FAFM.tex
v. 18/03/2001

# From Frobenius algebras to Frobenius manifolds

JOACHIM KOCK

## Introduction

A *Frobenius algebra* is a finite dimensional vector space equipped with a bilinear form and a multiplication — see 1.1 below for precise definition. A *Frobenius manifold* is a Riemannian manifold whose tangent spaces are all Frobenius algebras closely related to each other: In the coordinate description, the structure constants for the multiplication should be third derivatives of a single function (which satisfies WDVV). In a more intrinsic description, one constructs a connection out of the multiplication, and the requirement is that this connection should be flat. The aim of this note is to motivate and make precise these definitions from the view-point of Frobenius algebras, and explain how the algebraic notions turn into Riemannian geometry.

The first section recalls the notions of Frobenius algebras. Section 2 deals with the algebraic picture of Frobenius manifolds. From this view-point, a Frobenius manifold is a family of Frobenius algebras satisfying certain conditions: the bilinear form must be constant in the family, and the structure constants for the multiplication must be third derivatives of a single function which satisfies the WDVV equations.

In the geometric description, which is the subject of Section 3, the bilinear form corresponds to the Riemannian metric, and this metric must be *flat* in the sense that the corresponding Levi-Civita connection is flat. The multiplication must now be a multiplication on the tangent sheaf, and it must by compatible with the metric in the same way as the bilinear form of a Frobenius algebra is 'invariant under multiplication'. The multiplication is now a $(2,1)$ tensor and it can be used to deform the Levi-Civita connection. The connection thus obtained must be flat also — this is the coordinate free formulation of the existence of the potential with the WDVV condition.

Two examples of Frobenius manifolds constitute the red thread of this exposition: the *trivial Frobenius manifold* (for which all the tangent spaces are isomorphic as Frobenius algebras), and *quantum cohomology*. Analysing these two examples will lead to the good general definitions.

Important examples of Frobenius algebras are *graded*, for example cohomology rings. This condition is equivalent to quasi-homogeneity of the potential of the corresponding trivial Frobenius manifold. The corresponding condition on a general Frobenius manifold is the existence of an *Euler field*, a linear vector field which is a grading for the multiplication and which acts on the metric by conformal transformation. This is the content of Section 4.

For my own benefit I have included an appendix with the relevant notions from Riemannian geometry.

# 1 Frobenius algebras

Throughout, let $k$ be a commutative $\mathbb{Q}$-algebra. All tensor products will be over $k$. Typically, we will have $k = \mathbb{Q}$ or $k = \mathbb{C}$, and sometimes abusively we will talk of vector spaces where we really mean free $k$-modules of finite rank.

**1.1 Frobenius algebras.** A Frobenius algebra over $k$ is a free $k$-module $A$ of finite rank, equipped with a commutative and associative multiplication with 1-element, and a $k$-bilinear symmetric non-degenerate form $g : A \times A \to k$, which is associative in the sense that $g(xy, z) = g(x, yz)$.

The bilinearity means we can regard it as a map $A \otimes A \to k$ (a $(2, 0)$-tensor).

Having this non-degenerate $g$ is equivalent to having a linear form $\varepsilon : A \to k$ whose kernel contains no nontrivial ideals. Indeed, given $g$ we put $\varepsilon(x) := g(1, x)$, and conversely, given $\varepsilon$ we define $g(x, y) := \varepsilon(xy)$. This linear form is often called a trace map. At times we will denote it by $\int$, because of Example 1.3 below.

**1.2 Coordinates.** Let $T_0, \ldots, T_r$ be a basis for $A$. We can write $g$ in terms of tensor elements, $g_{ij} := g(T_i, T_j) = \varepsilon(T_i T_j)$. Since $g$ is non-degenerate, the matrix $(g_{ij})$ is invertible; let $(g^{ij})$ denote the inverse. We use this matrix to raise indices. Set also $g_{ijk} := \varepsilon(T_i T_j T_k)$. Now the multiplication reads

$$T_i T_j = \sum_{e,f} g_{ije}\, g^{ef} T_f. \tag{1}$$

**1.3 Cohomology rings.** Let $X$ be a homogeneous projective variety over $\mathbb{C}$ and consider its cohomology space $A = H^*(X, \mathbb{Q})$. Then the intersection product (the cup product) provides the algebra structure (over $\mathbb{Q}$). Since $X$ has a cellular decomposition (and all cells are affine spaces of even $\mathbb{R}$-dimension), there is no odd cohomology, so the cup product is commutative. The degree map $\int_X$ (integration over the fundamental class) plays the rôle of the trace map, making $A$ into a Frobenius algebra. Then we get the *Poincaré metric*

$$g_{ij} = g(T_i, T_j) := \int_X T_i \cup T_j.$$

(Other smooth projective varieties can be used as well, but if there is odd cohomology, $A$ will not be commutative but only super-commutative. All the constructions can be generalised to this setting, then the Frobenius manifolds are rather super-manifolds. The main difference is that a lot of care is needed to keep track of signs. See Manin [6] for details.)

**1.4 Example.** In the case $X = \mathbb{P}^2$, with basis $T_i = h^i$ (where $h$ is the class of a hyperplane), the matrices are

$$(g_{ij}) = (g^{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

## 2 Formal Frobenius manifolds — algebraic approach

**2.1 Set-up.** Throughout this section, let $H$ be a free $k$-module of finite rank with a non-degenerate symmetric bilinear pairing $g : H \otimes H \to k$. Set $K = k[[H^*]]$, the completed symmetric algebra on $H^*$, the dual module of $H$. In other words, for a fixed basis $T_0, \ldots, T_r$ of $H$, if $\mathbf{x} = (x^0, \ldots, x^r)$ are the coordinates of an element $\gamma = \sum x^i T_i \in H$, then $K = k[[\mathbf{x}]]$. (Think of $x^i$ as the linear form taking $\gamma \mapsto x^i$.) Put $H_K := H \otimes K$. Consider the $K$-linear extension of $g$ to a $K$-bilinear form $g : H_K \otimes H_K \to K$.

The following definition seems natural from the view-point of Frobenius algebras. However, to conform to the definitions of Manin [6], cf. 2.4 and 3.2, we should not require neither associativity nor existence of 1-element...

