Frobenius splitting of complete intersections in $\mathbb{P}^n$

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0 Introduction

The notion of Frobenius splitting was introduced in 1985 by Mehta and Ramanathan in their paper [M-R]: For a variety X of characteristic p, the absolute Frobenius morfism $F : X \to X$ is the identity on points, and $F^\sharp : \mathcal{O}_X \to F_*\mathcal{O}_X$ is locally the p'th power map $a \mapsto a^p$. Now X is said to be *Frobenius split* if $F^\sharp : \mathcal{O}_X \to F_*\mathcal{O}_X$ is a split injection. A consequence of Frobenius splitting is the vanishing theorem for ample line bundles, and for smooth varieties also the Kodaira vanishing theorem.

In a "fundamental isomorphism" established via the Cartier operator, the splittings were related to the global sections of the anti-canonical sheaf, and criteria for splitting in terms of these sections were deduced. Applying these criteria, Mehta and Ramanathan showed that Schubert varieties in characteristic p are Frobenius split, and by a semi-continuity argument showed that Schubert varieties in characteristic zero have cohomology vanishing for ample line bundles.

In a subsequent paper [R] Ramanathan refined the tools and proved, among other results, that Schubert varieties are cut out by quadrics and linear forms in any embedding given by an ample line bundle.

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The present paper treats complete intersections in $\mathbb{P}^n$ in a polynomial setup that makes explicit computations feasible, and brings to the fore the geometry. After reviewing the definitions of Frobenius splitting and establishing notation, the fundamental isomorphism is stated explicitly for $\mathbb{P}^n$ in terms of polynomials and the main result is established: All splittings of a complete intersection in $\mathbb{P}^n$ come from splittings of $\mathbb{P}^n$ (Proposition 2.6).

For hypersurfaces we get the following criterion: $Z(f) \subset \mathbb{P}^n$ of degree $n+1$ is split if and only if the coefficient of $(x_0 \cdots x_n)^{p-1}$ in f^{p-1} is nonzero. Such a $Z(f)$ is called a *splitting divisor*, and it is shown that every split complete intersection arises as intersection of components of a splitting divisor (Proposition 3.2).

This paper grew out of my masters dissertation at the University of Aarhus, and I am in debt to my patient advisor Niels Lauritzen, and also to the Department of Mathematics at the Federal University of Pernambuco, Brazil, where most of the work was done. Finally I would like to thank the referee for careful reading and constructive comments that have led to several improvements.

1 Review of Frobenius splitting

Throughout the paper k will be a fixed algebraically closed field of characteristic $p > 0$, and by a variety we will understand a equidimensional reduced quasi-projective scheme of finite type over k (but not necessarily irreducible).

Let X be a k-scheme. Let $F : X \to X$ be the absolute Frobenius morphism. It is the identity on the underlying set of points, and on sheaf level it is $\mathcal{O}_X \to F_*\mathcal{O}_X$, given locally as the Frobenius ring endomorphism $a \mapsto a^p$. By abuse of notation, we also use the symbol F to denote this homomorphism of sheaves. Note that F is not a k-homomorphism. For any $\mathcal{O}_X$-module M, the direct image F_*M is equal to M as an abelian group, but has $\mathcal{O}_X$-module structure given by $a.m = a^p m$ for all $a \in \mathcal{O}_X$ and $m \in M$.

Definition. — A k-scheme X is called *Frobenius split* (or just *split*) if the Frobenius homomorphism $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ has a splitting, i.e. an $\mathcal{O}_X$-module homomorphism $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_X$ is the identity map.

We will refer to $\mathcal{O}_X$-module homomorphisms $F_*\mathcal{O}_X \to \mathcal{O}_X$ as *pre-splittings* of X. A pre-splitting $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ is thus a splitting if and only if $\varphi(1) = 1$.

Remark. — A necessary condition for a scheme to be Frobenius split is that it is reduced: otherwise the Frobenius homomorphism fails to be injective.

The main tools for determining whether a variety is Frobenius split are the notion of *compatible splitting* that relates splitting to an ambient variety, and the "fundamental isomorphism" which gives criteria of splitting in terms of sections of the anticanonical sheaf.

