmapsto & a(L) \end{array}$$

is $-2\pi i$ times the induced cohomology map of:

$$\begin{array}{ccc} \mathcal{O}_X^* & \longrightarrow & \Omega^1_X \\ f & \longmapsto & \frac{1}{2\pi i} dlog\ f \end{array}$$

where $dlog\ f = \frac{df}{f}$. If we admit this, then the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r, "e^{2\pi i}"] \arrow[d, equal] & \mathcal{O}_X^* \arrow[r] \arrow[d, "\frac{dlog\ f}{2\pi i}"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O}_X \arrow[r, "d"'] & \Omega^1_X \arrow[r] & 0
\end{tikzcd}$$

induces a commutative square

$$\begin{array}{ccc} H^1(X, \mathcal{O}_X^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) \\ a \downarrow & & \downarrow \\ H^1(X, \Omega_X^1) & \longrightarrow & H^2(X, \mathbb{C}) \end{array}$$

where the lower horizontal map is in fact an isomorphism of 1-dimensional vector spaces (it is surjective because there is no cohomology beyond the dimension, and $h^1(X, \Omega_X^1) = 1$ follows from Serre duality). Thus the Atiyah class of a holomorphic line bundle can be thought of as the complexified Chern class.

We now proceed to check the claim above. If $\{\theta_{ij}\}$ are transition functions for L then

$$a(L) = \{\theta_{ij}^{-1} d\theta_{ij}\} \in H^1(X, \Omega_X^1).$$

Indeed, take two local splittings $L \to D(L)$ and look at the class of the difference. Choose trivialisations of L

$$\begin{array}{ccccc} U_i \times \mathbb{C} & \xrightarrow[\sim]{\varphi_i} & L_{|U_i} & & = L_i \\ \downarrow d & & \downarrow d_i & & \\ \Omega^1_{X|U_i} & \xrightarrow[\sim]{\overline{\varphi}_i} & (L_i \otimes \Omega_X^1)_{|U_i} & = & L_i \otimes \Omega_X^1 \end{array}$$

Here $\overline{\varphi}_i$ is just $\varphi_i \otimes \mathrm{Id}$ and $d_i : L_i \to L_i \otimes \Omega_X^1$ is simply the composition $d_i = \overline{\varphi}_i \circ d \circ \varphi_i^{-1}$.

Now set

$$\begin{array}{rcl} \psi_i : L_i & \longrightarrow & D(L_i) \\ s & \longmapsto & (s, d_i s) \end{array}$$

and check that this is an $\mathcal{O}_X$-linear map. Obviously it is a local splitting for $\mathcal{R}(L)$ on U_i. It is also clear that the class $a(L)$ is represented by $\{d_i - d_j\}$ on $U_i \cap U_j$. To see $a(L)$ as a section of Ω_X^1 (on $U_i \cap U_j$), set $b_{ij}(v) = \overline{\varphi}_i^{-1}(d_i - d_j)\varphi_i(v)$ for $v \in \mathcal{O}_X(U_i \cap U_j)$. One can check that $b_{ij} = -\theta_{ij}^{-1} d\theta_{ij}$ where $\theta_{ij} = \varphi_j^{-1} \circ \varphi_i$ are the transition functions of L.

This completes the proof of the Theorem of André Weil. $\square$

Remark. — It is a fact that if X is a smooth projective variety and V a holomorphic vector bundle on X which admits a holomorphic connection, then all its Chern classes vanish. Since in general, a holomorphic connection is not necessarily flat, we cannot talk of the associated holonomy representation. But it has been conjectured that if V is a holomorphic vector bundle which can be equipped with a holomorphic connection, then V can be equipped with a *flat* holomorphic connection (cf. [A]).

3 Theorem of Narasimhan and Seshadri

Throughout this section we shall be considering the association

$$\mathrm{Rep}(\pi, \mathrm{GL}(n,\mathbb{C})) \quad \xrightarrow{\Psi} \begin{Bmatrix} \text{isomorphism classes of} \\ \text{holomorphic vector bundles} \\ \text{of rank } n \text{ and degree } 0 \end{Bmatrix}.$$

The theorem of André Weil states that the image of Ψ are those bundles whose indecomposable components all have degree zero. In particular, Ψ is *not* surjective since we have already encountered the example $\mathcal{O}(-x_0) \oplus \mathcal{O}(x_0)$ $(x_0 \in X)$ which does not come from a representation.

A morphism in the category of representations is a linear transformation $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that the following diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\varphi"] \arrow[d, "\rho_1(\gamma)"'] & \mathbb{C}^n \arrow[d, "\rho_2(\gamma)"] \\ \mathbb{C}^n \arrow[r, "\varphi"'] & \mathbb{C}^n \end{tikzcd}$$

commutes for all $\gamma \in \pi$. An isomorphism is an invertible morphism — that is, two representations ρ_1 and ρ_2 are isomorphic if there exists $C \in \mathrm{GL}(n,\mathbb{C})$ such that $\rho_1 = C\rho_2 C^{-1}$. In which case we will say that the two representations are equivalent and write $\rho_1 \sim \rho_2$.

The association Ψ really goes down to $\mathrm{Rep}(\pi, \mathrm{GL}(n,\mathbb{C}))/\sim$; but even then it is not injective as we have already noted in the case of line bundles:

Example. — Take non trivial $\rho_1, \rho_2 \in \mathrm{Rep}(\pi, \mathbb{C}^*)$ such that the associated line bundles $L_{\rho_1} \simeq L_{\rho_2} \simeq \mathcal{O}_X$, but $\rho_1 \not\simeq \rho_2$. This is possible because the fibres of Ψ are big. Then we have $\mathrm{Hom}(\rho_1, \rho_2) = 0$, because any non-zero map would be an isomorphism. On the other hand, $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathbb{C}$. So non-equivalent representations may induce isomorphic vector bundles.

As in the case of line bundles (cf. Proposition 1.8), to get injectivity, we must restrict our attention to the space $\mathrm{Rep}(\pi, \mathrm{U}(n))$ of *unitary* representations . The image will then no longer be all bundles whose indecomposable components have degree zero, but only *polystable* bundles, as we now will see.

Stability

Definition. — A holomorphic vector bundle V is said to be *stable* (resp. *semistable*) if for any proper non-zero subsheaf $W \subset V$ we have

$$\frac{\deg W}{\mathrm{rk}\, W} < \frac{\deg V}{\mathrm{rk}\, V} \qquad (\text{resp. } \tfrac{\deg W}{\mathrm{rk}\, W} \leq \tfrac{\deg V}{\mathrm{rk}\, V}).$$

The sheaf W is not necessarily a subbundle, but it is a locally free sheaf and thus a vector bundle in its own right. Stability of a bundle can also be defined with respect to only subbundles. This can be seen to be equivalent to the definition given above.

3.1 Observation. — If V is semistable of degree zero, then all its indecomposable components also have degree zero. Indeed, if $V = \oplus V_i$ is the decomposition of V into its indecomposable components, we have: $\frac{\deg V_i}{\operatorname{rk} V_i} \leq \frac{\deg V}{\operatorname{rk} V}$ which in turn forces the degree of V_i to be zero for all i. Similarly, a stable bundle of degree zero is easily checked to be indecomposable.

