Frobenius splitting of complete intersections in $\mathbb{P}^n$

Joachim Kock

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Til familie og venner

Algebra beskæftiger sig med symboler og ligninger og er kendetegnet ved en meget høj grad af formalisme. Som komplement står geometrien, der er kendetegnet ved høj grad af intuition: vi kan tegne og "se" tingene for os. Algebraisk geometri er læren om forholdet mellem de to discipliner. Målet er at give algebraiske udtryk og formel sikkerhed om ting vi kan se, og omvendt give algebraen liv ved at give geometriske tolkninger af dens resultater.

De algebraiske objekter er essentielt polynomier, og de tilhørende geometriske objekter er *nulpunktsmængderne* for dem — det kunne være kurver eller flader.

Mit speciale handler om algebraisk geometri og giver gode eksempler på dette samspil. For eksempel finder jeg først et rent algebraisk kriterium for Frobenius-splitning (det specialet handler om), nemlig corollar 2.2.4 på side 20 der udtaler sig om polynomier f. Med det i hånden kan man afgøre hvorvidt visse kurver er Frobenius-split, forudsat altså man kender deres algebraiske beskrivelse: deres polynomium.

Derefter udleder jeg geometriske betingelser for egenskaben. Det vil sige, *uden* at kende de algebraiske beskrivelser af kurverne jeg har tegnet på side viii, men blot ved at se på dem, kan man afgøre at de tre første er Frobenius-split hvorimod de tre næste ikke er. De geometriske kriterier kan derimod ikke umiddelbart udtale sig om "et abstrakt algebraisk objekt" — ikke før vi har fået det tegnet, eller på anden måde fået hold på dets geometriske egenskaber.

JOACHIM

Forord

— Kan du splitte en elliptisk kurve? spurgte min specialevejleder Niels mig en dag, uden nogen egentlig anledning. Det vi ellers snakkede om på den tid, og det som det var meningen mit speciale skulle handle om, var Frobenius-splitning af Schubert-varieteter. Spørgsmålet om elliptiske kurver er faktisk lidt tricky, for det afhænger af primtallet p (se Example 2.2.9), og jeg sad i over en uge og kunne ikke finde ud af det. Det irriterede mig meget at det skulle være så svært, for det var jo "bare et polynomium". Jeg måtte erkende at jeg manglede en grundlæggende fornemmelse af hvad Frobenius splitning "egentlig er", og gav mig til at studere først keglesnit og så kubiske kurver. Lige så langsomt begyndte jeg at få lidt intuition og jeg var meget stolt den dag jeg gik op på Niels' kontor og tegnede seks kurver (se illustrationen side viii), og fortalte ham hvilke der var split og hvilke der ikke var.

Herefter tror jeg hverken Niels eller jeg var i tvivl om, at det var noget i den retning mit speciale skulle handle om. Og det gør det så, i den forstand at næsten alle resultaterne i det har deres oprindelse i de seks kurver.

Fra plane kurver blev det til hyperflader, og da jeg begyndte at vise et par resultater om dem, foreslog Niels mig at skrive en artikel. Det var sidste sommer, og kort tid efter tog jeg til Recife i Brasilien, hvor jeg så sad hele efteråret og puslede med hyperflader med henblik på en artikel. Jeg fik et par gode input fra mine vejledere dernede, og det gav også pote da jeg begyndte at køre en hel masse polynomier igennem Maple.

Henimod jul da jeg kom hjem til Århus, havde jeg opgraderet til complete intersections og fået formuleret mine resultater så klart at artiklen var ved at være klar til publicering. Den er her i begyndelsen af januar kommet ud som et preprint "Frobenius splitting of complete intersections in $\mathbb{P}^n$" og er netop blevet sendt til *Communications in Algebra.*

Specialet her hedder det samme, og Kapitel 2 er næsten ordret identisk med artiklen. Kapitel 1 er en mere udførlig fremstilling af de nødvendige forudsætninger end jeg kunne give i artiklen. Først og fremmest har jeg inkluderet en redegørelse for "the fundamental isomorphism", og iøvrigt indføjet et par udeladte beviser. Endvidere har jeg indføjet et appendix om Cartier-operatoren, der er et vigtigt aspekt ved dualitets-teorien.

Jeg vil gerne her benytte lejligheden til at udtrykke min varmeste tak først og fremmest til min vejleder Niels Lauritzen, for hans tålmodighed og hans entusiasme og energi, og dernæst til Departamento de Matemática ved Universidade Federal de Pernambuco, Brasil, for deres store gæstfrihed og støtte iøvrigt.

Og endelig vil jeg gerne takke mit gamle Mat1-hold: Benny, David, Frank, Henning, Jan, Jeppe, Lars, Peter, and Søren, hvis gode selskab på Første Del motiverede mig til at overholde studieplanen, istedet for at blive hængende og fordybe mig for meget i matematikken.

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Matematisk Institut
Aarhus Universitet

Joachim Kock

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Introduction

The notion of Frobenius splitting was introduced in 1985 by Mehta and Ramanathan in their paper [M-R]: For a variety X of characteristic p, the absolute Frobenius morphism $F : X \to X$ is the identity on points, and $F^\sharp : \mathcal{O}_X \to F_*\mathcal{O}_X$ is locally the p'th power map $a \mapsto a^p$. Now X is said to be *Frobenius split* if $F^\sharp : \mathcal{O}_X \to F_*\mathcal{O}_X$ is a split injection. The motivation for this definition was that Frobenius splitting implies the vanishing theorem for ample line bundles, and for smooth varieties also the Kodaira vanishing theorem.

In a "fundamental isomorphism" established via the Cartier operator, the splittings were related to the global sections of the anti-canonical sheaf, and criteria for splitting in terms of these sections were deduced. Applying these criteria, Mehta and Ramanathan showed that Schubert varieties in characteristic p are Frobenius split, and by a semi-continuity argument showed that Schubert varieties in characteristic zero have cohomology vanishing for ample line bundles.

A series of papers refined these techniques and established the projective normality of Schubert varieties in any embedding given by an ample line bundle [R-R], arithmetic Cohen-Macauley-ness of Schubert varieties [R:arit], and as a culmination of the work on Schubert varieties, Ramanathan [R] showed that Schubert varieties are cut out by quadrics and linear forms in any embedding given by an ample line bundle.

Later the techniques of Frobenius splitting were applied to other classes of varieties related to homogeneous spaces, but to my knowledge very little has been said about Frobenius splitting of varieties not arising in connection with group actions.

The present thesis takes a step to fill in this gap. Taking a classical point of view and looking at varieties cut out by polynomials, it turns out that quite a lot can be said about complete intersections, and that simple geometric properties are related to the notion of Frobenius splitting.

The organization of the exposition is as follows: In Chapter 1 we review the general theory of Frobenius splitting following [M-R] and [R], and establish the results needed in Chapter 2.

Most attention is paid to the fundamental isomorphism. Since we need it for not necessarily smooth varieties, we cannot use the original proof of [M-R] based on the Cartier operator, which is only defined for smooth varieties. Instead we launch the machinery of duality for a finite morphism, from which the fundamental isomorphism follows quite easily. Since this theory is not very well documented in the literature, we account for it here in some detail.

The Cartier operator point of view is however a useful complement to the duality proof since it allows for more explicit computations. In an appendix we give a brief account for the Cartier operator and its connections to Frobenius splitting.