Let $Y \subset X$ be a closed subvariety with ideal sheaf I. Since Frobenius commutes with any morphism, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "F"] & \mathcal{O}_X \arrow[r] \arrow[d, "F"] & \mathcal{O}_Y \arrow[r] \arrow[d, "\overline{F}"] & 0 \\
0 \arrow[r] & F_*I \arrow[r] & F_*\mathcal{O}_X \arrow[r] & F_*\mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

where the middle F is Frobenius for X, the left hand F is just its restriction to I, and $\overline{F}$ is induced by the two others and coincides with Frobenius for Y.

Definition. — Let $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ be a pre-splitting of X. If $\varphi(F_*I) \subset I$, then a pre-splitting $\overline{\varphi} : F_*\mathcal{O}_Y \to \mathcal{O}_Y$ of Y will be induced, and in that case φ is said to be *compatible with* Y. If furthermore φ is a splitting of X, then $\overline{\varphi}$ will be a splitting of Y and we will say that Y *is compatibly split in* X (Cf. [M-R], Definition 3).

The next two lemmas are basic. The first one allows all questions of compatibility to be settled locally:

1.1 Lemma. *(Cf. [M-R], Lemma 1.)* — *Let φ be a pre-splitting of X and let $Y \subset X$ be a closed subvariety with ideal sheaf I. Let U be an open subset in X touching all irreducible components of Y. If now the restriction of φ to U is compatible with $Y \cap U \subset U$, then φ is also compatible with the whole of Y. In other words,*

$$\varphi(F_*I)_{|U} \subset I_{|U} \Rightarrow \varphi(F_*I) \subset I.$$

1.2 Lemma. *(Cf. [R], 1.9 and 1.10.)* — *Let φ be a pre-splitting of X.*

(i) *φ is compatible with a closed subvariety if and only if it is compatible with each irreducible component.*

(ii) *If φ is compatible with two closed subvarieties then it is also compatible with their scheme theoretic intersection. In particular, if φ is a splitting then the scheme theoretic intersection is compatibly split and must therefore be reduced.*

1.3 Standard argument. — When checking whether an X-pre-splitting φ is compatible with an irreducible subvariety Y we can use Lemma 1.1 to do it locally — actually in the local ring of a point x which we can choose to be smooth in Y. That is, it suffices to show $\varphi(F_*I_x) \subset I_x$ in $\mathcal{O}_{X,x}$. And by faithful flatness of completion, it is enough to check this in the completion, so if X is smooth of dimension n and Y has codimension c, we can choose suitable coordinates and reduce to checking that $\varphi(F_*(x_1, \dots, x_c)) \subset (x_1, \dots, x_c)$ in $k[[x_1, \dots, x_n]]$.

The fundamental isomorphism on which all results on Frobenius splitting are based is the following:

1.4 Theorem. *(Cf. [M-R], Proposition 5.) — Let X be a Gorenstein variety, and let ω_X be the dualizing sheaf of X. Then there is a natural isomorphism of $\mathcal{O}_X$-modules*

$$F_*\omega_X^{1-p} \xrightarrow{\sim} \mathcal{H}om(F_*\mathcal{O}_X, \mathcal{O}_X).$$

The theorem is proved (for X smooth) exploiting the Cartier operator (as in [M-R]), or by means of relative duality for the Frobenius morphism (as in [R]). Relative duality also yields the general result (as sketched in [M-Sr]).

The general version of this theorem is only used once in the sequel, when we prove that all splittings of complete intersections come from compatible splittings of $\mathbb{P}^n$. The rest of the paper is concerned only with compatible splitting of varieties embedded in projective space, and it is convenient to restate the theorem in the case of $X = \mathbb{P}^n$:

2 Compatible splitting in $\mathbb{P}^n$

We will describe the isomorphism by means of the following variant of the Cartier operator:

Definition. — Let $C : F_*k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]$ be the $k[x_1, \dots, x_n]$-module homomorphism defined as $(x_1 \cdots x_n)^{p-1} \mapsto 1$, and $m \mapsto 0$ for all other monomials m without a p'th power factor. Monomials with a p'th power factor are dealt with by linearity: $C(g^p f) = gC(f)$.