Theorem 4 *(Narasimhan–Seshadri [NS2]) — Let X be a compact Riemann surface of genus $g \geq 2$. Then*

(i) A holomorphic bundle V of rank n and degree zero is stable if and only if $V \simeq V_\rho$ for some irreducible unitary representation ρ of the fundamental group of X.

(ii) The association $\rho \rightsquigarrow V_\rho$ establishes an equivalence of categories between the category of irreducible unitary representations of the fundamental group of X and the category of stable holomorphic vector bundles of degree zero on X.

Remark. — Since $\mathrm{U}(n)$ is compact, all representations are semi-simple; that is, they are direct sums of irreducible representations. Hence,

3.2 Corollary. — *A holomorphic vector bundle V of degree zero and rank n is isomorphic to V_ρ for some unitary representation ρ if and only if it decomposes as*

$$V \simeq \bigoplus_{i=1}^{k} V_i$$

where each V_i is stable of degree zero.

Definition. — A holomorphic vector bundle which can be decomposed as a direct sum of stable bundles of degree zero is called a *polystable* bundle.

3.3 Remark. — It easily follows from the definitions that a polystable bundle is always semistable, but not conversely. In fact, polystable bundles arise as specialisations of semistable bundles. As we shall see, the space of stable bundles is smooth but non-compact and is compactified by adding points corresponding to polystable bundles.

3.4 Corollary. — *Let ρ_1 and ρ_2 be unitary representations of the fundamental group of X. Then the associated vector bundles V_{ρ_1} and V_{ρ_2} are isomorphic if and only if the representations are equivalent.*

Here it doesn't matter if equivalence is taken to be in $\mathrm{GL}(n, \mathbb{C})$ or in $\mathrm{U}(n)$; the equivalence relation is the same.

Plan of proof: The proof has two parts. The first part shows that if ρ is an irreducible unitary representation, then the associated bundle V_ρ is stable. Further, it also shows that there is an isomorphism

$$\mathrm{Hom}(\rho_1, \rho_2) \simeq \mathrm{Hom}(V_{\rho_1}, V_{\rho_2}).$$

These two results rely on the principle of local constancy which we proceed to establish.

The second part (the surjectivity part) is the hard part: here it is shown that every stable bundle is associated to an irreducible unitary representation.

3.5 Principle of Local Constancy. (cf. [NS1]) — If ρ is unitary, then any section of V_ρ which vanishes at some point must vanish at all points.

Proof of the Principle. — Consider once again the pull-back diagram

$$\begin{array}{ccc} \widetilde{X} & \longleftarrow & \widetilde{X} \times \mathbb{C}^n \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \underset{\phi}{\longleftarrow} & V_\rho \end{array}$$

We have seen that global sections of V_ρ can be considered as π-equivariant holomorphic maps $\widetilde{X} \to \mathbb{C}^n$ — this is valid for any representation ρ. Now the claim is that when the representation is *unitary*, these π-equivariant holomorphic maps $\widetilde{X} \to \mathbb{C}^n$ are in fact *constant*. Let us denote by $(\mathbb{C}^n)^\pi$ the subspace of π-invariants for the π-action on $\mathbb{C}^n$ via ρ. This is clearly the same as π-equivariant constant maps from X into $\mathbb{C}^n$. So what we want to prove is that the natural map

$$(\mathbb{C}^n)^\pi \longrightarrow H^0(X, V_\rho)$$

is an isomorphism. The injectivity of this map is trivial. Let us show it is surjective: let $s \in H^0(X, V_\rho)$ and think of it as a π-equivariant holomorphic map $s : \widetilde{X} \to \mathbb{C}^n$. Let us denote by $s_i : \widetilde{X} \to \mathbb{C}$ the composition of s with the i-th co-ordinate function on $\mathbb{C}^n$. Consider now the continuous function

$$\begin{array}{rcl} h : \widetilde{X} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\sum_{i=1}^{n} |s_i(x)|^2 \end{array}$$

Since ρ is unitary and thus preserves the inner product, this is a π-invariant continuous function. So it goes down to a continuous function on X. But since X is compact, this implies that h attains its maximum in $\widetilde{X}$. Furthermore h is a subharmonic function (that is, $\Delta h \geq 0$) as the modulus of a holomorhic function and a finite sum of subharmonic functions is subharmonic. so we conclude that h is constant on $\widetilde{X}$. (This is a consequence of the "mean-value" property of subharmonic functions: $h(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta})\, d\theta$, cf. [Ahl])

From this one can deduce that the functions s_i's are constant (use the maximum modulus property of holomorphic functions) and hence, s is constant. That is to say, it comes from $(\mathbb{C}^n)^\pi$. This completes the proof of the local constancy principle. □

Part 1 of the proof of Theorem 4. — Let us prove the implication "ρ is an irreducible unitary representation $\Rightarrow V_\rho$ is stable". To show this, we first prove that if ρ is a unitary representation, then V_ρ is semistable. (In fact it will be polystable.) Suppose the contrary: thus there exists $W \subset V_\rho$ such that $\frac{\deg W}{\operatorname{rk} W} > \frac{\deg V_\rho}{\operatorname{rk} V_\rho} = 0$. Let $r = \operatorname{rk} W$ and $d = \deg W > 0$. Then $\bigwedge^r W$ is a line bundle of degree $d > 0$ - that is, $\bigwedge^r W \in \operatorname{Pic}^d X$. Now inside $\operatorname{Pic}^d X$, consider the space T_d of bundles with non-trivial global holomorphic sections. One can show that T_d has positive dimension. So twisting $\bigwedge^r W$ with some $L \in \operatorname{Pic}^\circ X$ we can guarantee the existence of a global section. More precisely, there is a non-zero section $s \in H^0(X, \bigwedge^r W \otimes L)$, where $L \simeq L_\chi$ for some unitary character χ, (cf. 1.8). Since the bundle has degree $d > 0$, s must have a zero. Now $\bigwedge^r W \otimes L_\chi$ is a subbundle in $\bigwedge^r V_\rho \otimes L_\chi \simeq V_{\wedge^r \rho \otimes \chi}$, so s is a global section of $V_{\wedge^r \rho\otimes\chi}$ having a zero. But $\wedge^r \rho \otimes \chi$ is unitary, so by the principle of local constancy, such a section must be the zero section — a contradiction. So V_ρ is semistable as claimed.

Next let us prove that if ρ is an *irreducible* unitary representation, then the associated bundle V_ρ is stable. Suppose not; then there exists a subbundle $W \subset V_\rho$ such that $\deg W \geq 0$. But since V_ρ has already been shown to be semistable, $\deg W = 0$.

Now, $\bigwedge^r W \subset \bigwedge^r V_\rho \simeq V_{\wedge^r \rho}$, and $\bigwedge^r \rho$ is a unitary representation (though not necessarily irreducible). So repeating the argument above we conclude that $\bigwedge^r V_\rho$ has a nowhere vanishing section s. Take a point $x_0 \in X$, then $s(x_0) = t_1 \wedge \cdots \wedge t_r$ where t_i belongs to the fibre of V_ρ at x_0. Or equivalently, pull back the section by p

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p"'] & \widetilde{X} \times \bigwedge^r \mathbb{C}^n \arrow[l] \arrow[d] \\ X \arrow[r, shift right, "s"'] & V_{\wedge^r \rho} \arrow[l, shift right, "\phi"'] \end{tikzcd}$$

to get $\tilde{s} = p^* s$ — a π-equivariant map $\widetilde{X} \to \bigwedge^r \mathbb{C}^n$. Since $\bigwedge^r \rho$ is also a unitary representation, such a section must be constant: $\operatorname{Im} \tilde{s} = v_1 \wedge \cdots \wedge v_r$, $v_i \in \mathbb{C}^n$.