**2.2 Formal pre-Frobenius manifolds.** The structure of a formal pre-Frobenius manifold on $(H, g)$ is given by a structure on $H_K$ of a Frobenius algebra over $K$, with bilinear form $g$.

In other words, we have a multiplication $*$ on $H_K$ such that $g(T_i * T_j, T_k) = g(T_i, T_j * T_k)$. We can also think of the structure as a family over $K$ of Frobenius algebra structures on $H$, all with the same bilinear form. Precisely, for each maximal ideal $\mathfrak{m} \subset K$ we get a 'point-wise' multiplication

$$\begin{aligned} \mu_{\mathfrak{m}} : H \otimes H &\to H \otimes k = H \\ x \otimes y &\mapsto (x \otimes 1) * (y \otimes 1) \mod \mathfrak{m} \end{aligned}$$

This is just multiplying in $H_K$ and then evaluating the variables $x^0, \ldots, x^r$ in the vector of $k$-scalars corresponding to $\mathfrak{m}$.

More generally, we could consider any $k$-algebra $R$ and require a Frobenius algebra structure over $R$ on $H_R := H \otimes R$ (such that $g : H_R \otimes H_R \to R$ is constant). Perhaps this is not so interesting. There should be some relation between $H$ and $R$ as in the case $R = K$. A better approach is to start with the $k$-algebra $R$ and then take $H$ to be the module of derivations $H = \mathrm{Der}_k(R)$. Then we require it to be a Frobenius algebra over $R$... Below we see that this is in fact what we did... For a short account of this view-point, see Dubrovin [2], Lecture 1, Remark 1.5.

**2.3 Example.** THE TRIVIAL PRE-FROBENIUS MANIFOLD. For any Frobenius algebra $H$, we can construct the trivial pre-Frobenius manifold, simply by extending $K$-linearly the multiplication of $H$ to $H_K$. (This way, the new variables $x^0, \ldots, x^r$ are just dead weight... )

For a pre-Frobenius manifold to be Frobenius we require that all the Frobenius algebra structures come from one single (formal) function $\Phi$. To state the requirement in the algebraic setting we need coordinates. In the next section we give the geometric description from which it will be clear that the definition does not depend on the chosen basis.

**2.4 Formal Frobenius manifolds.** (Cf. Manin [6], III.1.1.) A *formal Frobenius manifold* on $(H, g)$ is given by a formal power series, called a potential, $\Phi \in K$ (defined up to quadratic terms) satisfying the WDVV equations

$$\sum_{e,f} \Phi_{ije}\, g^{ef}\, \Phi_{fkl} = \sum_{e,f} \Phi_{jke}\, g^{ef}\, \Phi_{fil}. \tag{2}$$

Here $\Phi_i = \frac{\partial}{\partial x^i}\Phi$ is the partial derivative with respect to $x^i$, and the triple indices denote three times partial derivative with respect to the corresponding variables.

**2.5 WDVV = associativity.** Consider the multiplication $*$ on $H_K$ defined by

$$T_i * T_j := \sum_{e,f} \Phi_{ije}\, g^{ef}\, T_f.$$

Note that since the order of the partial derivatives is immaterial, $*$ is commutative. Now we claim that $*$ is associative if and only if $\Phi$ satisfies WDVV. To see this is just a question of writing out what associativity means. We have

$$\begin{aligned}(T_i * T_j) * T_k &= \sum_{e,f}\sum_{l,m} \Phi_{ije}\, g^{ef}\, \Phi_{fkl}\, g^{lm}\, T_m,\\ T_i * (T_j * T_k) &= \sum_{e,f}\sum_{l,m} \Phi_{jke}\, g^{ef}\, \Phi_{fil}\, g^{lm}\, T_m.\end{aligned}$$

Now since the $T_m$ are independent and the matrix $g^{lm}$ is invertible, we see that associativity is equivalent to th WDVV equations (2).

**2.6 Remark.** Observe that this multiplication is compatible with the metric. In fact

$$g(T_i * T_j, T_k) = \Phi_{ijk} = g(T_i, T_j * T_k). \tag{3}$$

This follows easily: $g(T_i * T_j, T_k) = g(\sum \Phi_{ije} g^{ef} T_f, T_k) = \sum \Phi_{ije} g^{ef} g_{fk} = \Phi_{ijk}$.

There is one annoying thing with this definition: the existence of a 1-element for $*$ is not guaranteed. So strictly speaking, a formal Frobenius manifold is not necessarily a formal pre-Frobenius manifold in the sense of 2.2. Therefore it is slightly sloppy to say (as Manin does) that the existence of $\Phi$ satisfying WDVV is equivalent to turning $H_K$ into a commutative associative $K$-algebra. . .

**2.7 Trivial Frobenius manifolds.** Given any Frobenius algebra $(H, g)$, we claim that the corresponding trivial pre-Frobenius manifold is in fact Frobenius. We need to exhibit a potential $\Phi \in K$ whose third derivatives are the constants $g_{ijk}$, (recall from (1) that $T_i T_j = \sum g_{ije} g^{ef} T_f$). The cubic polynomial

$$\Phi(\mathbf{x}) = \sum_{i,j,k} \frac{x^i x^j x^k}{6}\, g_{ijk}$$

clearly does the job: $\Phi_{ijk} = g_{ijk}$. (The factor $6 = 3!$ takes care of repetitions of the monomials.) Observe that we can modify $\Phi$ as we wish by adding terms of degree two or less without changing the third derivatives.

**2.8 Example.** When $A = H^*(X, \mathbb{Q})$ is the cohomology ring of $X$ then the potential of the corresponding trivial Frobenius manifold is called the classical potential of $X$.

As an easy example, consider the cohomology ring of $\mathbb{P}^2$ (cf. 1.4 above). The only non-zero triple top products in $H^*(\mathbb{P}^2)$ are $\int T_0 T_0 T_2 = 1$ and $\int T_0 T_1 T_1 = 1$, so the classical potential for $\mathbb{P}^2$ is

$$\Phi = \frac{x^0 x^0 x^2}{2!} + \frac{x^0 x^1 x^1}{2!}.$$

**2.9 Quantum cohomology.** (Cf. Fulton-Pandharipande [3].) This example is the starting point for the theory of Frobenius manifolds, and perhaps the most important one. Let $X$ be a projective homogeneous variety and let $H := H^*(X, \mathbb{Q})$ be the cohomology space, with cup multiplication, trace map and bilinear form $g$ as in Example 1.3. Let $\Phi$ be the genus zero Gromov-Witten potential of $X$. Then *the triple* $(H, g, \Phi)$ *is a formal Frobenius manifold.* The product $T_i * T_j := \sum_e \Phi_{ije} g^{ef} T_f$ is called the quantum product.