In Chapter 2 we leave these abstract theories behind, and come to our main object of study, complete intersections in $\mathbb{P}^n$. First the fundamental isomorphism is stated explicitly for $\mathbb{P}^n$ in terms of polynomials, and an elementary but quite tedious proof is given. This is one cornerstone of our exposition, since it allows for explicit computations.

The other cornerstone is the result that all splittings of a complete intersection come from splittings of $\mathbb{P}^n$ (Proposition 2.2.3). Combining these two results, the stage is set for a series of results of geometric nature.

For hypersurfaces we get the following criterion: $Z(f) \subset \mathbb{P}^n$ of degree $n+1$ is split if and only if the coefficient of $(x_0 \cdots x_n)^{p-1}$ in f^{p-1} is nonzero. Such a $Z(f)$ is called a *splitting divisor*, and it is shown that every split complete intersection arises as intersection of components of a splitting divisor (Proposition 2.2.7).

Direct computation readily gives that the three plane curves

are split, and that the following three are not:

Three central results of this paper state the exact content of this picture and make generalizations to higher dimensions: c components of a splitting divisor intersect in codimension c (Proposition 2.3.1). A divisor $D + L \subset \mathbb{P}^n$ with L linear is splitting if and only if $D \mid L \subset L = \mathbb{P}^{n-1}$ is splitting (Lemma 2.3.3). A codimension 1 singularity of a split hypersurface is analytically reducible (Proposition 2.4.3).

To the best of my knowledge, all the results on complete intersections are new and have been submitted to *Communications in Algebra*.

General conventions

Throughout, k will be a fixed algebraically closed field of characteristic $p > 0$, and by a variety we will understand a equidimensional reduced quasi-projective scheme of finite type over k (but not necessarily irreducible). We do not require our morphisms of varieties to be k-morphisms.

Chapter 1

General theory of Frobenius splitting

1.1 Definition of Frobenius splitting

Let X be a k-scheme. Let $F : X \to X$ be the absolute Frobenius morphism. It is the identity on the underlying set of points, and on sheaf level it is $\mathcal{O}_X \to F_*\mathcal{O}_X$, given locally as the Frobenius ring endomorphism $a \mapsto a^p$. By abuse of notation, we also use the symbol F to denote this homomorphism of sheaves. Note that F is not a k-homomorphism. When X is of finite type over k, then F is a finite morphism. For any $\mathcal{O}_X$-module M, the direct image F_*M is equal to M as an abelian group, but has $\mathcal{O}_X$-module structure given by $a.m = a^p m$ for all $a \in \mathcal{O}_X$ and $m \in M$. It follows that the functor F_* from $\mathfrak{Mod}(X)$ to itself is exact and faithful.

We will also need F^*. For an $\mathcal{O}_X$-module M, by definition $F^*M = M \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X{}'$ where the prime denotes that scalars must be applied on the right factor by usual ring multiplication $a.(m \otimes s) = m \otimes as$ for $a \in \mathcal{O}_X$ (and *not* $m \otimes a^p s$).

Definition. — A k-scheme X is called *Frobenius split* (or just *split*) if the Frobenius homomorphism $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ has a splitting, i.e. an $\mathcal{O}_X$-module homomorphism $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_X$ is the identity map.

We will refer to $\mathcal{O}_X$-module homomorphisms $F_*\mathcal{O}_X \to \mathcal{O}_X$ as *pre-splittings* of X. A pre-splitting $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ is thus a splitting if and only if

$\varphi(1) = 1$. Note that this last condition can be checked in the local ring at any point (provided X is connected).

Remark. — A necessary condition for a scheme to be Frobenius split is that it is reduced: otherwise the Frobenius homomorphism fails to be injective.

The main tools for determining whether a variety is Frobenius split are the notion of *compatible splitting* that relates splitting to an ambient variety, and the "fundamental isomorphism" which gives criteria of splitting in terms of sections of the anticanonical sheaf.

Let $Y \subset X$ be a closed subvariety with ideal sheaf I. Since Frobenius commutes with any morphism, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\ & & \big\downarrow F & & \big\downarrow F & & \big\downarrow \overline{F} & & \\ 0 & \longrightarrow & F_*I & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & F_*\mathcal{O}_Y & \longrightarrow & 0 \end{array}$$

where the middle F is Frobenius for X, the left hand F is just its restriction to I, and $\overline{F}$ is induced by the two others and coincides with Frobenius for Y.

Definition. — Let $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ be a pre-splitting of X. If $\varphi(F_*I) \subset I$, then a pre-splitting $\overline{\varphi} : F_*\mathcal{O}_Y \to \mathcal{O}_Y$ of Y will be induced, and in that case we will call the morphism φ *compatible with* Y. If furthermore φ is a splitting of X, then $\overline{\varphi}$ will be a splitting of Y and we will say that Y *is compatibly split in* X (Cf. [M-R], Definition 3).

The next two lemmas are basic. The first one allows all questions of compatibility to be settled locally:

1.1.1 Lemma. *(Cf. [M-R], Lemma 1.)* — *Let φ be a pre-splitting of X and let $Y \subset X$ be a closed subvariety with ideal sheaf I. Let U be an open subset in X touching all irreducible components of Y. If now the restriction of φ to U is compatible with $Y \cap U \subset U$, then φ is also compatible with the whole of Y. In other words,*

$$\varphi(F_*I)_{|U} \subset I_{|U} \Rightarrow \varphi(F_*I) \subset I.$$

Proof. — Showing that $\varphi(F_*I)$ is a subsheaf of I is a local question so we reduce to the case where X is affine $=$ Spec R. Now $U =$ Spec R_f for some $f \in R$, and the assumption on U means that f is in no associated prime of I. Now let $s \in \varphi(F_*I)$. Locally we have $\varphi(F_*I)_f \subset I_f$ so for some $n \geq 0$ we have $sf^n \in I$. Since now f is in no associated prime of I we conclude that $s \in I$. So $\varphi(F_*I) \subset I$. □

1.1.2 Lemma. *(Cf. [R], 1.9 and 1.10.) — Let φ be a pre-splitting of X.*

(i) *φ is compatible with a closed subvariety Y if and only if it is compatible with each irreducible component of Y.*

(ii) *If φ is compatible with two closed subvarieties then it is also compatible with their scheme theoretic intersection. In particular, if φ is a splitting then the scheme theoretic intersection is compatibly split and must therefore be reduced.*

Proof. — (i) Let I be the ideal sheaf of Y, and $I_1, \ldots, I_r$ the ideal sheaves of its irreducible components $Y_1, \ldots, Y_r$.

Suppose φ is compatible with Y so that $\varphi(F_*I) \subset I$: In the open subset U_i obtained by throwing away all components except Y_i we have $I_{|U_i} = I_{i|U_i}$, so φ restricted to U_i is compatible with Y_i. Now by Lemma 1.1.1 φ is compatible with Y_i globally as well.

Now suppose φ is compatible with each component: $\varphi(F_*I_i) \subset I_i$. Since Y is reduced we have $I = \cap I_i$, so we find that $\varphi(F_*I) = \varphi(F_*(\cap I_i)) \subset \cap I_i = I$.

(ii) Let I_1 and I_2 be the ideals of the two varieties. Then the ideal of the intersection is $I_1 + I_2$, and by the additivity of φ it is clear that $\varphi(F_*(I_1+I_2)) \subset I_1 + I_2$. □

1.1.3 Standard argument. — When checking whether an X-pre-splitting φ is compatible with an irreducible subvariety Y we can use Lemma 1.1.1 to do it locally — actually in the local ring of a point x which we can choose to be smooth in Y, i.e. it is enough to show $\varphi(F_*I_x) \subset I_x$ in $\mathcal{O}_{X,x}$. And by faithful flatness of completion, it is enough to check this in the completion, so if X is smooth of dimension n and Y has codimension c, we can choose suitable coordinates and reduce to checking that $\varphi(F_*(x_1, \ldots, x_c)) \subset (x_1, \ldots, x_c)$ in $k[[x_1, \ldots, x_n]]$.