If we can now show that $\operatorname{span}(v_1, \ldots, v_r)$ is a π-invariant subspace of $\mathbb{C}^n$ (note that it is of dimension r, $0 < r < n$), we will have established a contradiction to the assumption that ρ is irreducible. To this end, consider the Plücker embedding

$$\begin{array}{rcl} Gr(r, \mathbb{C}^n) & \hookrightarrow & \mathbb{P}(\bigwedge^r \mathbb{C}^n) \\ \operatorname{span}(v_1, \ldots, v_r) & \mapsto & [v_1, \wedge \cdots \wedge v_r] \end{array}$$

which is $\mathrm{GL}(n, \mathbb{C})$-equivariant. This proves $\operatorname{span}(v_1, \ldots, v_r)$ is π-invariant.

Now for (ii) of the theorem. Let ρ_1 and ρ_2 be two irreducible unitary representations. We want to show that the canonical map

$$\operatorname{Hom}(\rho_1, \rho_2) \hookrightarrow \operatorname{Hom}(V_{\rho_1}, V_{\rho_2})$$

is an isomorphism. The left hand side is $\operatorname{Hom}(\rho_1, \rho_2) \simeq [(\mathbb{C}^n)^* \otimes \mathbb{C}^n]^\pi$, with π acting via $\rho := {}^t\rho_1^{-1} \otimes \rho_2$.

The right hand side is $\operatorname{Hom}(V_{\rho_1}, V_{\rho_2}) \simeq H^0(X, V_{\rho_1}^* \otimes V_{\rho_2}) \simeq H^0(X, V_{{}^t\rho_1^{-1} \otimes \rho_2})$; but by the local constancy principle, $H^0(X, V_\rho) \simeq [(\mathbb{C}^n)^* \otimes \mathbb{C}^n]^\pi$, and the result follows.

This concludes the first part of the proof. $\square$

Now for the hard part of the proof: to show that every stable bundle of degree zero is isomorphic to a bundle associated to an irreducible unitary representation. The idea of proof is as follows: first we show that the space of equivalence classes of irreducible unitary representations of dimension n is a real analytic manifold of dimension $2(n^2(g-1)+1)$.

Next we invoke a theorem affirming that the isomorphism classes of stable holomorphic bundles of rank n and degree zero form the points of a connected complex manifold of (complex) dimension $n^2(g-1)+1$.

Finally, we show that the association of vector bundles to representations is a continuous surjective map between these two manifolds. As we have already shown that the map is injective, this will complete the proof of the theorem.

Theorem 5 — *The space* $\operatorname{Irr}\operatorname{Rep}(\pi, \mathrm{U}(n))/\sim$ *of equivalence classes of irreducible unitary representations of dimension* n *is a real analytic manifold of dimension*

$$2(n^2(g-1)+1).$$

Let F be the free group on $2g$ letters $a_1, \ldots, a_g, b_1, \ldots, b_g$. Then the fundamental group of X can be presented as:

$$\pi = \frac{F}{\prod_{i=1}^{g}[a_i, b_i]},$$

so that giving a representation $\rho : \pi \to \mathrm{U}(n)$ is equivalent to giving $2g$ elements $A_1, \ldots, A_g$, $B_1, \ldots, B_g \in \mathrm{U}(n)$ satisfying the relation $\prod_{i=1}^{g}[A_i, B_i] = I$.

Let us set $\Omega = \mathrm{U}(n)^{2g}$. Consider the real analytic map

$$\begin{array}{rcl} f : \Omega & \longrightarrow & \mathrm{SU}(n) \\ (A_1, \ldots, A_g, B_1, \ldots, B_g) & \longmapsto & \displaystyle\prod_{i=1}^{g}[A_i, B_i] \end{array}$$

then we have $\operatorname{Rep}(\pi, \mathrm{U}(n)) = f^{-1}(I)$ as sets.

3.6 Proposition. — *The map* f *is submersive at* $\rho \in \Omega$ *(i.e.* df_ρ *has maximal rank) if and only if* ρ *is irreducible*

Proof. — The proof amounts to computing the dimension of $\operatorname{Ker} df_\rho$.

Denote by $\operatorname{ad}\rho$ the composite representation

$$\pi \xrightarrow{\rho} \mathrm{U}(n) \xrightarrow{\mathrm{ad}} \operatorname{Aut}_{\mathbb{R}}(\mathfrak{u}(n))$$

where $\mathfrak{u}(n)$ is the Lie algebra of $\mathrm{U}(n)$ and ad is the adjoint representation.

Note that any representation of π can be composed with the natural projection $F \to \pi$ to give a representation of F.

Denote by $Z^1(F, \operatorname{ad}\rho)$ the space of 1-cocycles $\{h : F \to \mathfrak{u}(n) \mid h(\gamma_1 \cdot \gamma_2) = \gamma_1.h(\gamma_2) + h(\gamma_1)\}$ of the F module $\mathfrak{u}(n)$ and by $B^1(F, \operatorname{ad}\rho)$ the space of 1-coboundaries $\{h : F \to \mathfrak{u}(n) \mid h(\gamma) = \gamma \cdot A - A \text{for some} A \in \mathfrak{u}(n).\}$ Then $Z^1(F, \operatorname{ad}\rho) \supset B^1(F, \operatorname{ad}\rho)$ and we denote by $H^1(F, \operatorname{ad}\rho)$ the cohomology group $\frac{Z^1(F, \operatorname{ad}\rho)}{B^1(F, \operatorname{ad}\rho)}$. Note that the space $H^0(\pi, \operatorname{ad}\rho)$ can also be thought of as the subspace $\mathfrak{u}(n)^\pi$ of π-invariants for the $\operatorname{ad}\rho$-action of π on $\mathfrak{u}(n)$.

Since a $h \in Z^1(F, \operatorname{ad}\rho)$ is uniquely determined by its value on the generators of F, giving a 1-cocycle h is equivalent to giving $2g$ elements in $\mathfrak{u}(n)$. That is,

$$Z^1(F, \operatorname{ad}\rho) \simeq \bigoplus_{2g \text{ copies}} \mathfrak{u}(n)$$

Now let us compute $\operatorname{Ker} df_\rho$. Let $\mathbf{v} \in \mathsf{T}_\rho\Omega$; by a right translation we have the identification: $\mathbf{v} = (A_1, \ldots, A_g, B_1, \ldots, B_g)$ with $A_i, B_i \in \mathfrak{u}(n)$. We then have an isomorphism

$$\begin{array}{rcl} \mathsf{T}_\rho\Omega & \xrightarrow{\sim} & Z^1(F, \operatorname{ad}\rho) \\ \mathbf{v} & \mapsto & \delta_{\mathbf{v}} \end{array}$$

where $\delta_{\mathbf{v}} : F \to \mathfrak{u}(n)$ is the 1-cocycle determined by $\mathbf{v}$: $a_i, b_i \mapsto A_i, B_i$. It is not difficult to verify that $\mathbf{v} \in \mathsf{T}_\rho\Omega$ belongs to $\operatorname{Ker} df_\rho$ if and only if $\delta_{\mathbf{v}}(\prod[a_i, b_i]) = 0$ (see [NS1]).