Let us now explain these notions. The potential $\Phi$ is the generating function for the Gromov-Witten invariants, which roughly count rational curves in $X$ subject to incidence conditions:

STABLE MAPS. There is a *stack of stable maps* $\overline{M}_{0,n}(X, \beta)$ parametrising isomorphism classes of $n$-pointed stable maps $\mu : C \to X$ whose image is of degree $\beta$. There are evaluation maps $\overline{M}_{0,n}(X, \beta) \xrightarrow{\nu_i} X$ sending the class of a map $\mu$ to $\mu(p_i)$, the image of the mark $p_i$.

GROMOV-WITTEN INVARIANTS. The *Gromov-Witten invariants* are defined as

$$I_\beta^X(\gamma_1 \cdots \gamma_n) := \int_{[\overline{M}_{0,n}(X,\beta)]} \nu_1^* \gamma_1 \cup \cdots \nu_n^* \gamma_n.$$

We can regard $I_\beta^X$ as a multi-linear map from $\operatorname{Sym} H$ to $\mathbb{Q}$. This invariant roughly count rational curves in $X$ of degree $\beta$ which are incident to $n$ general subvarieties of classes (Poincaré dual to) $\gamma_i$. (The number $n$, and the classes $\gamma_i$ must be adjusted such that only a finite number of such curves exist — see Equation (6) below for the precise statement.)

THE GROMOV-WITTEN POTENTIAL is the generating function for the Gromov-Witten invariants. Precisely,

$$\begin{aligned}
\Phi &:= \sum_\beta q^\beta I_\beta^X \big( \exp(\gamma) \big) \\
&= \sum_\beta q^\beta \sum_{n=0}^\infty \frac{1}{n!} I_\beta^X (\gamma^{\otimes n}) \\
&= \sum_\beta q^\beta \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I_\beta^X (T_0^{a_0} \cdots T_r^{a_r}).
\end{aligned}$$

In the first line, $\gamma \in H^*(X, \mathbb{Q})$ is a generic element, and the expression exp is more or less defined by the following equality. In the last line we use coordinates $\gamma = \sum x^i T_i$; the inner sum is over all vectors $\mathbf{a} = (a_0, \ldots, a_r)$ of non-negative integers, and usual multi-index notation is understood.

The parameter $q$ and the symbol $q^\beta$ are of rather technical nature, cf. Getzler [4] for the precise definition. For now, to get the idea of its meaning, think of the case $X = \mathbb{P}^r$, so that the degree $\beta$ is just a natural number.

It turns out the only non-zero Gromov-Witten invariants with $a_0 \neq 0$ are those with $\beta = 0$ and $n = 3$, and they agree with the usual cup products: $\Phi_{0jk} = g_{0jk} = g_{jk}$. It follows from this observation that $T_0$ is the identity for the quantum product.

# 3 Frobenius manifold — geometric approach

From now on, we adopt Einstein summation convention, summing over repeated indices. So for instance, instead of $\sum_{e,f} \Phi_{ije}\, g^{ef}\, \partial_f$ we just write $\Phi_{ije}\, g^{ef}\, \partial_f$.

**3.1 Schematic account of the passage to Riemannian geometry.** With $H$ and $K$ defined as in 2.1, let $M$ be the formal scheme obtained by completing the affine space $H$ at the origin. Precisely, since $K = k[[x^0, \ldots, x^r]]$ is the completion of $\operatorname{Sym} H^*$, we are setting

$$M := \operatorname{Spec} K,$$

and the structure sheaf is $\mathcal{O}_M = \widetilde{K}$ (in the sense of Hartshorne [5], Ch. II). The tangent bundle of $M$ is then, by general principles, represented by the module of derivations $\operatorname{Der}_k(K)$. In other words, its sheaf of sections $\mathcal{T}_M$ (the tangent sheaf, or the sheaf of vector fields) is $\mathcal{T}_M = (\operatorname{Der}_k(K))^{\sim}$.

For $i = 0, \ldots, r$, let $\partial_i : K \to K$ denote partial derivative with respect to $x^i$. Then $\operatorname{Der}_k(K)$ is the free $K$-module generated by $\partial_0, \ldots, \partial_r$. These are now interpreted as the (constant) vector fields corresponding to the coordinate frame $x^0, \ldots, x^r$. Since $M$ is just a linear space, it is natural to identify all tangent spaces with the vector space $H$ itself. Under this identification, the frame $\partial_0, \ldots, \partial_r$ corresponds to the basis $T_0, \ldots, T_r$. In this way the module of derivations becomes identified with $H \otimes K$, so we have

$$\mathcal{T}_M = (H \otimes K)^{\sim}.$$

Suppose now that $(H, g)$ is a formal pre-Frobenius manifold (say in the sense of 2.2). We can interpret this in terms of $M$ and $\mathcal{T}_M$. The bilinear form $g$ defines a metric on $M$ (denoted $\langle\ |\ \rangle$)

$$\langle\, \partial_i \,|\, \partial_j \,\rangle := g_{ij},$$

which is flat (cf. A.3) since $g$ is constant with respect to the variables $x^i$. The multiplication $*$ on $H_K$ becomes a multiplication on $\mathcal{T}_M$ which we will denote by $\circ$, and we have $\langle\, X \circ Y \,|\, Z \,\rangle = \langle\, X \,|\, Y \circ Z \,\rangle$.

These considerations lead to the general geometric definition of a Frobenius manifold. In this geometrisation we will take two steps. First, we will not bother about the formalness. (Since anyway we are considering a scheme $M = \operatorname{Spec} K$, we might as well take any $k$-algebra $K$ and use it as structure sheaf for our manifold.) Second, we require only locally that our object to look like the example above.