1.2 Duality for the Frobenius morphism

The fundamental isomorphism on which all results on Frobenius splitting are based (Theorem 1.2.6 below) states that pre-splittings of X correspond to global sections of the sheaf ω_X^{1-p}. The theorem is a simple consequence of the duality theory for the finite morphism F, and since this theory is poorly documented in the literature we give it here. It is mostly distilled from [RD].

1.2.1. Let $f : X \to Y$ be an affine morphism of schemes. Then $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras and $X \simeq \mathbf{Spec}(f_*\mathcal{O}_X)$. This gives an equivalence of categories from $\mathfrak{Qco}(X)$ to $\mathfrak{Qco}(f_*\mathcal{O}_X)$, where the last is the category of quasi-coherent $\mathcal{O}_Y$-modules having additional structure of $f_*\mathcal{O}_X$-module. Given such a sheaf N, the corresponding $\mathcal{O}_X$-module is given by gluing together the sheaves $\widetilde{N(V)}$ which are $f_*\mathcal{O}_X(V)$-modules and thus $\mathcal{O}_X(f^{-1}(V))$-modules, and we will denote it $\widetilde{N}$. The usual f_* is also a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Qco}(f_*\mathcal{O}_X)$ and considered as such it is the inverse to $\widetilde{\ }$.

Definition. — Given quasi-coherent $\mathcal{O}_Y$-module N then $\mathcal{H}om_Y(f_*\mathcal{O}_X, N)$ is naturally a $f_*\mathcal{O}_X$-module. Let now $f^!N$ denote the corresponding $\mathcal{O}_X$-module $\mathcal{H}om_Y(f_*\mathcal{O}_X, N)^{\sim}$.

We will show that $f^!$ is right adjoint to f_*.

1.2.2. By definition and since f_* and $\widetilde{\ }$ are inverses to each other, we get for any quasi-coherent $\mathcal{O}_Y$-module N an isomorphism

$$f_*f^!N \stackrel{\sim}{\to} \mathcal{H}om_Y(f_*\mathcal{O}_X, N).$$

Composing with the evaluation-at-1 morphism $\mathcal{H}om_Y(f_*\mathcal{O}_X, N) \to N$ we get a natural morphism

$$\begin{aligned} Tr_f : f_*f^!N &\longrightarrow N \\ \sigma &\longmapsto \sigma(1) \end{aligned}$$

which we will call the *trace map*.

1.2.3. For any two $\mathcal{O}_X$-modules M and M' we have a natural morphism of $\mathcal{O}_Y$-modules

$$f_*\mathcal{H}om_X(M, M') \to \mathcal{H}om_Y(f_*M, f_*M').$$

Setting $M' = f^! N$, where N is a quasi-coherent $\mathcal{O}_Y$-module, and composing with the trace map in the second variable, we get for any $\mathcal{O}_X$-module M a natural morphism of $\mathcal{O}_Y$-modules

$$\begin{aligned} \theta_f : f_* \mathcal{H}om_X(M, f^! N) &\longrightarrow \mathcal{H}om_Y(f_* M, N) \\ \sigma &\longmapsto [\, m \mapsto \sigma(m)(1)] \end{aligned}$$

1.2.4 Proposition. *Duality for an affine morphism. — The morphism θ_f is an isomorphism.*

Proof. — The question is local on Y so we reduce to the case where Y is affine, say $Y = \mathrm{Spec}\ A$. Since f is an affine morphism, X is also affine, say $X = \mathrm{Spec}\ B$, and $f : X \to Y$ corresponds to a ring homomorphism $\varphi : A \to B$. Now the statement of the proposition amounts to the well-known fact that

$$\varphi^* \mathrm{Hom}_B(M, \mathrm{Hom}_A(\varphi^* B, N)) \xrightarrow{\sim} \mathrm{Hom}_A(\varphi^* M, N)$$

is an isomorphism, where φ^* denotes restriction of scalars along φ. (This is just a way of saying that $\mathrm{Hom}_A(\varphi^* B, -)$ is right adjoint to φ^*.) $\square$

1.2.5 Proposition. *Residue isomorphism for a finite morphism. — Let $f : X \to Y$ be a finite morphism of projective Gorenstein varieties of dimension n over k. Let (ω_X, tr_X) and (ω_Y, tr_Y) be their respective dualizing sheaves with fixed trace maps. Then there is an $\mathcal{O}_X$-module homomorphism*

$$tr_f : f_* \omega_X \to \omega_Y$$

canonically given from the trace maps tr_X and tr_Y.

The $\mathcal{O}_X$-module homomorphism

$$res_f : \omega_X \xrightarrow{\sim} f^! \omega_Y$$

corresponding to tr_f under the duality of Proposition 1.2.4 is an isomorphism.

Remark. — This proposition is Exercise III.7.2 in [H], and also, in a more general setup, Proposition III.8.4 of [RD], but one notes however that these to references require f to be a k-morphism.

Since we are going to use the proposition on the absolute Frobenius morphism which is not a k-morphism, we can not make this requirement.

Instead we make the following slightly weaker requirement on our morphism f, which is easily seen to be satisfied by the Frobenius and which suffices to make the proof work.

Extra condition on f. — We require that there is an automorphism of k giving a morphism $f_k : \text{Spec } k \to \text{Spec } k$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g_X"'] & Y \arrow[d, "g_Y"] \\ \text{Spec } k \arrow[r, "f_k"'] & \text{Spec } k \end{tikzcd}$$

commutes, where g_X and g_Y are the structure morphisms.

With this condition we have for any quasi-coherent $\mathcal{O}_X$-module M an isomorphism of k-modules

$$H^i(Y, f_*M) \simeq f_{k*}H^i(X, M). \tag{1.1}$$

Note that if f_k is the identity (which is equivalent to saying that f is a k-morphism), then we recover the usual k-linear map $H^i(Y, f_*M) \simeq H^i(X, M)$, valid when f is affine.

The map (1.1) is established as the composition of the following five k-linear maps:

$$\begin{array}{r} H^i(Y, f_*M) \simeq R^i g_{Y*}(R^0 f_*M) \simeq R^i(g_{Y*}f_*)(M) \\ \big| \simeq \quad \\ f_{k*}H^i(X, M) \simeq R^0 f_{k*}R^i g_{X*}(M) \simeq R^i(f_{k*}g_{X*})(M) \end{array}$$

where the second and the second-to-last $\simeq$'s are valid since f and f_k are affine. (It is a degenerate case of the Leray spectral sequence.)