This means that we have a commutative diagram

$$\begin{tikzcd} \mathsf{T}_\rho\Omega \arrow[r] & Z^1(F, \operatorname{ad}\rho) \\ \operatorname{Ker} df_\rho \arrow[u, hook] \arrow[r, "\sim"] & Z^1(\pi, \operatorname{ad}\rho) \arrow[u, hook] \end{tikzcd}$$

and here what we are interested in is the isomorphism $\operatorname{Ker} df_\rho \xrightarrow{\sim} Z^1(\pi, \operatorname{ad}\rho)$. We now have

$$\begin{aligned} \dim \operatorname{Ker} df_\rho = \dim Z^1(\pi, \operatorname{ad}\rho) &= \dim H^1(\pi, \operatorname{ad}\rho) + \dim B^1(\pi, \operatorname{ad}\rho) \\ &= h^1(\pi, \operatorname{ad}\rho) + \dim \mathfrak{u}(n) - \dim \mathfrak{u}(n)^\pi \end{aligned}$$

by the very definition of $B^1(\pi, \operatorname{ad}\rho)$. So we get

$$\dim \operatorname{Ker} df_\rho = h^1(\pi, \operatorname{ad}\rho) + n^2 - h^0(\pi, \operatorname{ad}\rho).$$

Now we relate these cohomology numbers to singular cohomology on X with coefficients in local systems. Consider the universal covering $p : \widetilde{X} \to X$. The representation $\operatorname{ad}\rho : \pi \to \operatorname{Aut}(\mathfrak{u}(n))$ gives rise to a local system on X, say L, of dimension n^2. As a sheaf of real vector spaces we have $L = p_*^\pi \mathfrak{u}(n)$, that is, if $U \subset X$ is an open set, then $p_*^\pi \mathfrak{u}(n)(U)$ consists of the π-equivariant continuous maps from $p^{-1}(U)$ to $\mathfrak{u}(n)$.

One can define Singular Cohomology with values in a local system. Then we have:

$$H^i(X, L) \simeq H^i(\pi, \operatorname{ad}\rho)$$

(cf. [M]. Here we must assume $g \geq 2$, to ensure that $\widetilde{X}$ is contractible). To compute these cohomology numbers we make use of the Euler-Poincaré formula:

$$n^2(2 - 2g) = \chi(L) = h^0(X, L) - h^1(X, L) + h^2(X, L),$$

since L is a local system of real vector spaces of dimension n^2 (cf. [B]).

Now by the Poincaré duality theorem for cohomology with coefficients in the local system L (see [B]), we have

$$H^2(X, L) \simeq H^0(X, L^*),$$

(here L^* is the *dual local system*). But here $L \simeq L^*$: for, $A \mapsto \operatorname{Tr}(AA^*)$ defines a nondegenerate quadratic form on $\mathfrak{u}(n)$ which is left invariant by the adjoint action of $\mathrm{U}(n)$ on $\mathfrak{u}(n)$. As π acts via the adjoint representation, we have $\operatorname{ad}\rho \sim {}^t(\operatorname{ad}\rho)^{-1}$, which in turn, implies $L \simeq L^*$.

These two observations give

$$h^1(X, L) = 2h^0(X, L) + n^2(2g - 2).$$

So, summarising, we have

$$\dim \operatorname{Ker} df_\rho = 2gn^2 - (n^2 - h^0(\pi, \operatorname{ad} \rho)).$$

But $df\rho$ has maximal rank when this number is equal to

$$\dim \Omega - \dim \mathrm{SU}(n) = 2gn^2 - (n^2 - 1).$$

This happens exactly when $h^0(\pi, \operatorname{ad} \rho) = 1$. But by Schur's lemma, $h^0(\pi, \operatorname{ad} \rho) = 1$ if and only if ρ is irreducible.

This concludes the proof of the Proposition. □

3.7 Corollary. — *The space* $\operatorname{Irr} \operatorname{Rep}(\pi, \mathrm{U}(n))$ *admits a natural structure of a real analytic manifold of dimension* $2(n^2(g - 1) + 1) + (n^2 - 1)$ *and the tangent space at* ρ, $\mathsf{T}_\rho(\operatorname{Irr} \operatorname{Rep}(\pi, \mathrm{U}(n))) \simeq Z^1(\pi, \operatorname{ad} \rho)$.

Conclusion of the Proof of Theorem 5. — Consider the action of $\mathrm{U}(n)$ on $\operatorname{Irr} \operatorname{Rep}(\pi, \mathrm{U}(n))$ by conjugation. Note that the centre of $\mathrm{U}(n)$ acts trivially. Again by Schur's lemma, the centre is precisely the isotropy. This means that the action goes down to a *free* $\mathrm{PU}(n)$ action on $\operatorname{Irr} \operatorname{Rep}(\pi, \mathrm{U}(n))$ and since $\mathrm{PU}(n)$ is compact, the quotient

$$Q = \frac{\operatorname{Irr} \operatorname{Rep}(\pi, \mathrm{U}(n))}{\mathrm{PU}(n)}$$

is a real analytic manifold (non-compact) of dimension $2(n^2(g - 1) + 1)$. (It follows furthermore that $\mathsf{T}_{[\rho]}(Q) \simeq H^1(\pi, \operatorname{ad} \rho)$, but we will not need this result.) □

We now turn our attention to the space of isomorphism classes of stable bundles.

Definition. — A *holomorphic family* of holomorphic vector bundles on X, parametrised by a complex manifold T (denoted by $\{W_t\}_{t \in T}$), is a holomorphic vector bundle W on $X \times T$. (Here $W_t := W_{|X \times \{t\}}$.)

Remark. — Note that the rank and the degree of W_t , $t \in T$, remain constant if T is connected. The fact that $\deg W_t$ remains constant follows from the homotopy property of vector bundles.

Now we state without proof the following "analytic version" of a theorem of Mumford (cf. [NS2]). For a brief sketch of its proof, see §4.

Theorem 6 — *There exists a complex manifold* $M(n, 0)$ *such that*

(i) The points on $M(n, 0)$ *parametrise isomorphism classes of stable holomorphic bundles of rank* n *and degree* 0.

(ii) The manifold $M(n,0)$ is connected and is of (complex) dimension $n^2(g-1)+1$.

(iii) The complex structure on $M(n,0)$ is natural. That is, if $\{W_t\}_{t\in T}$ is a family of stable holomorphic bundles of rank n and degree 0, parametrised by a complex manifold T, then the classifying map

$$\begin{array}{ccc} T & \longrightarrow & M(n,0) \\ t & \longmapsto & [W_t] \end{array}$$

is holomorphic.

Remark. — In the case when $n=1$, this complex manifold is just $\operatorname{Pic}^\circ X$.

* * *

We shall now continue with the proof of Theorem 4, and show that every stable bundle of degree zero is isomorphic to a bundle associated to an irreducible unitary representation.