Following Manin [6], we break up the definition steps, introducing the notion of pre-Frobenius manifold, and subsequently impose the condition of integrability and WDVV.

**3.2 Pre-Frobenius manifolds.** (Cf. Manin [6], I.1.3.) Let $M$ be a Riemannian manifold with structure sheaf $\mathcal{O}_M$, tangent sheaf $\mathcal{T}_M$, and flat metric $\langle\ |\ \rangle$. By definition, $M$ is a pre-Frobenius manifold if it is equipped with a commutative multiplication $\mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{T}_M$, denoted $X \otimes Y \mapsto X \circ Y$, satisfying

$$\langle\, X \circ Y \,|\, Z \,\rangle = \langle\, X \,|\, Y \circ Z \,\rangle.$$

(Manin's definition doesn't require the multiplication to have a 1-element. . . )

We are now going to interpret the Frobenius condition (integrability and WDVV) in these terms.

Start out with the Frobenius manifold $(H, g, \Phi)$ in the algebraic description. We have the multiplication $\circ$ defined by

$$\partial_i \circ \partial_j = \Phi_{ije}\, g^{ef} T_f.$$

It follows that $\langle\, \partial_i \circ \partial_j \,|\, \partial_k \,\rangle = \Phi_{ije}\, g^{ef}\, g_{fk} = \Phi_{ijk}$. More generally, for flat vector fields $X, Y, Z$ we have

$$\langle\, X \circ Y \,|\, Z \,\rangle = XYZ\Phi = \langle\, X \,|\, Y \circ Z \,\rangle\,.$$

Define a new connection on $\mathcal{T}_M$ by taking its Christoffel symbols to be ${A_{ij}}^k := \Phi_{ije} g^{ek}$, (cf. A.1). Since ${A_{ij}}^k$ are also the structure constants for the multiplication in the tangent sheaf, we are setting

$$\nabla_X Y = \nabla_X^{\mathrm{LC}} Y + X \circ Y \tag{4}$$

so we see that the multiplication (which is a tensor) constitutes a perturbation of the Levi-Civita connection $\nabla^{\mathrm{LC}}$, (cf A.2).

Let us compute the curvature. By A.3, we have

$${R_{ijk}}^m = \underbrace{{A_{ik}}^e {A_{je}}^m - {A_{jk}}^e {A_{ie}}^m}_{\text{(ii)}} + \underbrace{\partial_j {A_{ik}}^m - \partial_i {A_{jk}}^m}_{\text{(i)}}.$$

Now ${A_{jk}}^f = \Phi_{jke} g^{ef}$, and $\Phi_{jke}$ are the partial derivatives of the (formal) function $\Phi$, so we have $\partial_i \Phi_{jke} = \Phi_{ijke}$. Therefore part (i) of the expression vanishes. Since furthermore $\Phi$ satisfies WDVV, part (ii) vanishes as well, so we conclude that the new connection is flat.

So one could be tempted to put as definition of Frobenius manifold that the connection of (4) be flat. However, this would not be what we want, because it might happen that the curvature vanishes without (i) and (ii) are zero — they might cancel each other. Manin's solution to this problem is to introduce an extra parameter $\lambda \in k$ in the connection. Then (i) will be linear in $\lambda$ while (ii) will be quadratic, and thus they cannot cancel for all $\lambda$. Precisely we define a pencil of (torsion-free) connections

$$\nabla_X^\lambda Y = \nabla_X^{\mathrm{LC}} Y + \lambda (X \circ Y)$$

Now the vanishing of curvature for all $\lambda$ is equivalent to the simultaneous vanishing of (i) and (ii) in the expression for the curvature of $\nabla^\lambda$:

$$\underbrace{\lambda^2 \left( {A_{ik}}^e {A_{je}}^m - {A_{jk}}^e {A_{ie}}^m \right)}_{\text{(ii)}} + \underbrace{\lambda \left( \partial_j {A_{ik}}^m - \partial_i {A_{jk}}^m \right)}_{\text{(i)}}.$$

Here the vanishing of (i) is equivalent to the existence of a local potential $\Phi$ (see Manin [6], I.1.5. for the proof of this equivalence), and the vanishing of (ii) is equivalent to the WDVV equation for $\Phi$ as we already noted.

In other words, flatness of $\nabla^\lambda$ for all $\lambda$ is equivalent to the Frobenius condition (existence of $\Phi$ satisfying WDVV).

We have been led to the following general definition.

**3.3 Frobenius manifolds.** On a pre-Frobenius manifold $(M, g)$, define a pencil of torsion-free connections $\nabla^\lambda$ by

$$\nabla^\lambda_X Y = \nabla^{\mathrm{LC}}_X Y + \lambda(X \circ Y), \qquad \lambda \in k$$

where $\nabla^{\mathrm{LC}} = \nabla^0$ is the Levi-Civita connection of $g$. Now $M$ is said to be a Frobenius manifold if $\nabla^\lambda$ is flat for all $\lambda$.

**3.4 Flat identity.** An identity vector field is $e \in \mathcal{T}_M$ such that $e \circ X = X$ for all $X \in \mathcal{T}_M$. (From the view-point of Frobenius algebra, the existence of this field should be part of the definition of Frobenius manifold.) This may or may not be flat (i.e., independent of $x$, cf. A.4). In the example of the trivial Frobenius manifold as well as in quantum cohomology we saw that it was indeed flat, in fact $e = \partial_0$. In general, if $e$ is flat, we can do a change of (flat) coordinates so that $e$ is part of the coordinate frame and let us say equal to the new $\partial_0$. In Dubrovin's definition of Frobenius manifold, having flat identity is required.

Provided $e$ is a flat field, then for flat fields $X, Y$ we have

$$eXY\Phi = \langle\, X \,|\, Y \,\rangle$$

In coordinates, if $e = \partial_0$ then equivalently

$$\Phi_{0ij} = g_{ij}.$$

This holds in the example of quantum cohomology, where $e = \partial_0$ is the field corresponding to the fundamental class $T_0$ of $X$.

# 4 Grading, quasi-homogeneity, and Euler field

**4.1 Sloppiness of Frobenius algebras.** Let us be given a vector space $A$ which we equip with Frobenius algebra structure. That is: we define a $k$-linear multiplication map $\mu : A \otimes A \to A$ and a $k$-linear form $\varepsilon : A \to k$ (both maps subject to the usual conditions).