Proof. — First we construct tr_f. The natural pairing (Yoneda pairing)

$$\mathcal{H}om_Y(f_*\omega_X, \omega_Y) \times H^n(Y, f_*\omega_X) \to H^n(Y, \omega_Y)$$

is a perfect pairing (by Serre duality for Y), which is equivalent to saying that we have an isomorphism

$$\text{Hom}_Y(f_*\omega_X, \omega_Y) \simeq \text{Hom}_k(H^n(Y, f_*\omega_X), H^n(Y, \omega_Y)), \tag{1.2}$$

so to give a canonical element on the left hand side is equivalent to give a canonical element on the right. Now by (1.1) we have $H^n(Y, f_*\omega_X) \simeq f_{k*}H^n(X, \omega_X)$, and from $f_{k*}H^n(X, \omega_X)$ to $H^n(Y, \omega_Y)$ the best map we can come up with is the "ratio" of the two trace maps, i.e. the unique k-linear map that makes the diagram

$$\begin{tikzcd}
f_{k*}H^n(X, \omega_X) \arrow[r] \arrow[d, "tr_X"'] & H^n(Y, \omega_Y) \arrow[d, "tr_Y"] \\
f_*k & k \arrow[l, "f_k"]
\end{tikzcd}$$

commute. Via (1.2) this gives our morphism $tr_f : f_*\omega_X \to \omega_Y$.

For the second assertion ,set $res_f = \theta_f^{-1}(tr_f) : \omega_X \to f^!\omega_Y$.

To show that res_f is an isomorphism we first show that $f^!\omega_Y$ is a dualizing sheaf for X. Let M be any coherent $\mathcal{O}_X$-module and consider the diagram

$$\begin{tikzcd}
f_{k*}\mathrm{Hom}_X(M, f^!\omega_Y) \times f_{k*}H^n(X, M) \arrow[r] \arrow[dd, "\theta_f \times id"'] & f_{k*}H^n(X, f^!\omega_Y) \arrow[d, "id"] \\
 & H^n(Y, f_*f^!\omega_Y) \arrow[d, "Tr_f"] \\
\mathrm{Hom}_Y(f_*M, \omega_Y) \times H^n(Y, f_*M) \arrow[r] & H^n(Y, \omega_Y) \arrow[d, "tr_Y"] \\
 & k \arrow[d, "f_k"] \\
 & f_{k*}k
\end{tikzcd}$$

where the rows are the natural pairings, and the vertical id's are the k-linear maps from (1.1). The diagram is commutative since θ_f is defined via Tr_f.

Since f is finite, the direct image f_*M is also coherent, and so by Serre duality the bottom row, followed by tr_Y, is a duality pairing. Thus, the top row followed by all the vertical maps on the right

$$f_{k*}\mathrm{Hom}_X(M, f^!\omega_Y) \times f_{k*}H^n(X, M) \to f_{k*}H^n(X, f^!\omega_Y) \to f_{k*}k$$

must also be a duality pairing. Now we can remove the f_{k*}-twist on the four modules, since f_k is just an automorphism (and therefore f_{k*} is full and faithful), and we see that $f^!\omega_Y$ together with the trace map $tr_Y \circ Tr_f \circ id : H^n(X, f^!\omega_Y) \to k$ has the characterizing property of a dualizing sheaf for X. Now by uniqueness of dualizing sheaves it must be isomorphic to ω_X.

And so, since res_f is a non-zero map between two isomorphic line bundles, it must itself be an isomorphism. □

Remark. — We will call tr_f a trace morphism, since via the isomorphism res_f it is a special case of the general trace map Tr_f of No. 1.2.2, and is given as

$$f_*\omega_X \xrightarrow{res_f} f_*f^!\omega_Y \xrightarrow{Tr_f} \omega_Y.$$

Now we return to the Frobenius $F : X \to X$, which is finite for all varieties.

1.2.6 Theorem. *("The Fundamental Isomorphism") (Cf. [M-R], Proposition 5.) — Let X be a Gorenstein variety and let ω_X be the dualizing sheaf of X. Then there is a natural isomorphism of $\mathcal{O}_X$-modules*

$$F_*\omega_X^{1-p} \xrightarrow{\sim} \mathcal{H}om(F_*\mathcal{O}_X, \mathcal{O}_X).$$

Proof. — If X is not projective we will let ω_X denote the sheaf induced from the dualizing sheaf on the projective closure. So it suffices to prove the theorem for X projective. Now since F is finite, all the above applies.

Taking F_* on the isomorphism res_F^{-1} and tensoring with ω_Y^{-1} we get an isomorphism

$$\mathcal{H}om(F_*\mathcal{O}_X, \mathcal{O}_X) \simeq \omega_X^{-1} \otimes F_*\omega_X$$

and by the projection formula this last module is isomorphic to $F_*(F^*\omega_X^{-1} \otimes \omega_X)$. Now the result follows from the observation that for any invertible sheaf L we have $F^*L \simeq L^{\otimes p}$. □

1.2.7. — Let us walk back through the isomorphisms to see what a global section $\sigma \in F_*\omega_X^{1-p}$ looks like when considered a morphism $F_*\mathcal{O}_X \to \mathcal{O}_X$: On some open set U where ω is a generator for ω_X we can write $\sigma = \omega^{-p} \otimes a\omega$, for some $a \in \mathcal{O}_X(U)$. Back in $F_*(F^*\omega_X^{-1} \otimes \omega_X) \simeq \omega_X^{-1} \otimes F_*\omega_X$ we get $\omega^{-1} \otimes a\omega$ and taking res_F we get $\omega^{-1} \otimes res_F(a\omega) \in \omega_X^{-1} \otimes \mathcal{H}om(F_*\mathcal{O}_X, \omega_X)$.

Now $res_F(a\omega)$ expressed in terms of tr_F is the morphism $s \mapsto tr_f(sa\omega)$ (where $s \in F_*\mathcal{O}_X$) so we end up with the morphism

$$\begin{array}{rcl} F_*\mathcal{O}_X & \to & \mathcal{O}_X \\ s & \mapsto & tr_F(sa\omega)\omega^{-1}. \end{array}$$

Now what we need is knowledge of the trace map tr_F, but in general it is not easily computable.

However, when X is a smooth variety, the *Cartier operator* C is a nonzero element in $\mathrm{Hom}(F_*\omega_X, \omega_X)$, and since this space is only one-dimensional, we can take C as our trace map.

In the Appendix to this Chapter we give a brief account of the Cartier operator, and show how it combined with 1.2.7 gives a local criterion for Frobenius splitting. These techniques are central in [M-R], but will not be further explored in the present exposition. In the next chapter we specialize to $\mathbb{P}^n$ and give a description from scratch of the space of pre-splittings.

Appendix: The Cartier operator

Here we give a brief account of the Cartier operator and its connections to Frobenius splitting.

Let X be a variety of dimension n, let F be the Frobenius morphism, and let $d : \mathcal{O}_X \to \Omega_X^1$ be the universal derivation. Applying the functor F_* to these two modules we get an $\mathcal{O}_X$-linear map $d : F_*\mathcal{O}_X \to F_*\Omega_X^1$; indeed $d(a.f) = d(a^p f) = pa^{p-1}da + a^p df = a.df$ for $a \in \mathcal{O}_X$ and $f \in F_*\mathcal{O}_X$.

This makes the twisted deRham complex

$$F_*\mathcal{O}_X \to F_*\Omega_X^1 \to \ldots \to F_*\Omega_X^n \to 0$$

a complex of coherent $\mathcal{O}_X$-modules and $\mathcal{O}_X$-module homomorphisms.

Let $Z^i \subset F_*\Omega_X^i$ be the sheaf of closed i-forms, let $B^i \subset F_*\Omega_X^i$ be the sheaf of exact i-forms, and let $\mathcal{H}^i(F_*\Omega_X^\bullet)$ be the cohomology sheaf Z^i/B^i. So for an open affine $U \subset X$, the space $H^0(U, \mathcal{H}^i(F_*\Omega_X^\bullet))$ consists of all closed differential i-forms on U modulo the exact ones. For a section $\omega \in Z_X^i$ let $\overline{\omega}$ denote its class in $\mathcal{H}^i(F_*\Omega_X^\bullet)$.