For a polynomial $f(x_1, \dots, x_n)$ of degree $\leq (n+1)(p-1)$ we call the term $a(x_1 \cdots x_n)^{p-1}$ the *critical term*, and then $C(f) = \sqrt[p]{a}$. $(a \in k.)$

2.1 Proposition. — *There is an isomorphism of k-vector spaces*

$$H^0(\mathbb{P}^n, F_*\mathcal{O}_{\mathbb{P}^n}((n+1)(p-1)) \xrightarrow{\sim} \mathrm{Hom}(F_*\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})$$

where a form s of degree $(n+1)(p-1)$ is sent to the homomorphism φ_s given locally (on the standard open set $U_0 = \{x_0 \neq 0\}$, say) as

$$\varphi_s(m) = C(sm),$$

where both $m \in \mathcal{O}_{\mathbb{P}^n}$ and s are dehomogenized with respect to x_0 in the usual way.

Henceforth, by abuse of language, we will refer to forms s of degree $(n+1)(p-1)$ as pre-splittings (resp. splittings) when we actually mean the φ_s corresponding to s under the isomorphism of Proposition 2.1.

Note that for such a form s, the coefficient $C(s)$ is the same whether we consider x_0 as a variable or dehomogenize and neglect it.

Note also that up to scalars $C(s)$ is independent of automorphisms of $\mathbb{P}^n$, so we can choose coordinates as we please.

2.2 Corollary. — *$\mathbb{P}^n$ is Frobenius split: Splittings of $\mathbb{P}^n$ are forms s of degree $(n+1)(p-1)$ with $C(s) \neq 0$. That is, the term $(x_0 \cdots x_n)^{p-1}$ occurs with nonzero coefficient.*

Next we give a criterion of compatibility in $\mathbb{P}^n$, involving only s and not φ_s. Our s will be globally defined to start with, so it suffices to work with $\mathbb{A}^n$ (by Lemma 1.1).

Definition. — For an ideal I in a noetherian ring, we denote by $I^{[p]}$ the ideal generated by all p'th powers of elements of I.

2.3 Proposition. *(Cf. [F] 1.12.) — Let s be a polynomial in $x_1, \ldots, x_n$ of degree $\leq (n+1)(p-1)$ and let φ_s be the pre-splitting for $\mathbb{A}^n$ induced by s via the fundamental isomorphism. Let Y be a closed subvariety of $\mathbb{A}^n$ with ideal I. Then φ_s is compatible with Y if and only if $s \cdot I \subset I^{[p]}$*

Proof. — The "if" implication is trivial. For the other implication: Let c be the codimension of Y, choose a smooth point of Y and reduce to the case $I = (x_1, \ldots, x_c)$ as in 1.3. Remark that $I^{[p]} = (x_1^p, \ldots, x_c^p)$, and that a polynomial belongs to I or $I^{[p]}$ if and only if each monomial does. So we reduce to the case where s is a monomial, say $s = \prod x_i^{q_i}$. Suppose φ_s is compatible with Y, i.e. $\varphi_s(I) \subset I$. If $q_i \geq p$ for some $i = 1, \ldots, c$, there is nothing to prove. So assume $0 \leq q_i \leq p-1$ for each $i = 1, \ldots, c$.

Now let m be the monomial that fills up the exponents to nearest $p-1$ (mod p), that is, $m = \prod x_i^{a_i}$ where $0 \leq a_i \leq (p-1)$ and $q_i + a_i = (p-1) + pb_i$, and by the assumption on q_i we have $b_i = 0$ for $i \leq c$. Now apply φ_s to m:

$$\varphi_s(m) = C(sm) = C\left(\prod x_i^{q_i+a_i}\right) = C\left((x_1 \cdots x_n)^{p-1} \prod_{i=c+1}^{n} x_i^{pb_i}\right) = \prod_{i=c+1}^{n} x_i^{b_i}.$$

This is not in I, so we conclude that m is not in I either, and therefore the first c q_i's are already equal to $p-1$, which in turn implies $s \cdot I \subset I^{[p]}$. $\square$

2.4 Corollary. *(Cf. [L-M-P]) — Let Y be a closed subvariety of codimension c in $\mathbb{P}^n$. If a pre-splitting s vanish to order $c(p-1)$ along Y, then it is compatible with Y, and if s vanishes to higher order, it cannot be a splitting.*

Proof. — Continuing the notation of the proof above, the ideal of Y is $I = (x_1, \ldots, x_c)$. If $s \in I^{c(p-1)}$ then each monomial term of s either has a p'th power factor of some x_i, or it is the term $(x_1 \cdots x_c)^{p-1}$. In either case follows $s \cdot I \subset I^{[p]}$. If $s \in I^{c(p-1)+1}$, then each monomial term has a p'th power factor, and so the critical term does not occur, and s can not be a splitting. $\square$

When Y is a hypersurface the converse also holds:

2.5 Corollary. — *A pre-splitting $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((n+1)(p-1)))$ of $\mathbb{P}^n$ is compatible with a hypersurface $Z(f)$ if and only if f^{p-1} is factor in s.*

Indeed, locally the ideal of $Z(f)$ is (f). By Proposition 2.3, compatibility means $sf \in (f^p)$ which is equivalent to $s \in (f^{p-1})$.