Given the trace map $\varepsilon$ we can define a new one by multiplying with some invertible scalar $r \in k$, that is, composing with some $r : k \to k$. That gives a new trace map, just as good. Of course this affects the metric with the same factor $r$.

To have a bilinear form on a vector space, given only up to a scalar is called a conformal structure.

There is also a rescaling possibility in the multiplication map. Namely, given $\mu = \mu_1$, then for each invertible $r \in k$ we define $\mu_r$ as $\mu$ followed by multiplication by $r$. This is again a commutative and associative multiplication map, but note that the 1-element changes from 1 to $1/r$. This implies that the $k$-algebra structure changes! In other words, rescaling is not a $k$-algebra map but only a $k$-linear map.

The goal of this section is to show that these two sloppinesses are generated by a field, the Euler field.

**4.2 Graded Frobenius algebras.** A graded Frobenius algebra is one $A = \oplus_{i=0}^d A_i$ such that $A_i \otimes A_j \to A_{i+j}$. We require $A_0 \simeq k$, but we do not assume that $A$ is generated in degree 1. Assume the basis elements $T_0, \dots, T_r$ are homogeneous, and let their degrees be denoted $p_i := \deg T_i$. We always assume that $p_0 = 0$ and that the last graded piece (the socle) is of degree $d := p_r$. With structure constants $g_{ij}^k := \sum_e g_{ije} g^{ek}$ for the multiplication, we have

$$p_i + p_j = p_k \quad \text{or} \quad g_{ij}^k = 0$$

and in particular

$$p_i + p_j = d \quad \text{or} \quad g_{ij} = 0. \tag{5}$$

**4.3 Example.** If $A = H^*(X, \mathbb{Q})$ is the cohomology space of $X$ then $d = \dim_{\mathbb{C}} X$, and $p_i$ is such that $T_i \in H^{2p_i}(X)$.

Example,

draw the cohomology of the Grassmannian $\mathrm{Gr}(2, 4)$.

**4.4 Trivial Frobenius manifolds.** Consider now the classical potential $\Phi(t)$, cf. 2.7. By grading the variables in a smart way we can make it homogeneous, or more precisely quasi-homogeneous. There are many possible choices for the grading; the following convention is motivated by quantum cohomology, cf. 4.5 below.

$$d_i := \deg x^i := 1 - p_i.$$

With this grading, the classical potential $\Phi$ is quasi-homogeneous of degree $d_\Phi := 3 - d$. (Note that the grading of $x^i$ is chosen such that for all $i$, the element $x^i T_i$ has degree 1… maybe this observation is useless???)

Quasi-homogeneous means that for every constant $c \in k$ we have

$$\Phi(c^{d_0} x^0, \dots, c^{d_r} x^r) = c^{d_\Phi} \Phi(x^0, \dots, x^r).$$

In general, for a true homogeneous functions $F$ of degree $d_F$, we have Eulers identity $\sum x^i \partial_i F = d_F F$. Similarly for quasi-homogeneous functions, and for our $\Phi$ in particular, we have

$$\sum_{i=0}^{r} d_i x^i \partial_i \Phi = d_\Phi \Phi$$

We will express this in terms of a vector field $E := \sum_{i=0}^r d_i x^i \partial_i$. Now the quasi-homogeneity relation reads

$$E(\Phi) = d_\Phi \Phi.$$

**4.5 Quantum cohomology.** QUASSI-HOMOGENEITY OF THE GROMOV-WITTEN POTENTIAL. Let us show that also the Gromov-Witten potential $\Phi$ is quasi-homogeneous of degree $d_\Phi = 3 - d$. (The degrees $d_0, \dots, d_r$ and $d = \dim X$ are the same as in the trivial Frobenius manifold of the cohomology ring, cf. 4.4.) First recall that dimension of $\overline{M}_{0,n}(X, \beta)$ is

$$\dim \overline{M} = d - 3 + \int_\beta c_1(X) + n.$$

On the other hand, a product $T_0^{a_0} \dots T_r^{a_r}$ has a total codimension of $\sum a_i p_i$. Since $n = \sum a_i$ is the number of marks, the Gromov-Witten invariant $I_\beta^X(T_0^{a_0} \dots T_r^{a_r})$ is non-zero only if the following relation holds (the selection rule)

$$\sum a_i(p_i - 1) = d - 3 + \int_\beta c_1(X). \tag{6}$$

Now recall that the Gromov-Witten potential is

$$\Phi = \sum_{\beta \geq 0} q^\beta \sum_{\mathbf{a}} \frac{\mathbf{x}^\mathbf{a}}{\mathbf{a}!} I_\beta^X(T_0^{a_0} \dots T_r^{a_r})$$

Continuing the convention introduced for the trivial Frobenius manifold of $H^*(X, \mathbb{Q})$ we set $d_i := \deg x^i := 1 - p_i$, and we also set

$$\deg q^\beta := \int_\beta c_1(X). \tag{7}$$

Now for any fixed $\beta$ and $\mathbf{a}$, the corresponding term in $\Phi$ has degree $\int_\beta c_1(X) + \sum a_i d_i$. By the selection rule, this is exactly $3-d$. (Note that it is the selection rule that effectively dictates the gradings.)