Now the paring

$$\mathcal{H}^i(F_*\Omega_X^\bullet) \times \mathcal{H}^j(F_*\Omega_X^\bullet) \xrightarrow{\wedge} \mathcal{H}^{i+j}(F_*\Omega_X^\bullet)$$

induced by the exterior product of differential forms is $\mathcal{O}_X$-bilinear and makes $\bigoplus \mathcal{H}^i(F_*\Omega_X^\bullet)$ a graded differential $\mathcal{O}_X$-algebra.

Theorem. *(Cartier) — There is a unique homomorphism of graded $\mathcal{O}_X$-algebras*

$$c^{-1} : \bigoplus \Omega_X^i \longrightarrow \bigoplus \mathcal{H}^i(F_*\Omega_X^\bullet)$$

such that $c^{-1}(df) = \overline{f^{p-1}df}$, for $f \in \omega_X^0 = \mathcal{O}_X$.

If X is smooth, then c^{-1} is an isomorphism.

Remark. — The symbol c^{-1} for this map is historically motivated, and is not meant to imply that it always has an inverse. In the smooth case, Cartier originally constructed c, the inverse to c^{-1}. The following construction of c^{-1} is due to Deligne (cf. [K]).

Proof. — That c^{-1} is a homomorphism of graded $\mathcal{O}_X$- algebras means that it is a homomorphism of $\mathcal{O}_X$-modules, and that $c^{-1}(1) = 1$ and $c^{-1}(\sigma \wedge \rho) = c^{-1}(\sigma) \wedge c^{-1}(\rho)$.

Now it suffices to construct the map in degree 1, and then let the rule $c^{-1}(\sigma \wedge \rho) = c^{-1}(\sigma) \wedge c^{-1}(\rho)$ define the map in higher degrees. In degree 0 the map is automatically defined just by the rule $c^{-1}(1) = 1$ and linearity: the closed sections of $F_*\mathcal{O}_X$ are the p'th power sections $\mathcal{O}_X^p$ and c^{-1} is just the Frobenius homomorphism $F : \mathcal{O}_X \to \mathcal{O}_X^p \subset F_*\mathcal{O}_X$.

By the universality of the derivation d we have a natural isomorphism $\mathrm{Der}(\mathcal{O}_X, \mathcal{H}^1(F_*\Omega_X^\bullet)) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{H}^1(F_*\Omega_X^\bullet))$, so giving c^{-1} is equivalent to giving a derivation $\delta : \mathcal{O}_X \to \mathcal{H}^1(F_*\Omega_X^\bullet)$ with $\delta(a) = \overline{a^{p-1}da}$. This last condition already defines the map δ; it remains to show that it is a derivation.

It does satisfy the product rule:

$$\delta(xy) = \overline{(xy)^{p-1}d(xy)} = \overline{x^{p-1}y^p dx + x^p y^{p-1} dy} = y\delta x + x\delta(y)$$

The additivity is a bit more tricky: For any two sections $x, y \in \mathcal{O}_X$ we must show that $(x+y)^{p-1}d(x+y) - x^{p-1}dx - y^{p-1}dy$ is exact.

Consider the polynomial $\Phi(X,Y) = \left(\frac{(X+Y)^p - X^p - Y^p}{p}\right)$ in the two symbols X and Y: note that is has *integer* coefficients, and that its formal derivative is $d(\Phi(X,Y)) = (X+Y)^{p-1}d(X+Y) - X^{p-1}dX - Y^{p-1}dY$.

Interpreting this in $\mathcal{O}_X$ by substituting x for X and y for Y, we see that the section $\Phi(x,y) \in \mathcal{O}_X$ is the primitive we are looking for. So δ is a derivation and c^{-1} is constructed.

Now suppose X is smooth: To check that c^{-1} is an isomorphism it is enough to check that in the completion at each closed point the induced map is an isomorphism. So we reduce to the case $X = \mathrm{Spec}\, k[[x_1, \ldots, x_n]]$, and the checking becomes quite straightforward. Just to give the idea, we work through the top class $\omega_X = \Omega_X^n$:

Let $\mathbf{dx}$ denote its generator $dx_1 \wedge \ldots \wedge dx_n$. Now the p^n elements

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n}\, \mathbf{dx}, \qquad \alpha_i = 0, \ldots, p-1, \tag{1.3}$$

constitute a $\mathcal{O}_X$-basis for $Z_X^n = F_*\Omega_X^n$, but they are all exact except the element

$$(x_1 \cdots x_n)^{p-1}\, \mathbf{dx},$$

so $\overline{(x_1 \cdots x_n)^{p-1}\, \mathbf{dx}} = c^{-1}(\mathbf{dx})$ is a generator for $\mathcal{H}^n(F_*\Omega_X^\bullet)$, and so c^{-1} is an isomorphism of $k[[x_1, \ldots, x_n]]$-modules. □

Definition. — For a smooth variety the *Cartier operator* $C : Z_X^i \to \Omega_X^i$ is defined as the composition $Z_X^i \longrightarrow \mathcal{H}^i(F_*\Omega^\bullet) \xrightarrow{c} \Omega_X^i$, where c denotes the inverse to the c^{-1} from the theorem above.

Corollary. — *The Cartier operator is the unique $\mathcal{O}_X$-module homomorphism $C : Z_X^i \to \Omega_X^i$ satisfying the three rules*

(i) $C(f^{p-1}df) = df$
(ii) $C(\sigma \wedge \rho) = C(\sigma) \wedge C(\rho)$
(iii) $C(d\sigma) = 0.$

Remark. — Rule (iii) accounts for the uniqueness: If C' is another $\mathcal{O}_X$-homomorphism satisfying the three rules, then by (iii) it factors through $\mathcal{H}^i(F_*\Omega^\bullet)$ and then by the uniqueness assertion of the theorem it must be equal to C.

Note that Z_X^i is defined as a submodule in $F_*\Omega_X^i$, so it comes with twisted $\mathcal{O}_X$-structure, and the linearity therefore means $C(f^p\sigma) = C(f.\sigma) = fC(\sigma)$.

For the purposes of Frobenius splitting we are mostly interested in the Cartier operator on the top classes, $C : F_*\omega_X \to \omega_X$. Since this map is a nonzero element of $\operatorname{Hom}(F_*\omega_X, \omega_X)$ and this space is just a 1-dimensional vector space, we can use it as our trace map tr_F of Section 1.2.

Now the fundamental isomorphism (Theorem 1.2.6) can be described quite explicit in local coordinates: In the completion at a closed point P (continuing the notation from the proof above), $F_*\omega_X$ is the free $k[[x_1, \ldots, x_n]]$-module generated by the p^n sections in (1.3), and any section can be written locally as a linear combination

$$f_0 \,.\, (x_1 \cdots x_n)^{p-1}\, \mathbf{dx} \;+\; \sum_\alpha f_\alpha \,.\, (x_1^{\alpha_1} \cdots x_n^{\alpha_n})\, \mathbf{dx}$$

where f_0 and f_α are power series and α runs through all "exact" n-tuples $(\alpha_1, \ldots, \alpha_n)$, i.e. all the tuples with $0 \leq \alpha_i \leq p-1$ except the tuple $(p-1, \ldots, p-1)$, cf. (1.3).