Part 2 of the Proof of Theorem 4. — We have to show that the map

$$\begin{array}{ccc} \dfrac{\operatorname{Irr}\operatorname{Rep}(\pi,\mathrm{U}(n))}{\mathrm{PU}(n)} & \xrightarrow{\Phi} & M(n,0) \\ \rho & \longmapsto & [V_\rho] \end{array}$$

is surjective. Because $M(n,0)$ is connected, it is enough to show that $\operatorname{Im}\Phi$ is both open and closed. We have already seen that Φ is an injection; let us check that Φ is continuous. Let $R' := \operatorname{Rep}(\pi,\mathrm{GL}(n,\mathbb{C}))$ and $R := \operatorname{Irr}\operatorname{Rep}(\pi,\mathrm{GL}(n,\mathbb{C}))$. consider now the map

$$\begin{array}{ccc} \mathrm{GL}(n,\mathbb{C})^{2g} & \longrightarrow & \mathrm{SL}(n,\mathbb{C}) \\ (A_i,B_i) & \longmapsto & \prod[A_i,B_i]. \end{array}$$

An argument similar to that of the proof of Theorem 5 shows that R acquires a natural structure of a complex manifold and it can be proved that there is a *holomorphic family* $(V_\rho)_{\rho\in R}$ of vector bundles on X parametrised by R. Set

$$R^s = \{\rho\in R \mid V_\rho \text{ is stable}\}.$$

This subset is non-empty and is in fact an *open subset* in R. (In any family of bundles, the set of points representing stable (resp. semistable) bundles is open. See [NS2] for details). By the naturality of the complex structure of $M(n,0)$, there is a holomorphic classifying map $R^s\to M(n,0)$. We have already shown that irreducible unitary representations give rise to stable bundles. So, we also have a continuous injection $\operatorname{Irr}\operatorname{Rep}(\pi,\mathrm{U}(n))\hookrightarrow R^s$. The composition

$$\operatorname{Irr}\operatorname{Rep}(\pi,\mathrm{U}(n))\hookrightarrow R^s\to M(n,0)$$

is therefore a *continuous* map, and it goes down to a continuous map Φ of real manifolds on the quotient. Since this map Φ is also injective, by Brouwer's theorem on invariance of domains, it follows that $\operatorname{Im}\Phi$ is open.

Let us show $\operatorname{Im}\Phi$ is also closed. Let $U := \operatorname{Rep}(\pi, \mathrm{U}(n))$; it is compact as $U = f^{-1}(I) \subset \Omega$ (cf. discussion preceding Proposition 3.6). Set $R'^{ss} = \{\rho \in R' \mid V_\rho \text{ is semi stable}\}$. It is clear that $U \subset R'^{ss}$. Now define $K \subset R'^{ss} \times M(n,0)$ to be

$$K := \{(\rho, t) \mid \dim \operatorname{Hom}(V_\rho, W_t) \neq 0\}.$$

(here W_t stands for the bundle represented by the point $t \in M(n,0)$). By the theorem of semicontinuity, K is closed. Let $q : R'^{ss} \times M(n,0) \to M(n,0)$ be the natural projection. Then $q(K \cap U \times M(n,0)) \subset M(n,0)$ is closed, since K is closed and U is compact. We have

$$q((K \cap U \times M(n,0)) = \{t \in M(n,0) \mid \exists \rho \in U \text{ with } \operatorname{Hom}(V_\rho, W_t) \neq 0\}.$$

But now the following lemma shows that any non-zero morphism $V_\rho \to W_t$ must be an isomorphism and hence $q(K \cap U \times M(n,0)) = \operatorname{Im}\Phi$. Thus $\operatorname{Im}\Phi$ is closed. This concludes the proof. $\square$

3.8 Lemma. — *Let V and W be vector bundles of same rank and degree; suppose V is semistable and W is stable. Then any non-zero map $\varphi : V \to W$ is an isomorphism.*

Proof. — Note first that the notion of semistability (resp. stability) can be defined equivalently in terms of quotient sheaves instead of subsheaves: a bundle V is semistable when for all proper non-zero quotient sheaves Q we have $\frac{\deg Q}{\operatorname{rk} Q} \geq \frac{\deg V}{\operatorname{rk} V}$.

Suppose that φ is not surjective; then $\operatorname{Im}\varphi \simeq V/\operatorname{Ker}\varphi$ is a proper non-zero quotient of V. So we have

$$\frac{\deg V}{\operatorname{rk} V} \leq \frac{\deg \operatorname{Im}\varphi}{\operatorname{rk} \operatorname{Im}\varphi}.$$

On the other hand, since W is stable and $\operatorname{Im}\varphi$ is a non-zero proper subsheaf of V, we also have

$$\frac{\deg \operatorname{Im}\varphi}{\operatorname{rk} \operatorname{Im}\varphi} < \frac{\deg W}{\operatorname{rk} W}.$$

A contradiction. So φ is surjective and since the ranks of V and W are equal, it is injective as well. $\square$

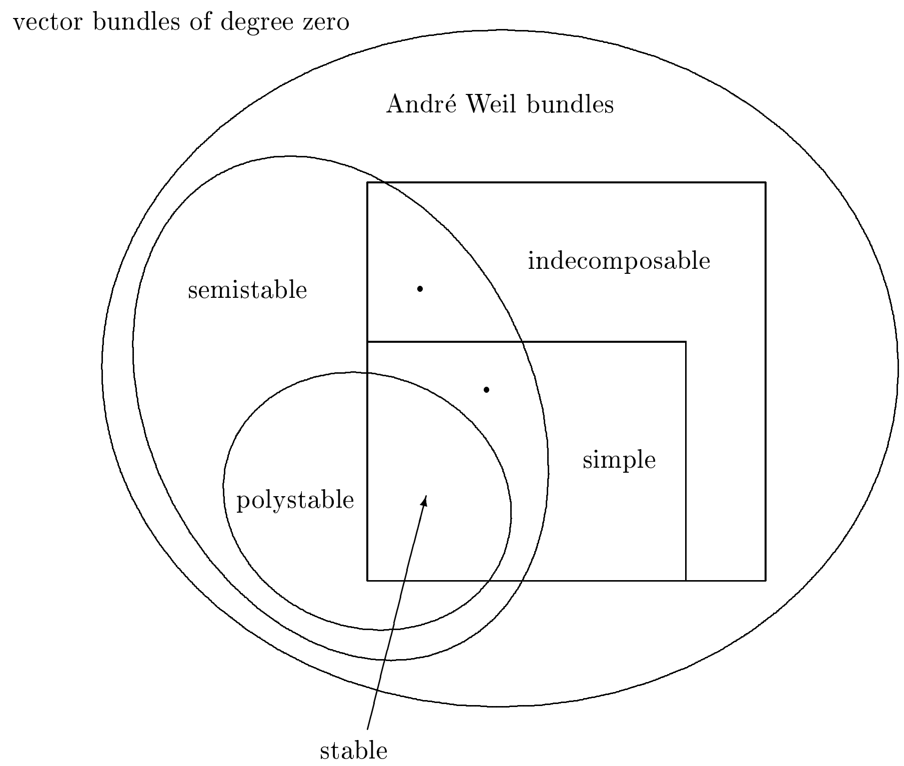
Simple Bundles and Examples

Definition. — A vector bundle V on X is called *simple* if $\dim . H^0(X, \operatorname{End} V) = 1$.

3.9 Observations. — A simple bundle is always indecomposable (indeed, if it were decomposable: $V \simeq V_1 \oplus V_2$, then the space of global endomorphisms of V would be of dimension at least two).