**4.6 Quantum cohomology.** Euler field. Now in order to express the quasi-homogeneity of $\Phi$ in terms of a field $E$, we must take care of those symbols $q^\beta$, which we haven't defined properly. Whenever the symbol $\beta$ appears as an exponent, think of it as the number $\int_\beta c_1(X)$. In this way if we simply put put $\deg q = 1$ then we obtain $\deg q^\beta = \int_\beta c_1(X)$. Applying directly Eulers identity we get

$$\big(q \frac{\partial}{\partial q} + \sum_i d_i x^i \partial_i\big)\Phi = d_\Phi \Phi.$$

To express this in terms of a vector field we have to eliminate $q$. This is accomplished using the *divisor equation*. The divisor equation states (cf. Fulton-Pandharipande [3], Equation 40) that for $p_k = 1$, so that $T_k$ is a divisor class, we have

$$I_\beta^X(\gamma_1 \cdots \gamma_n \cdot T_k) = \textstyle\int_\beta T_k \cdot I_\beta^X(\gamma_1 \cdots \gamma_n). \tag{8}$$

There are some exceptions for $\beta = 0$ but these correspond to low degree terms of $\Phi$ that we don't need to care about. In terms of the potential we have

$$\partial_k \Phi = \sum_\beta \textstyle\int_\beta T_k I_\beta^X(\exp(\gamma)).$$

Consider now the effect of the operator $q \frac{\partial}{\partial q}$ we found as first term above:

$$q \frac{\partial}{\partial q}\Phi = \sum_\beta \textstyle\int_\beta c_1(X) I_\beta^X(\exp(\gamma)).$$

We see that (at least up to quadratic terms)

$$q\frac{\partial}{\partial q} = \sum_i r_i \partial_i$$

where the constants $r_i$ are the coordinates of the first Chern class, $c_1(X) = \sum_i r_i T_i$ (The only non-zero terms in this sum are then obviously those in degree 1, that is, those $i$ with $p_i = 1$.)

So in conclusion, the Euler field is

$$E = (d_i x^i + r_i)\partial_i$$

with the found constants $r^i$. For a high-tech proof, see Getzler [4], Prop. 5.3.

For example, for $X = \mathbb{P}^2$ we have

$$E = x^0\partial_0 + 3\partial_1 - x_2\partial_2.$$

Note that the constant term $r_i\partial_i = 3\partial_1$ is concentrated where $i = 1$, which is also exactly where $d_i = 0$.

**4.7 Euler field. First definition.** In general for a Frobenius manifold $(M, g, \Phi)$, the potential $\Phi$ is only defined up to a quadratic polynomial, so the homogeneity should be similarly defined. We require only $E(\Phi) = d_\Phi \Phi$ up to a quadratic polynomial.

Definition. A vector field $E$ is called an Euler field if it is of degree at most one,

$$E = (d_i x^i + r_i)\partial_i,$$

and we have

$$E\Phi = d_\Phi \Phi$$

modulo a quadratic polynomial.

There can be at most one choice of these constants $d_0, \dots, d_r, d_\Phi$, except for a common constant factor. Supposing $d_0 \neq 0$, we will always normalise so that $d_0 = 1$ as in the examples above.

We are now going to express this condition in terms of $\nabla$ and $g$, rather than $\Phi$.

**4.8 Lie derivative of the metric.** Let $g$ denote the metric tensor. Its Lie derivative $L_Z(g)$ is given by the formula (cf. A.5, A.6)

$$L_Z(g)(X, Y) = L_Z(g(X, Y)) - g([Z, X], Y) - g(X, [Z, Y]). \tag{9}$$

*Definition.* A vector field $Z$ is called *conformal* if there is a constant $D$ such that $L_Z(g) = Dg$.

**4.9 Example.** If the metric is constant (or just flat?) then for a vector field $Z$ to be conformal it must be linear, i.e., of the form $Z = (d_i x^i + r_i)\partial_i$.

**4.10 Some remarks.** Let $E = (d_i x^i + r_i)\partial_i$ be an Euler field for the Frobenius manifold $(M, g, \Phi)$. Then we have

$$L_E(\partial_j) = -d_j \partial_j.$$

This is just to compute $[(d_i x^i + r_i)\partial_i, \partial_j] = -d_j \partial_j$; here the constants $r_i$ play no rôle.

It follows that $E\Phi_j = (d_\Phi - d_j)\Phi_j$. Indeed, since $\Phi_j = \partial_j \Phi$ is really a contraction, we can compute $E(\partial_j \Phi) = L_E(\partial_j)\Phi + \partial_j E\Phi = -d_j \Phi_j + d_\Phi \Phi_j$. Similarly,

$$E\Phi_{ijk} = (d_\Phi - d_i - d_j - d_k)\Phi_{ijk}.$$

**4.11 Quantum cohomology.** The effect of the Euler field on the metric. Continuing the case of quantum cohomology (or the trivial Frobenius manifold), we now claim that the Euler field $E = (d_i x^i + r_i)\partial_i$ acts on the Poincaré metric $g$ by conformal transformation. In other words, the Euler field is conformal in the Poincaré metric:

$$L_E(g) = D \cdot g, \qquad \text{with } D = 2 - d.$$

Proof. Plugging $X = \partial_i$ and $Y = \partial_j$ into Equation (9), we get

$$\begin{aligned} L_E(g)(\partial_i, \partial_j) &= L_E(g_{ij}) - g([E, \partial_i], \partial_j) - g(\partial_i, [E, \partial_j]) \\ &= -g(-d_i \partial_i, \partial_j) - g(\partial_i, -d_j \partial_j) \\ &= d_i g_{ij} + d_j g_{ij} \\ &= (2 - d) g_{ij}. \end{aligned}$$

Here in the second line we used that $L_E(g_{ij}) = 0$ since $g_{ij}$ are constants. In the last line we used that $d_i + d_j = 2 - d$ or otherwise $g_{ij} = 0$ according to Equation (5).

**4.12 Quantum cohomology.** The effect of the Euler field on the multiplication tensor. Similar arguments show that the effect on the multiplication $\mu$ is

$$L_E(\mu) = \mu$$

In other words, the Euler field acts trivially on the Frobenius algebras. (rescaling by $1 = d_0$). Proof.

$$L_E(\mu)(\partial_i, \partial_j) = L_E\big(\sum \Phi_{ije} g^{ef} \partial_f\big) - L_E \partial_i \circ \partial_j - \partial_i \circ L_E \partial_j$$

Now the last two terms give $(d_i + d_j)(\partial_i \circ \partial_j)$. As to the first term:

$$\begin{aligned} L_E\big(\Phi_{ije} g^{ef} \partial_f\big) &= L_E \Phi_{ije} g^{ef} \partial_f + \Phi_{ije} g^{ef} L_E \partial_f \\ &= (d_\Phi - d_i - d_j - d_e - d_f) \Phi_{ije} g^{ef} \partial_f \end{aligned}$$

using the remarks above. Now for $g^{ef}$ (as well as for $g_{ef}$) we have

$$g^{ef} = 0 \quad \text{ or } \quad p_e + p_f = d.$$

Which is the same as saying $d_e + d_f = 2 - d = d_\Phi - 1$. So combining the contributions we get

$$\Phi_{ije} g^{ef} \partial_f = \partial_i \circ \partial_j$$

as claimed.