Now the Cartier operator is described as

$$C\left(f_0 \,.\, (x_1 \cdots x_n)^{p-1}\, \mathbf{dx} \;+\; \sum_\alpha f_\alpha \,.\, (x_1^{\alpha_1} \cdots x_n^{\alpha_n})\, \mathbf{dx}\right) = f_0\, \mathbf{dx}$$

With all these notational preparations it is easy to give the following local criterion for Frobenius splitting:

Proposition. *(Cf. [M-R], Proposition 6.) — Let X be a smooth variety. Let σ be a global section of $F_*\omega_X^{1-p}$ and let $\varphi_\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ be the homomorphism corresponding to σ under the fundamental isomorphism 1.2.6. Let $P \in X$ be any point.*

Now with the notation as above, φ_σ is a splitting if and only if in the local description of σ at P the coefficient f_0 of $(x_1 \cdots x_n)^{p-1}/(\mathbf{dx})^{p-1}$ is equal to 1.

Proof. — The pre-splitting φ_σ is a splitting if and only if $\varphi_\sigma(1) = 1$ locally at P. Now by 1.2.7 we have

$$\varphi_\sigma(1) = C\left(f_0 \,.\, (x_1 \cdots x_n)^{p-1}\,\mathbf{dx} + \sum_\alpha f_\alpha \,.\, (x_1^{\alpha_1} \cdots x_n^{\alpha_n})\,\mathbf{dx}\right) / \,\mathbf{dx} = f_0,$$

whence the result. $\square$

Chapter 2

Frobenius splitting of complete intersections in $\mathbb{P}^n$

We now come to our main object of study, complete intersections in $\mathbb{P}^n$, where we are going to get our hands dirty with some explicit calculations. Our method is first to get hold of all splittings of $\mathbb{P}^n$, and then show that all splittings of complete intersections are induced from them by compatibility.

2.1 The fundamental isomorphism for $\mathbb{P}^n$

We start out by restating the fundamental isomorphism for $\mathbb{P}^n$ in terms of polynomials, and give a new elementary proof. The proof is somewhat tedious but gives new insight into the practical problems associated with Frobenius splitting which are not made explicit by the duality proof.

First we need some notation:

Definition. — Let $C : F_*k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ be the $k[x_1, \ldots, x_n]$-module homomorphism defined as $(x_1 \cdots x_n)^{p-1} \mapsto 1$, and $m \mapsto 0$ for all other monomials m without a p'th power factor. Monomials with a p'th power factor are dealt with by linearity: $C(g^p f) = gC(f)$.

For a polynomial $f(x_1, \ldots, x_n)$ of degree $\leq (n+1)(p-1)$ we call the term $a(x_1 \cdots x_n)^{p-1}$ the *critical term*, and then $C(f) = \sqrt[p]{a}$. ($a \in k$.)

Remark. — Since for the affine space $\mathbb{A}^n$ the canonical sheaf $\omega_{\mathbb{A}^n}$ is isomorphic to $k[x_1, \ldots, x_n]$, this C is of course just the Cartier operator in disguise.

In the following, for each $i = 0, \ldots, n$, let $U_i \subset \mathbb{P}^n$ be the standard open set $\{x_i \neq 0\}$.

2.1.1 Proposition. — *There is an isomorphism of k-vector spaces*

$$\Phi : H^0(\mathbb{P}^n, F_*\mathcal{O}_{\mathbb{P}^n}((n+1)(p-1)) \xrightarrow{\sim} \mathrm{Hom}(F_*\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})$$

where a form f of degree $(n+1)(p-1)$ is sent to the homomorphism φ_f given locally (say on U_0) as

$$\varphi_f(m) = C(fm),$$

where both f and m are dehomogenized with respect to x_0 in the usual way.

Proof. — As Φ is described, it is at first only a homomorphism to $\mathrm{Hom}(F_*\mathcal{O}_{U_0}, \mathcal{O}_{U_0})$. We need to check that φ_f does extend to a global homomorphism. In *Step 1* we describe the vector space $\mathrm{Hom}(F_*\mathcal{O}_{U_0}, \mathcal{O}_{U_0})$ and find a necessary and sufficient condition on sections herein to extend to global homomorphisms. In *Step 2* we show that our φ_f's are indeed extendable, so Φ is well defined. And finally in *Step 3* we construct the inverse to Φ.

Step 1. — The p^n monomials $\prod(x_i/x_0)^{a_i}$ (where $0 \leq a_i \leq p-1$) form a basis of $F_*\mathcal{O}_{U_0}$ as an $\mathcal{O}_{U_0}$-module, so the p^n homomorphisms

$$\begin{array}{rcl} \prod(x_i/x_0)^{a_i} & \mapsto & 1 \\ \text{others} & \mapsto & 0 \end{array}$$

constitute a basis of $\mathrm{Hom}(F_*\mathcal{O}_{U_0}, \mathcal{O}_{U_0})$ as an $\mathcal{O}_{U_0}$-module. A k-basis is then

$$\begin{array}{rcl} \prod(x_i/x_0)^{a_i} & \mapsto & \prod(x_i/x_0)^{b_i} \\ \text{others} & \mapsto & 0 \end{array} \tag{2.1}$$

where the b_i are any nonnegative integers. There are infinitely many choices of these, but not all of them extend to global homomorphisms:

Suppose $\varphi \in \mathrm{Hom}(F_*\mathcal{O}_{U_0}, \mathcal{O}_{U_0})$ is given by $\prod(x_i/x_0)^{a_i} \mapsto \prod(x_i/x_0)^{b_i}$.

Define s to be the smallest integer such that $\sum a_i \leq ps$, that is,

$$ps - (p-1) \leq \sum a_i \leq ps. \tag{2.2}$$

Now consider $(x_0/x_j)^s.(x_i/x_0)^{a_i} = (x_0/x_j)^{ps}(x_i/x_0)^{a_i} \in \Gamma(U_0 \cap U_j, F_*\mathcal{O}_{U_0})$: By construction of s the x_0-denominators are cleared, and the section thus extends to U_j. By linearity of φ we have

$$\left(\frac{x_0}{x_j}\right)^{ps} \prod \left(\frac{x_i}{x_0}\right)^{a_i} \mapsto \left(\frac{x_0}{x_j}\right)^{s} \prod \left(\frac{x_i}{x_0}\right)^{b_i}. \tag{2.3}$$

Since the left hand side is a section over U_j, φ can only extend to U_j when the right hand side is a section over U_j as well, which means that $\sum b_i \leq s$. To see that φ does indeed extend, note that the left hand side, by the minimality of s, is in fact one of the basis vectors for $F_*\mathcal{O}_{U_j}$, so if just $\sum b_i \leq s$, then the rule 2.3 actually *defines* an element in $\mathrm{Hom}(F_*\mathcal{O}_{U_j}, \mathcal{O}_{U_j})$, which obviously agrees with φ on the intersection $U_0 \cap U_j$ and thus extends φ.

In other words, in Equation 2.1 the choices of b_i's that give extendable homomorphisms are exactly those with $\sum b_i \leq s$.