Also, a stable bundle is simple. For, the lemma 3.8 above shows that any non-zero endomorphism of a stable bundle must be an isomorphism, that is, $H^0(X, \operatorname{End} V)$ is a division algebra over $\mathbb{C}$. Since this division algebra (whose centre contains $\mathbb{C}$) is also a finite extension of $\mathbb{C}$ and since $\mathbb{C}$ is algebraically closed, it must be the trivial extension. That is, $H^0(X, \operatorname{End} V) = \mathbb{C}$.

The relations between the various types of degree zero bundles are represented by the following picture:



By André Weil bundles we mean bundles all of whose indecomposable components have degree zero.

Let us show by examples that the two dotted regions in the picture are non-empty.

3.10 Example. — Let L be a line bundle of degree zero such that $L^2 := L \otimes L \not\simeq \mathcal{O}_X$. Define a degree zero vector bundle V of rank 2 by a non-split extension

$$0 \to L \to V \to L^* \to 0. \tag{$*$}$$

(Such an extension exists because $\mathrm{Ext}^1(L^*, L) \simeq H^1(X, L^2) \neq 0$, by Serre duality theorem.) Then V is simple and semistable but *not* stable.

Proof. — Let us first see that V is simple. Dualising $(*)$ we get $0 \to L \to V^* \to L^* \to 0$. Tensoring this with V and taking global sections we get the exact sequence

$$0 \to H^0(X, L \otimes V) \to H^0(X, \mathrm{End}\, V) \to H^0(X, L^* \otimes V).$$

Now we claim that $H^0(X, L \otimes V) = 0$ and $H^0(X, L^* \otimes V) \simeq \mathbb{C}$. From this it follows that $H^0(X, \mathrm{End}\, V) \simeq \mathbb{C}$, proving the simplicity of V.

To prove the claim, consider a morphism $\varphi : L^* \to V$ and think of it as an element in $H^0(X, L \otimes V)$.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & V \arrow[r, "p"] & L^* \arrow[r] & 0 \\
 & & L^* \arrow[u, "\varphi"] \arrow[ur] & &
\end{tikzcd}$$

The composition $L^* \xrightarrow{\varphi} V \xrightarrow{p} L^*$ must be scalar multiplication and since the extension is non-split, it must be the zero morphism. So φ factors through L. The lift $\overline{\varphi} : L^* \to L$ can be thought of as a section of L^2 and so by assumption is the zero map. So φ is also zero. This shows that $H^0(X, L \otimes V) = 0$.

To compute $H^0(X, L^* \otimes V)$: tensoring $(*)$ with L^* we get

$$0 \to \mathcal{O}_X \to L^* \otimes V \to (L^*)^2 \to 0.$$

Now $H^0(X, (L^*)^2) = 0$, so from the long exact cohomology sequence of the above short exact sequence we get

$$H^0(X, L^* \otimes V) \simeq \mathbb{C}.$$

Next, we check that V is semistable. Let M be a line bundle of positive degree and suppose that there is a map $\varphi : M \to V$ (the following argument is similar to the one just carried out). In the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & V \arrow[r, "p"] & L^* \arrow[r] & 0 \\
 & & M \arrow[u, "\varphi"] \arrow[ur] & &
\end{tikzcd}$$

the composition $M \xrightarrow{\varphi} V \xrightarrow{p} L^*$ can be thought of as a section of $M^* \otimes L^*$. Since $M^* \otimes L^*$ has negative degree, such a section must be the zero section. So φ factors through L giving a section of $M^* \otimes L$, which again must be zero, whence φ is the zero map. So there are no inclusions of positive degree line bundles in V. That is, V is semistable. Finally, we note that V is not stable as it has a degree zero sub-bundle L. $\square$

Note that the bundle V considered above is associated to a $GL(n, \mathbb{C})$-representation of the fundamental group (since it is indecomposable of degree zero) but not to *any* unitary representation.

3.11 Example. — Consider a non-split extension of type

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{i} V \xrightarrow{p} \mathcal{O}_X \longrightarrow 0 \qquad (*)$$

Then V is semistable and indecomposable, but *not* simple (in fact, $\dim H^0(X, \operatorname{End} V) = 2$).

Proof. — The semistability follows from the same argument as in the previous example. Suppose V is not indecomposable, i.e. $V \simeq L_1 \oplus L_2$. Since V is semistable, L_1 and L_2 must be line bundles of degree zero. Now in the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X \arrow[r] & V \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ & & L_i \arrow[u, hook, "\varphi_i"] \arrow[ur] & & \end{tikzcd}$$

the composition $L_i \overset{\varphi_i}{\hookrightarrow} V \to \mathcal{O}_X$ must either be an isomorphism or zero. If it is zero then φ_i factors through the left hand $\mathcal{O}_X$, and this lift must then be an isomorphism. In any case, $L_i \simeq \mathcal{O}_X$, contradicting the initial assumption that $(*)$ is non-split.

To see that V is not simple, let us first note that $H^0(X,V) \simeq \mathbb{C}$: Indeed, taking global sections in the sequence $(*)$ gives

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X,V) \to H^0(X,\mathcal{O}_X).$$

An element in $H^0(X,V)$ can be thought of as a map $s : \mathcal{O}_X \to V$ and its image in $H^0(X,\mathcal{O}_X)$ is the composition $\mathcal{O}_X \xrightarrow{s} V \xrightarrow{p}$, which must be zero: otherwise s could be scaled to give a splitting to p. So $H^0(X,\mathcal{O}_X) \to H^0(X,V)$ must be an isomorphism.

Now dualising $(*)$, tensoring with V, and taking global sections yields

$$0 \longrightarrow H^0(X,V) \longrightarrow H^0(X, \operatorname{End} V) \xrightarrow{q} H^0(X,V).$$

Here q is non-zero, because q evaluated on the identity element in $H^0(X, \operatorname{End} V)$ gives i, considered as an element in $H^0(X,V)$. So q is a surjection, and this makes $H^0(X, \operatorname{End} V)$ an extension of the two 1-dimensional vector spaces $H^0(X,V)$. □

4 Deformations of Vector Bundles and Local Moduli

A good reference for the material of this section is [NS2]. See also [S] which contains an extensive list of further references.

Let $\{V_t\}_{t\in T}$ be a family of vector bundles on X parametrised by a complex manifold T. (That is, $V \to X \times T$ is a holomorphic vector bundle.) Let $t_0 \in T$ be a fixed point and set $V_0 = V_{t_0}$. We say that $\{V_t\}_{t\in T}$ is a *deformation* of V_0 over T. These data will remain fixed in what follows.

If $X \times T \overset{q}{\to} X$ is the natural projection onto the first factor, then the *trivial* deformation of V_0 over T is the pullback q^*V_0.

Two natural questions that now arise are: how far is a given deformation from being trivial and how to measure it?

A related question is: can one extend the identity map

$$V_{|X\times\{t_0\}} = V_0 = q^*V_{0|X\times\{t_0\}}$$

to an isomorphism

$$V_{|X\times U} \simeq q^*V_{0|X\times U}$$

over a neighbourhood $U \subset T$ of t_0?

Locally on X, this is always possible: there exists an open cover $\{U_i\}$ of X, a neighbourhood $U \ni t_0$ and isomorphisms $\psi_i : V_{|U_i \times U} \xrightarrow{\sim} q^* V_{0|U_i \times U}$ extending the identity map $V_{|U_i \times \{t_0\}} = q^* V_{0|U_i \times \{t_0\}}$

The construction of these maps goes as follows: around each point of $X \times \{t_0\}$, there is a cubic open set $U_i \times W_i \subset X \times T$ such that V can be trivialised on it. Cover $X \times \{t_0\}$ with these trivialising open sets. Since X is compact we can take this cover to be finite. Now if we set $U = \cap W_i$, then U is open. Using the trivialisations, one can in an obvious fashion lift the identity on $U_i \times \{t_o\}$ to an isomorphism ψ_i as above.