We are now going to give a new definition of an Euler field. The following proposition shows it agrees with the first definition.

**4.13 Euler field.** (Cf. Dubrovin [2], Lecture 1, Definition 1.2; Manin [6], I.2.2.1.) On a Frobenius manifold $(M, g, \Phi)$, a conformal vector field $E$ is said to be an Euler field, if it acts on the multiplication tensor by rescaling. Precisely, we require the existence of a constant $d_0$ such that $L_E(\mu) = d_0\, \mu$. In other words, for vector fields $X, Y$ we require

$$L_E(X \circ Y) - [E, X] \circ Y - X \circ [E, Y] = d_0\ X \circ Y.$$

**4.14 Proposition.** *(Cf. Manin [6], I.2.2.2.) A conformal vector field $E$ on a Frobenius manifold $(M, g, \Phi)$ is Euler in the sense of 4.13 if and only if we have*

$$E\Phi = d_\Phi \Phi$$

*modulo quadratic terms. Here $d_\Phi = D + d_0$ where $D$ is the conformal constant, and $d_0$ is the rescaling constant.*

*Proof.* To have $E\Phi = d_\Phi \Phi$ modulo quadratic terms is equivalent to having, for all flat vector fields $X, Y, Z$ the relation

$$d_\Phi XYZ\Phi = XYZE\Phi.$$

On the right hand side, move $E$ to the left and correct for commutators, getting

$$= EXYZ\Phi - XY[E, Z]\Phi - X[E, Y]Z\Phi - [E, X]YZ\Phi$$

Now write $XYZ\Phi = \langle\, X \circ Y \,|\, Z\,\rangle$, and similarly for the other terms

$$= E\ \langle\, X \circ Y \,|\, Z\,\rangle\ -\ \langle\, X \circ Y \,|\, [E, Z]\,\rangle\ -\ \langle\, X \circ [E, Y] \,|\, Z\,\rangle\ -\ \langle\, [E, X] \circ Y \,|\, Z\,\rangle\,.$$

As we know from Equation (9), we have in general $E\ \langle\, X \circ Y \,|\, Z\,\rangle\ = L_E(g)(X \circ Y, Z) + \langle\, [E, X \circ Y] \,|\, Z\,\rangle\ +\ \langle\, X \circ Y \,|\, [E, Z]\,\rangle$. Filling this in, we get (two terms cancel out)

$$= L_E(g)(X \circ Y, Z) + \langle\, [E, X \circ Y] \,|\, Z\,\rangle\ -\ \langle\, X \circ [E, Y] \,|\, Z\,\rangle\ -\ \langle\, [E, X] \circ Y \,|\, Z\,\rangle\,.$$

On the first term we can use the fact that $E$ is conformal, and the last three terms are just $\langle\, L_E(\mu)(X, Y) \,|\, Z\,\rangle$:

$$= D\ \langle\, X \circ Y \,|\, Z\,\rangle\ +\ \langle\, L_E(\mu)(X, Y) \,|\, Z\,\rangle\,.$$

Now this last term is equal to $d_0 XYZ\Phi$ if (and only if?) furthermore $E$ acts as scaling transformation as it is required to be Euler. $\square$

## List of notations (Frobenius manifold of quantum cohomology)

The Euler field acts on the structures as follows, provided $d_0 = 1$:

$$\begin{aligned} L_E(\circ) &= \circ \\ L_E(g) &= D \cdot g \\ L_E\Phi &= (D+1)\Phi \end{aligned}$$

| | |
|---|---|
| $T_i$ | basis for the Frobenius algebra $H = H^*(X, \mathbb{Q})$ |
| $\partial_i$ | flat vector fields on $M$ (corresponding to $T_i$) |
| $x^i$ | formal variables corresponding to $T_i$; flat coordinates on $M$ |
| $p_i$ | the degree of $T_i$ (half the degree in cohomology) |
| $d := p_r$ | the degree of $T_r$, the socle, (dimension of the space $X$) |
| $d_i := 1 - p_i$ | degree of $x^i$ |
| $d_\Phi := 3 - d$ | degree of the Gromov-Witten potential $\Phi$ |
| $D := 2 - d$ | conformal constant (provided $d_0 = 1$) |

# Appendix: Riemannian geometry

My reference is do Carmo [1].

Let $M$ be a Riemannian manifold (over $k$) with metric $g$. We do not required $g$ to be positive definite, so more precisely perhaps it's a pseudo-Riemannian metric.

**A.1 Connections.** A (linear) connection is a $k$-derivation $\nabla : \mathcal{T}_M \to \mathrm{End}_{\mathcal{O}_M}(\mathcal{T}_M)$, denoted $Y \mapsto \nabla Y := [X \mapsto \nabla_X Y]$. which is a derivation. Precisely, $\nabla$ is linear in the lower argument, $\nabla_{fX+f'X'}Y = f\nabla_X Y + f'\nabla_{X'}Y$, for $f, f' \in \mathcal{O}_M$ and $X, X', Y \in \mathcal{T}_M$. And satisfies Leibniz' rule in the other argument: $\nabla_X(f.Y) = X(f).\nabla_X Y + f.\nabla_X Y$.

In a coordinate system $x^0, \ldots, x^r$, with $\partial_i = \frac{\partial}{\partial x^i}$, the Christoffel symbols of the connection $\nabla$ are the functions ${\Gamma_{ij}}^k \in \mathcal{O}_M$ defined as $\nabla_{\partial_i}\partial_j = {\Gamma_{ij}}^k\,\partial_k$. If $X = X^i\partial_i$ and $Y = Y^j\partial_j$ are two vector fields, then the rules for connections give the following coordinate expression.

$$\nabla_X Y = \big(X^i Y^j {\Gamma_{ij}}^k + X^i \partial_i(Y^k)\big)\partial_k.$$

On an affine manifold, there is a one-to-one correspondence between connections and Christoffel symbols, i.e., collections of $n^3$ functions. Observe that a connection is not a tensor since it is not linear in the second argument. However, given one connection, then all other connections are obtained by adding tensors to the Christoffel symbols.