2.6 Proposition. — *Let $Y = Z(f_1, \ldots, f_c)$ be a complete intersection in $\mathbb{P}^n$ of dimension ≥ 1. Then all pre-splittings (resp. splittings) of Y come from pre-splittings (resp. splittings) of $\mathbb{P}^n$ compatible with the hypersurface $Z = Z(f_1 \cdots f_c)$.*

Proof. — Let I be the ideal sheaf of Y and let $d_i = deg(f_i)$ and $d = \sum d_i$. Recall that since Y is a complete intersection I/I^2 is locally free of rank c and has a filtration with quotients $\mathcal{O}_{\mathbb{P}^n}(-d_i) \otimes \mathcal{O}_Y$, so taking highest exterior power we get $\bigwedge^c(I_Y/I_Y^2) \simeq \mathcal{O}_{\mathbb{P}^n}(-d) \otimes \mathcal{O}_Y$. The dualizing sheaf ω_Y is then $\omega_{\mathbb{P}^n} \otimes \bigwedge^c(I_Y/I_Y^2)^{\check{}} \simeq \mathcal{O}_Y(-n-1+d)$, (e.g. by [H], Ch. III, Proposition 7.11) and so by Theorem 1.4 the space of pre-splittings is isomorphic to $H^0(Y, \mathcal{O}_Y((n+1-d)(p-1)))$.

As we have just seen, pre-splittings of $\mathbb{P}^n$ compatible with Z are of the form $(f_1 \cdots f_c)^{p-1} \cdot g$, where g is homogeneous of degree $(n+1-d)(p-1)$, so the space of Z-compatible pre-splittings of $\mathbb{P}^n$ is isomorphic to $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((n+1-d)(p-1)))$.

Now any pre-splitting of Z is also compatible with Y (by Lemma 1.2) and the induction of Z-compatible pre-splittings of $\mathbb{P}^n$ to pre-splittings of Y is just the natural map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((n+1-d)(p-1))) \longrightarrow H^0(Y, \mathcal{O}_Y((n+1-d)(p-1)))$$

which is surjective since Y is a complete intersection and has dimension ≥ 1 (see [H], Ch. III, Exercise 5.5). So pre-splittings of Y come from Z-compatible pre-splittings of $\mathbb{P}^n$.

Let now $\overline{\sigma} : F_* \mathcal{O}_Y \to \mathcal{O}_Y$ be a pre-splitting of Y induced by $\sigma : F_* \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}$. If furthermore $\overline{\sigma}$ is a splitting, then we have $\overline{\sigma}(1) = 1$, and so we must have $\sigma(1) = 1 \pmod{(f)}$. But since any pre-splitting sends 1 to a scalar, we conclude that $\sigma(1) = 1$, and σ is a splitting. So splittings of Y come from Y-compatible splittings of $\mathbb{P}^n$. $\square$

Remark. — It is *not* true that all pre-splittings of $\mathbb{P}^n$ compatible with Y are also compatible with Z; what the proposition says is that there *exists* a Z-compatible pre-splitting which induces the given Y-pre-splitting.

This result allows us to handle all questions of splitting of complete intersections in the more manageable setup of compatible splitting in $\mathbb{P}^n$, and henceforth we will use the words *split* and *compatibly split in* $\mathbb{P}^n$ interchangeably for complete intersections. Also we have reduced all questions of complete intersections to questions of hypersurfaces.

2.7 Corollary. *(Cf. [F], Prop. 2.1.) — Let $Y = Z(f_1, \ldots, f_c) \subset \mathbb{P}^n$ be a complete intersection, and let $d = \sum deg(f_i)$. If $d > n+1$ then Y is not split. If $d = n+1$, then*

$$Y \text{ is split } \Leftrightarrow Z(f_1 \cdots f_c) \text{ is split } \Leftrightarrow C((f_1 \cdots f_c)^{p-1}) \neq 0.$$

3 Splitting divisors

Next we show that a compatible split hypersurface is always contained in a $Z(f)$ of degree $n+1$.