The obstruction to patching these ψ_i together to get a global map is now measured by the extent to which the automorphisms

$$\theta_{ij} = \psi_i \circ \psi_j^{-1} : q^* V_{0|(U_i \cap U_j) \times U} \longrightarrow q^* V_{0|(U_i \cap U_j) \times U}$$

fail from being the identity automorphism over $(U_i \cap U_j) \times U$. These automorphisms satisfy the cocycle rule $\theta_{ij} \circ \theta_{jk} = \theta_{ik}$ on $(U_i \cap U_j \cap U_k) \times U$ and hence define an element in $Z^1(X \times U, \mathrm{Aut}(q^* V_0))$. ($\mathrm{Aut}(q^* V_0)$ is the sheaf of automorphisms of $q^* V_0$; note that it is a sheaf of non-abelian groups). Note also that the identity automorphism of V_0 can be extended to a global (that is, on $X \times U$) isomorphism of V with $q^* V_0$ *if and only if* $\{\theta_{ij}\}$ is a coboundary $\delta\{\phi_i\}$ with the property that $\phi_i(-, t_0)$ is the identity automorphism of $V_{0|U_i}$. Assume now that U is an open disc Δ in the complex plane and $t_0 = 0$, so that we are looking only at a 1-parameter deformation. The Taylor expansion of $\theta_{ij}(t)$ in t around the origin gives

$$\theta_{ij}(t) = \mathrm{Id} + h_{ij} t + \text{ higher order terms}$$

where $h_{ij} = \frac{d}{dt} \theta_{ij}(t) \mid_{t=0}$ are local endomorphisms of V_0. Since $\{\theta_{ij}\}$ satisfies the multiplicative cocycle rule for all $t \in \Delta$, we get $h_{ij} + h_{jk} = h_{ik}$ over $U_i \cap U_j$, so that

$$\{h_{ij}\} \in Z^1(X, \mathrm{End}\, V_0).$$

The image θ of this cocycle in the first cohomology is really the first order infinitesimal obstruction to a formal (that is, ignoring convergence questions) 1-parameter extension of the identity automorphism of V_0 to an isomorphism of V with $q^* V_0$ over $X \times \Delta$. For, if h_{ij} is a 1-coboundary, then any first order lift of the identity can be modified in an obvious fashion to satisfy the cocycle rule.

As it turns out, a lot of interesting information is already encoded in the first order term; for example, see Theorem 7 below.

Let $\mathsf{T}_{t_0}(T)$ denote the tangent space at t_0; for each tangent vector $\mathbf{v} \in \mathsf{T}_{t_0}(T)$ one can ask if a formal extension (as above) of the identity automorphism of V_0 to a 1-parameter family is possible along the direction $\mathbf{v}$. This defines the *Kodaira-Spencer map* for the family $\{V_t\}_{t \in T}$ at t_0:

$$\eta_{t_0} : \mathsf{T}_{t_0}(T) \longrightarrow H^1(X, \mathrm{End}\, V_{t_0}),$$

which is a $\mathbb{C}$-linear map taking $\mathbf{v}$ to $\theta(\mathbf{v})$, the first order obstruction to extending (as above) the identity map of V_0 any one parameter family of deformations of V_{t_0} defined by a holomorphic curve through t_0 fitting the tangent vector $\mathbf{v}$. It measures how far the family is from being trivial in the direction of a given tangent vector $\mathbf{v} \in \mathsf{T}_{t_0}(T)$.

Remark. — In the language of schemes, the above can be reformulated as follows: Let $D = \mathbb{C}[\varepsilon]/\varepsilon^2$ be the ring of dual numbers. A *(first order) infinitesimal deformation* of V_0 is an *algebraic* vector bundle W on $X \times \operatorname{Spec} D$ such that the fibre over the closed point is isomorphic to V_0.

Now a Zariski tangent vector $\mathbf{v}$ of T at t_0 is the same as a morphism $\mathbf{v} : \operatorname{Spec} D \to T$ that sends the closed point to t_0. Hence, for each such $\mathbf{v}$ we can get an infinitesimal deformation from our original deformation by base change via the morphism $\mathbf{v}$:

$$\begin{tikzcd} V_{|\mathbf{v}} \arrow[r] \arrow[d] & V \arrow[d] \\ X \times \operatorname{Spec} D \arrow[r, "\mathrm{Id} \times \mathbf{v}"'] & X \times T \end{tikzcd}$$

(here $V_{|\mathbf{v}}$ is a short hand notation for the pullback $V \times_{X \times T} (X \times \operatorname{Spec} D)$.)

One can now prove that there is a bijection

$$\left\{\begin{array}{l}\text{isomorphism classes of}\\ \text{first order infinitesimal}\\ \text{deformations of } V_0\end{array}\right\} \longleftrightarrow H^1(X, \operatorname{End} V_0).$$

To see this, given an infinitesimal deformation $W \to X \times \operatorname{Spec} D$, consider the short exact sequence of sheaves on $X \times \operatorname{Spec} D$:

$$0 \to V_0 \to W \xrightarrow{\rho} V_0 \to 0$$

where ρ is restriction to the closed point and the kernel of ρ can again be identified with V_0. Thinking of this as an exact sequence of vector bundles on X, we get an extension class in $\operatorname{Ext}^1(V_0, V_0) \simeq H^1(X, \operatorname{End} V_0)$ and vice-versa.

Now the Kodaira-Spencer map can be described as the map sending a tangent vector $\mathbf{v} \in \mathsf{T}_{t_0}(T)$ to the isomorphism class of the infinitesimal deformation $V_{|\mathbf{v}}$.

Definition. — A family of vector bundles $\{V_t\}_{t \in T}$ (as above) is said to be *regular* at t_0 if $\dim H^1(X, \operatorname{End} V_t)$ is constant in a neighbourhood of t_0.

Theorem 7 — *Let $\{V_t\}_{t \in T}$ be a family of vector bundles on X regular at $t_0 \in T$. If for each t in some neighbourhood of t_0, the Kodaira-Spencer map $\eta_t : \mathsf{T}_t(T) \to H^1(X, \operatorname{End} V_t)$ is the zero map, then $\{V_t\}_{t \in T}$ is isomorphic to the the trivial deformation of V_0 over this neighbourhood.*

Note that the converse is an immediate consequence of the definition of the Kodaira-Spencer map. A natural question at this stage is:

Question. — Given $v \in H^1(X, \operatorname{End} V_0)$, can we "integrate" v? That is, can we find a 1-parameter deformation $\{V_t\}_{t \in \Delta}$ of V_0 such that $\eta_0(\frac{d}{dt}\,|_{t=0}) = v$?

Or, what is the same: do all infinitesimal deformations come from 1-parameter deformations?