On a manifold with constant metric, the Euclidean connection is the one whose Christoffel symbols are all zero.

**A.2 The Levi-Civita connection.** The *torsion* tensor of a connection $\nabla$ is $\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$. Writing out in coordinates, you see that the torsion tensor is identically zero if and only if the Christoffel symbols are symmetric in the lower indices; in that case the connection is called torsion free.

The *Levi-Civita connection* $\nabla^{\mathrm{LC}}$ is the (unique) torsion free connection $\nabla$ compatible with the metric $g$ in the sense that $Z \langle X \,|\, Y \rangle = \langle \nabla_Z X \,|\, Y \rangle + \langle X \,|\, \nabla_Z Y \rangle$.

**A.3 Curvature.** The curvature tensor of a connection is the map $R : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathrm{End}(\mathcal{T}_M)$ given by

$$R(X,Y)Z := \nabla_Y \nabla_Y X - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

In some books, the sign is the opposite; in any case it is just a convention.

In a coordinate neighbourhood, the tensor elements ${R_{ijk}}^m$ of the curvature are defined by $R(\partial_i, \partial_j)\partial_k = {R_{ijk}}^m \partial_m$. If ${\Gamma_{ij}}^k$ are the Christoffel symbols of the connection, then

$${R_{ijk}}^m = {\Gamma_{ik}}^e {\Gamma_{je}}^m - {\Gamma_{jk}}^e {\Gamma_{ie}}^m + \partial_j {\Gamma_{ik}}^m - \partial_i {\Gamma_{jk}}^m.$$

A connection is flat if its curvature is identical zero. A metric is flat if its Levi-Civita connection is flat.

**A.4 Flat vector fields.** The Levi-Civita connection acts on vector fields producing tensors $\nabla X : \mathcal{T}_M \to \mathcal{T}_M$. A vector field is said to be flat if $\nabla X = 0$. In flat coordinates, and equivalent condition is that if $X = X^i \partial_i$ then $X^i$ are constant functions.

There is a sub-sheaf of $\mathcal{T}_M$ whose sections are exactly the flat vector fields...

**A.5 Lie derivatives.** Let $Z$ denote a vector field. Its action as derivative on functions can be extended to the whole tensor algebra as follows — the so-called Lie derivative $L_Z$. The infinitesimal definition of the Lie derivative is this: given any tensor $S$, then $L_Z S$ is defined as the tensor (of same type)

$$\frac{d}{dt}(\phi_t^* S)_{|t=0}$$

where $\phi_t$ is the 1-parameter group corresponding to $Z$ (the local flow of $Z$). So pull back the tensor along the flow of $Z$, and take the limit of this as the amount of flow goes to zero... This is the analytic construction.

Here comes the more functorial definition. The Lie derivative $L_Z$ is the unique operator on the tensor algebra with the following properties.

(i) $L_Z f = Z(f)$ for $f \in \mathcal{O}_M$
(ii) $L_Z X = [Z,X]$ for $X \in \mathcal{T}_M$
(iii) $L_Z(S \otimes T) = L_Z S \otimes T + S \otimes L_Z T$
(iv) $L_Z(c(S)) = c(L_Z S)$

The Leibniz rule of number (iii) says that $L_Z$ is a derivative on the tensor algebra. Rule (iv) says that $L_Z$ is compatible with any contraction $c$.

To see how it works in practice, let us work through this example;

**A.6 Lie derivative of the metric.** Let $g$ denote the metric tensor. Its Lie derivative $L_Z(g)$ is given by the formula

$$L_Z(g)(X,Y) = L_Z(g(X,Y)) - g([Z,X],Y) - g(X,[Z,Y]).$$

Proof: We have $g : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M$ and we want to describe $L_Z(g) : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M$. We describe it by saying what it does to $X \otimes Y \in \mathcal{T}_M \otimes \mathcal{T}_M$. Now $g(X,Y) \in \mathcal{O}_M$ is really a contraction

$$\begin{aligned} c : (\mathcal{T}_M \otimes \mathcal{T}_M)^* \otimes (\mathcal{T}_M \otimes \mathcal{T}_M) \quad &\to \quad \mathcal{O}_M \\ g \otimes X \otimes Y \quad &\mapsto \quad g(X,Y) \end{aligned}$$

So we can use rule (iv) to compute $L_Z$ on the function $g(X,Y) \in \mathcal{O}_M$.

$$L_Z(g(X,Y)) = L_Z(c(g \otimes X \otimes Y)) = c(L_Z(g \otimes X \otimes Y))$$

Now we can use the Leibniz rule on this tensor product to get

$$L_Z(g \otimes X \otimes Y) = L_Z(g) \otimes X \otimes Y + g \otimes L_Z X \otimes Y + g \otimes X \otimes L_Z Y$$

and this sum we have to contract. We get then

$$L_Z(g(X,Y)) = L_Z(g)(X,Y) + g(L_Z X, Y) + g(X, L_Z Y)$$

whence the result, by rule (ii).

# References


[1] Manfredo Perdigão do Carmo. *Geometria Riemanniana.* Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1988. English translation: Riemannian Geometry, Birkhäuser, Boston, 1992.

[2] Boris Dubrovin. *Geometry of 2D topological field theories.* In *Integrable systems and quantum groups*, no. 1620 in Lecture Notes in Mathematics, pp. 120–348. Springer-Verlag, New York, 1996. (hep-th/9407018).

[3] William Fulton and Rahul Pandharipande. *Notes on Stable Maps and Quantum Cohomology.* In J. Kollár, R. Lazarsfeld and D. Morrison, editors, *Algebraic Geometry, Santa Cruz 1995*, vol. 62, II of Proc. Symp. Pure. Math., pp. 45–96. (alg-geom/9608011).

[4] Ezra Getzler. *The Virasoro conjecture for Gromov-Witten invariants.* In *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, vol. 241 of Contemp. Math., pp. 147–176. Amer. Math. Soc., Providence, RI, 1999. (math.AG/9812026).

[5] Robin Hartshorne. *Algebraic Geometry.* No. 52 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1977.

[6] Yuri I. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces.* AMS Colloquium Publications, Providence, RI, 1999.