Definition. — A *full splitting* of $\mathbb{P}^n$ is a splitting of form f^{p-1} where f has degree $n+1$. The divisor of zeros of such an f will be called a *splitting divisor.*

3.1 Lemma. — *Let $g(x_0, \ldots, x_n)$ be homogeneous of degree $n(p-1)$, and let $x_0^{q_0} \cdots x_n^{q_n}$ be of degree $p-1$, i.e. $\sum q_i = p-1$. If $(x_0 \cdots x_n)^{p-1}$ occurs in $g \cdot x_0^{q_0} \cdots x_n^{q_n}$, then there exists a linear form $h = A_0 x_0 + \cdots + A_n x_n$ such that $(x_0 \cdots x_n)^{p-1}$ also occurs in $g \cdot h^{p-1}$.*

Proof. — The set of all h is $\mathbb{P}^{n*}$ with homogeneous coordinates $(A_0 : \ldots : A_n)$. Recall that

$$h^{p-1} = \sum \left(\frac{(p-1)!}{\prod s_i!} \right) \prod_{i=0}^{n} A_i^{s_i} x_i^{s_i}$$

where the sum is taken over all $n+1$-tuples $\underline{s}=(s_0,\dots,s_n)$ of nonnegative integers such that $\sum_{i=0}^n s_i = p-1$.

Given $\prod x_i^{s_i}$ let $c_{\underline{s}}$ be the coefficient in g of $\prod x_i^{p-1-s_i}$ (the fill-up-to-$(p-1)$-factor). Now we have that the coefficient of $(x_0\cdots x_n)^{p-1}$ in $g\cdot h^{p-1}$ is

$$\sum c_{\underline{s}} \left(\frac{(p-1)!}{\prod s_i!} \right) \prod_{i=0}^{n} A_i^{s_i}.$$

This is a homogeneous polynomial in A_i, so the set of bad choices of h is a Zariski closed subset of $\mathbb{P}^{n*}$. Now we only need to check that it is not the whole space, i.e. that not all the $c_{\underline{s}}$'s are zero. But since $(x_0\cdots x_n)^{p-1}$ occurs in $g\cdot x_0^{q_0}\cdots x_n^{q_n}$, the value of $c_{\underline{s}}$ corresponding to $x_0^{q_0}\cdots x_n^{q_n}$ is nonzero and we are through. □

3.2 Proposition. — *If a complete intersection $Y = Z(f_1,\dots,f_c)$ is compatibly split in $\mathbb{P}^n$, then it can also be split by a splitting divisor. In fact it can be split by a splitting divisor of the form $Z(f_1)+\cdots Z(f_c)+H_d+\cdots+H_n$, where $d=\sum deg(f_i)$ and the H_i's are hyperplanes.*

Proof. — By Proposition 2.6, we can find a splitting of form $(f_1\cdots f_c)^{p-1}g$, such that g is homogeneous of degree $(n+1-d)(p-1)$, and such that $(x_0\cdots x_n)^{p-1}$ occurs. Without loss of generality we can assume that g is a monomial in $x_0,\dots,x_n$. Now successive use of the lemma replaces g with $(p-1)$'th powers of linear factors and establishes the result. □

So all split hypersurfaces are subvarieties in splitting divisors, and all split complete intersections arise as intersections of components of splitting divisors.

References

[M-R] V.B. Mehta and A. Ramanathan: Frobenius splitting and cohomology vanishing for Schubert varieties, *Ann. of Math.* **122** (1985), 27–40.

[R] A. Ramanathan: Equations defining Schubert varieties and Frobenius splitting of diagonals, *Publ. Math. I.H.E.S.* **65** (1987), 61–90.

[M-Sr] V.B. Mehta and V. Srinivas: Normal F-pure surface singularities, *J. Algebra* **143** (1991), 130–143.

[L-M-P] V. Lakshmibai, V.B. Mehta and A.J. Parameswaran: Frobenius splittings and blow-ups, *preprint, T.I.F.R. Bombay, 1995.*

[F] R. Fedder: F-purity and rational singularity, *Trans. Amer. Math. Soc.* **278** (1983), 461–480.

[H] R. Hartshorne: *Algebraic Geometry*, GTM 52, Springer-Verlag, New York, 1977.