Let us try to answer this question at a formal level. That is, given $\{h^1_{ij}\} \in Z^1(X, \operatorname{End} V_0)$, when does there exist a formal power series

$$\theta_{ij}(t) = 1 + h^1_{ij}t + h^2_{ij}t^2 + ...$$

satisfying $\theta_{ij}(t).\theta_{jk}(t) = \theta_{ik}(t)$? To answer this, if we equate the coefficients of t^2, we get one of the necessary conditions: the 2-cocycle $\{h_{ijk}\}$, where $h_{ijk} := h^1_{ij} \cdot h^1_{jk}$, is a coboundary. In a similar fashion, one can check that the coefficients of various powers of t in the expression $\theta_{ij}(t).\theta_{jk}(t) - \theta_{ik}(t)$ are all forced to be 2-coboundaries. Conversely, if we extend $1 + h^1_{ij}t$ arbitrarily to a power series $\theta_{ij}(t)$, we would like to have multiplicative cocycle rule for θ_{ij}. Again, if we isolate the coefficients of t^n $(n > 1)$, in the expression $\theta_{ij}(t).\theta_{jk}(t) - \theta_{ik}(t)$, we find that they are all 2-cocycles. If we knew that $H^2(X, \operatorname{End} V_0) = 0$, these would be coboundaries. We could then modify the extension in an obvious way to get a different extension of $1 + h^1_{ij}t$ to a formal power series satisfying the multiplicative cocycle rule. This shows that a given element $\{h^1_{ij}\} \in Z^1(X, \operatorname{End} V_0)$ can be "formally integrated" if $H^2(X, \operatorname{End} V_0) = 0$.

A natural extension of the above question is: can one integrate along all "directions" in $H^1(X, \operatorname{End} V_0)$ at once? That is, does there exist a deformation $\{V_t\}_{t\in T}$ of V_0 (over some pointed complex manifold (T, t_0)), such that the Kodaira-Spencer map of $\{V_t\}_{t\in T}$ at t_0 is surjective? (Once it is surjective, the base (T, t_0) can easily be modified so that this map becomes an isomorphism.) The existence of such a (T, t_0) implies that all infinitesimal deformations of V_0 can be obtained from one single deformation.

The answer to this second question is *yes* and is a consequence of the fact that $H^2(X, \operatorname{End} V_0) = 0$. This is the theorem of existence (see [AB] for a simple proof).

Theorem 8 *(Theorem of Existence) — Given a vector bundle V_0 on X, there exists a complex manifold T, $t_0 \in T$ and a holomorphic family $\{V_t\}_{t\in T}$ such that $V_{t_0} \simeq V_0$ and the Kodaira-Spencer map η_{t_0} at t_0 to this family is an isomorphism.*

Definition. — A deformation $\{V_t\}_{t\in T}$ of V_0 is said to be *locally complete* at t_0 if any other deformation of V_0 can be obtained locally as a pullback of $\{V_t\}_{t\in T}$. More precisely, given *any* deformation $W := \{W_s\}_{s\in S}$ of V_0 (over some pointed complex manifold (S, s_0)), then there exists a neighbourhood U_{s_0} of s_0 in S, a neighbourhood U_{t_0} of t_0 in T and a holomorphic map

$$\begin{aligned} \psi : U_{s_0} &\longrightarrow U_{t_0} \\ s_0 &\longmapsto t_0 \end{aligned}$$

with the property that $(\operatorname{Id} \times \psi)^* V \simeq W$ on $X \times U_{s_0}$.

Note that the map ψ is not required to be unique!

Roughly speaking, the family V is locally complete at t_0 if all "small" deformations of the given bundle already occur in V. The Theorem of Completeness, which we state here without proof, gives a condition on the Kodaira-Spencer map which will ensure that the given family is locally complete at a given point.

Theorem 9 *(Theorem of Completeness) — If the Kodaira-Spencer map of a deformation $\{V_t\}_{t\in T}$ of V_0 is surjective at t_0, then $\{V_t\}_{t\in T}$ is locally complete at t_0.*

The formal aspects of the proof of this theorem naturally lead to "algebraic" notions of formal deformations, *versal* families and the Theorem of Schlessinger (see [S]).

Definition. — Let (T, t_0) be a deformation space for V_0 such that the Kodaira-Spencer map at t_0 is an isomorphism. The germ at t_0 of T is called the *local moduli* of V_0.

Observations.(cf. [S])

1. This local moduli is unique upto non-canonical isomorphisms.
2. A family determined by a representative of the local moduli need not be *injectively parametrised* (we say that a family the is injectively parametrised if for pair of distinct points in the base, the corresponding bundles in the family are non-isomorphic). Here is a simple example:
   Let $X = \mathbb{P}^1$ be the projective line, and set $V_0 = \mathcal{O}(-1) \oplus \mathcal{O}(1)$.
   The tautological exact sequence
   $$0 \to \mathcal{O}(-1) \to \mathbb{P}^1 \times \mathbb{C}^2 \to \mathcal{O}(1) \to 0 \qquad (*)$$
   is non-split. This defines a non-zero element in $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) \simeq H^1(\mathbb{P}^1, \mathcal{O}(-2))$ which is just $\mathbb{C}$, by Serre duality. So for each $\lambda \in \mathbb{C}$ there is an extension, and $\lambda = 0$ corresponds to the split extension.
   Now the family $\{V_\lambda\}_{\lambda\in\mathbb{C}}$ can be seen to be:
   $$\begin{aligned} V_0 &\simeq \mathcal{O}(-1) \oplus \mathcal{O}(1) \\ V_\lambda &\simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \simeq \mathbb{P}^1 \times \mathbb{C}^2 \qquad \text{for } \lambda \neq 0 \end{aligned}$$
   Clearly this family is *not* injectively parametrised. We claim that this is in fact a representative for the local moduli for V_0. For, note that the local moduli of V_0 has dimension $= h^1(\mathbb{P}^1, \mathrm{End}\, V_0) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) + h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) + h^1(\mathbb{P}^1, \mathcal{O}(2)) + h^1(\mathbb{P}^1, \mathcal{O}(-2)) = 1$. Now, by writing explicitly a set of transition functions for the family $\{V_\lambda\}$, one can check that the Kodaira-Spencer map at 0 to this family is non-zero. This, in view of the Theorem of Completeness, proves our claim. Note also that this family is *not* regular at 0.
3. If the bundle V_0 is *simple*, then it can be shown that the local moduli of V_0 is always injectively parametrised. See [NS2] for a proof of this fact.

Suppose now that V_0 is a *simple* bundle of rank n. By an application of Riemann-Roch theorem we obtain
$$h^1(X, \mathrm{End}\, V_0) = n^2(g-1) + 1$$
and furthermore, by (3) above, the local classifying map (eg. the map ψ of the definition after Theorem 8) is *unique*. Consider now the set $S(n,d)$ of isomorphism classes of simple bundles of rank n and degree d on X. Now, around each point in $S(n,d)$, we get a complex coordinate chart determined by the local moduli of a bundle representing the point. Note that these charts are compatible and holomorphic on overlaps since the local moduli spaces are injectively parametrised. This gives a manifold structure on $S(n,d)$ which can be checked to have the naturality properties mentioned in Theorem 6, § 3. This proves (see [NS2] for details)

Theorem 10 — *The set $S(n,d)$ of isomorphism classes of simple bundles of rank n and degree d admits a structure of a complex manifold of dimension $n^2(g-1)+1$.*

Now the fact that the set of stable bundles of rank n and degree d forms an open subset of $S(n,d)$ completes a sketch of the proof of Theorem 6, § 3. Note also that when $n=1$ and $d=0$, this manifold is just $\mathrm{Pic}^\circ X$.

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