Step 2. — It is obvious that the map $f \mapsto \varphi_f$ is a vector space homomorphism locally on U_0. To show that is is well defined globally, we must check that φ_f is indeed extendable. So let $f = \prod x_i^{q_i}$ be the dehomogenization with respect to x_0 of a monomial of degree $(n+1)(p-1)$, and let us calculate $\varphi_f(m)$ for some $m = \prod_{i=1}^n x_i^{a_i}$:

$\varphi_f(m) = C(\prod x_i^{q_i} \prod x_i^{a_i})$, which is equal to 0, unless each a_i satisfy an equation of the form

$$q_i + a_i = (p-1) + pb_i, \tag{2.4}$$

so that $\prod x_i^{q_i} \prod x_i^{a_i} = (x_1 \cdots x_n)^{p-1} \prod x_i^{pb_i}$ and in this case $\varphi_f(m) = \prod x_i^{b_i}$. Since $0 \leq a_i \leq (p-1)$ for each i, the a_i's and the b_i's are uniquely determined by the equations 2.4. To show that this is an extendable homomorphism we must show that $\sum b_i \leq s$, where s is defined by Equation 2.2. Summing up the equations 2.4, we get

$$p \sum b_i = \sum q_i + \sum a_i - n(p-1),$$

and since $\sum q_i \leq (n+1)(p-1)$ and $\sum a_i \leq ps$, we get

$$p \sum b_i \leq ps + (p-1) < ps + p,$$

so that $\sum b_i < s+1$, or as we prefer to put it: $\sum b_i \leq s$. So φ_f extends to a global homomorphism, and Φ is well defined.

Step 3. — Let us construct the inverse to Φ: Given a basis vector of $\mathrm{Hom}(F_*\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})$, that is, a homomorphism φ defined over U_0 by $\prod x_i^{a_i} \mapsto \prod x_i^{b_i}$ satisfying $ps - (p-1) \leq \sum a_i \leq ps$ and $\sum b_i \leq s$, then we set $\Phi^{-1}(\varphi) = \prod_{i=0}^n x_i^{q_i}$ where for each $i = 1, \dots, n$ q_i is defined by Equation 2.4, and we set $q_0 = (n+1)(p-1) - \sum_{i=1}^n q_i$, just to homogenize. For this to make sense, we must verify that q_0 is nonnegative, i.e. that $\sum_{i=1}^n q_i \leq (n+1)(p-1)$:

We get $\sum q_i = p \sum b_i + n(p-1) - \sum a_i \leq ps + n(p-1) - (ps - (p-1)) = (n+1)(p-1)$, so $\Phi^{-1}(\varphi)$ is indeed in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((n+1)(p-1)))$, and Φ^{-1} is well defined. It is trivial to see that Φ and Φ^{-1} are inverses to each other.

We are through. □

Henceforth, by abuse of language, we will refer to forms f of degree $(n+1)(p-1)$ as pre-splittings (resp. splittings) when we actually mean the φ_f corresponding to f under the isomorphism of Proposition 2.1.1.

Note that for such a form f, the coefficient $C(f)$ is the same whether we consider x_0 as a variable or dehomogenize and neglect it.

Note also that up to scalars $C(f)$ is independent of automorphisms of $\mathbb{P}^n$, so we can choose coordinates as we please.

2.1.2 Corollary. — *$\mathbb{P}^n$ is Frobenius split: Splittings of $\mathbb{P}^n$ are forms f of degree $(n+1)(p-1)$ with $C(f) \neq 0$. That is, the term $(x_0 \cdots x_n)^{p-1}$ occurs with nonzero coefficient.*

2.2 Compatible splitting in $\mathbb{P}^n$

Using Proposition 2.1.1, we now proceed to give a criterion of compatibility in $\mathbb{P}^n$, involving only f and not φ_f. Since our f is globally defined to start with, it suffices to work with $\mathbb{A}^n$ (by Lemma 1.1.1).

Definition. — Given an ideal I in a noetherian ring. By $I^{[p]}$ we denote the ideal generated by all p'th powers of elements of I.

2.2.1 Proposition. — *Let f be a polynomial in $x_1, \dots, x_n$ of degree $\leq (n+1)(p-1)$ and let φ_f be the pre-splitting for $\mathbb{A}^n$ induced by f via the fundamental isomorphism. Let Y be a closed subvariety of $\mathbb{A}^n$ with ideal I. Then φ_f is compatible with Y if and only if $f \cdot I \subset I^{[p]}$*

Proof. — The "if" implication is trivial. For the other implication: Let c be the codimension of Y, choose a smooth point of Y and reduce to the case $I = (x_1, \ldots, x_c)$ as in 1.1.3. Remark that $I^{[p]} = (x_1^p, \ldots, x_c^p)$, and that a polynomial belongs to I or $I^{[p]}$ if and only if each monomial does. So we reduce to the case where f is a monomial, say $f = \prod x_i^{q_i}$. Suppose φ is compatible with Y, i.e. $\varphi_f(I) \subset I$. If $q_i \geq p$ for some $i = 1, \ldots, c$, there is nothing to prove. So assume $0 \leq q_i \leq p-1$ for each $i = 1, \ldots, c$.

Now let m be the monomial that fills up the exponents to nearest $p-1 \pmod p$, that is, $m = \prod x_i^{a_i}$ where $0 \leq a_i \leq (p-1)$ and $q_i + a_i = (p-1) + pb_i$, and by the assumption on q_i we have $b_i = 0$ for $i \leq c$. Now apply φ_f to m:

$$\varphi_f(m) = C(fm) = C\left(\prod x_i^{q_i+a_i}\right) = C\left((x_1 \cdots x_n)^{p-1} \prod_{i=c+1}^{n} x_i^{pb_i}\right) = \prod_{i=c+1}^{n} x_i^{b_i}.$$

This is not in I, so we conclude that m is not in I either, and therefore the first c q_i's are already equal to $p-1$, which in turn implies $f \cdot I \subset I^{[p]}$. □

2.2.2 Lemma. — *A pre-splitting $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((n+1)(p-1)))$ of $\mathbb{P}^n$ is compatible with a hypersurface $Z(f)$ if and only if f^{p-1} is factor in s.*

Proof. — Locally the ideal of $Z(f)$ is (f). By Lemma 2.2.1 compatibility means $sf \in (f^p)$ which is equivalent to $s \in (f^{p-1})$. □

2.2.3 Proposition. — *Let $Y = Z(f_1, \ldots, f_c)$ be a complete intersection in $\mathbb{P}^n$ of dimension ≥ 1. Then all pre-splittings (resp. splittings) of Y come from pre-splittings (resp. splittings) of $\mathbb{P}^n$ compatible with the hypersurface $Z = Z(f_1 \cdots f_c)$.*

Proof. — Let I be the ideal sheaf of Y and let $d_i = deg(f_i)$ and $d = \sum d_i$. Recall that since Y is a complete intersection I/I^2 is locally free of rank c and has a filtration with quotients $\mathcal{O}_{\mathbb{P}^n}(-d_i) \otimes \mathcal{O}_Y$, so taking highest exterior power we get $\bigwedge^c(I_Y/I_Y^2) \simeq \mathcal{O}_{\mathbb{P}^n}(-d) \otimes \mathcal{O}_Y$. The dualizing sheaf ω_Y is then $\omega_{\mathbb{P}^n} \otimes \bigwedge^c(I_Y/I_Y^2)^\vee \simeq \mathcal{O}_Y(-n-1+d)$, (e.g. by [H], Ch. III, Proposition 7.11) and so by Theorem 1.2.6 the space of pre-splittings is isomorphic to $H^0(Y, \mathcal{O}_Y((n+1-d)(p-1)))$.

As we have just seen, pre-splittings of $\mathbb{P}^n$ compatible with Z are of the form $(f_1 \cdots f_c)^{p-1} \cdot g$, where g is homogeneous of degree $(n+1-d)(p-1)$, so the space of Z-compatible pre-splittings of $\mathbb{P}^n$ is isomorphic to $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((n+1-d)(p-1)))$.

Now any pre-splitting of Z is also compatible with Y (by Lemma 1.1.2) and the induction of Z-compatible pre-splittings of $\mathbb{P}^n$ to pre-splittings of Y is just the natural map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((n+1-d)(p-1))) \longrightarrow H^0(Y, \mathcal{O}_Y((n+1-d)(p-1)))$$

which is surjective since Y is a complete intersection and has dimension ≥ 1 (see [H], Ch. III, Exercise 5.5). So pre-splittings of Y come from Z-compatible pre-splittings of $\mathbb{P}^n$.

Let now $\overline{\sigma} : F_*\mathcal{O}_Y \to \mathcal{O}_Y$ be a pre-splitting of Y induced by $\sigma : F_*\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}$. If furthermore $\overline{\sigma}$ is a splitting, then we have $\overline{\sigma}(1) = 1$, and so we must have $\sigma(1) = 1 \pmod{(f)}$. But since any pre-splitting sends 1 to a scalar, we conclude that $\sigma(1) = 1$, and σ is a splitting. So splittings of Y come from Y-compatible splittings of $\mathbb{P}^n$. $\square$

Remark. — It is *not* true that all pre-splittings of $\mathbb{P}^n$ compatible with Y are also compatible with Z; what the proposition says is that there *exists* a Z-compatible pre-splitting which induces the given Y-pre-splitting.

This result allows us to handle all questions of splitting of complete intersections in the more manageable setup of compatible splitting in $\mathbb{P}^n$, and henceforth we will use the words *split* and *compatibly split in* $\mathbb{P}^n$ interchangeably for complete intersections. Also we have reduced all questions of complete intersections to questions of hypersurfaces.

2.2.4 Corollary. — *Let $Y = Z(f_1, \ldots, f_c) \subset \mathbb{P}^n$ be a complete intersection, and let $d = \sum deg(f_i)$. If $d > n+1$ then Y is not split. If $d = n+1$, then*

$$Y \text{ is split } \Leftrightarrow Z(f_1 \cdots f_c) \text{ is split } \Leftrightarrow C((f_1 \cdots f_c)^{p-1}) \neq 0.$$

Next we show that a compatible split hypersurface is always contained in a $Z(f)$ of degree $n+1$.

Definition. — A *full splitting* of $\mathbb{P}^n$ is a splitting of form f^{p-1} where f has degree $n+1$. The divisor of zeros of such an f will be called a *splitting divisor*.

2.2.5 Example. — A splitting divisor for $\mathbb{P}^1$ consists of two distinct points.

2.2.6 Lemma. — *Let $g(x_0, \ldots, x_n)$ be homogeneous of degree $n(p-1)$, and let $x_0^{q_0} \cdots x_n^{q_n}$ be of degree $p-1$, i.e. $\sum q_i = p-1$. If $(x_0 \cdots x_n)^{p-1}$ occurs in $g \cdot x_0^{q_0} \cdots x_n^{q_n}$, then there exists a linear form $h = A_0x_0 + \cdots + A_nx_n$ such that $(x_0 \cdots x_n)^{p-1}$ also occurs in $g \cdot h^{p-1}$.*

Proof. — The set of all h is $\mathbb{P}^{n*}$ with homogeneous coordinates $(A_0 : \ldots : A_n)$. Recall that

$$h^{p-1} = \sum \left(\begin{array}{c} (p-1)! \\ \prod s_i! \end{array} \right) \prod_{i=0}^{n} A_i^{s_i} x_i^{s_i}$$

where the sum is taken over all $n+1$-tuples $\underline{s} = (s_0, \ldots, s_n)$ of nonnegative integers such that $\sum_{i=0}^{n} s_i = p-1$.

Given $\prod x_i^{s_i}$ let $c_{\underline{s}}$ be the coefficient in g of $\prod x_i^{p-1-s_i}$ (the fill-up-to-$(p-1)$-factor). Now we have that the coefficient of $(x_0 \cdots x_n)^{p-1}$ in $g \cdot h^{p-1}$ is

$$\sum c_{\underline{s}} \left(\begin{array}{c} (p-1)! \\ \prod s_i! \end{array} \right) \prod_{i=0}^{n} A_i^{s_i}.$$

This is a homogeneous polynomial in A_i, so the set of bad choices of h is a Zariski closed subset of $\mathbb{P}^{n*}$. Now we only need to check that it is not the whole space, i.e. that not all the $c_{\underline{s}}$'s are zero. But since $(x_0 \cdots x_n)^{p-1}$ occurs in $g \cdot x_0^{q_0} \cdots x_n^{q_n}$, the value of $c_{\underline{s}}$ corresponding to $x_0^{q_0} \cdots x_n^{q_n}$ is nonzero and we are through. $\square$

2.2.7 Proposition. — *If a complete intersection $Y = Z(f_1, \ldots, f_c)$ is compatibly split in $\mathbb{P}^n$, then it can also be split by a splitting divisor. In fact it can be split by a splitting divisor of the form $Z(f_1) + \cdots Z(f_c) + H_d + \cdots + H_n$, where $d = \sum deg(f_i)$ and the H_i's are hyperplanes.*

Proof. — By Proposition 2.2.3, we can find a splitting of form $(f_1 \cdots f_c)^{p-1} g$, such that g is homogeneous of degree $(n+1-d)(p-1)$, and such that $(x_0 \cdots x_n)^{p-1}$ occurs. Without loss of generality we can assume that g is a monomial in $x_0, \ldots, x_n$. Now successive use of the lemma replaces g with $(p-1)$'th powers of linear factors and establishes the result. $\square$

So all split hypersurfaces are subvarieties in splitting divisors, and all split complete intersections arise as intersections of components of splitting divisors.

2.2.8 Arithmetic questions. — When a splitting divisor $X = Z(f) \subset \mathbb{P}^n$ is smooth, then it has $\omega_X \simeq \mathcal{O}_X$. In this case it is Frobenius split if and only if the action of Frobenius on cohomology $H^{n-1}(X, \mathcal{O}_X) \to H^{n-1}(X, F_*\mathcal{O}_X)$ is injective, which is one possible definition of being *ordinary*, see [M-Sr,2]. The questions of Frobenius splitting thus become arithmetic, i.e. depending on the prime p, and do not have much geometric content:

2.2.9 Example. *An elliptic curve.* — Using 2.2.4 and a computer, one find that the curve $Z(y^2 - x^3 - 6x + 2) \subset \mathbb{A}^2$ (which is elliptic for $p \geq 5$) is split (i.e. ordinary) for all primes $p < 400$ except $p = 101$ and $p = 251$.

2.2.10 Example. — *The Fermat surface* $Z(x^4+y^4+z^4+w^4) \subset \mathbb{P}^3$ is smooth for $p \geq 3$. Now looking for the critical term in $(x^4 + y^4 + z^4 + w^4)^{p-1}$ we find that the surface is split exactly for $p \equiv 1 \pmod 4$.

Henceforth we shall concentrate on the geometric aspects, which occur when the splitting divisor is reducible (or otherwise singular).

2.3 Geometry of splitting divisors

2.3.1 Proposition. — *Let $Z(f_1 \cdots f_r)$ be a splitting divisor of $\mathbb{P}^n$, the f_i's its irreducible components. Let S be the intersection of c components. Then $codim(S) = c$ in $\mathbb{P}^n$.*

Proof. — Induction on c. Obviously one component has codimension 1. Now for the general case: Let Y be an irreducible component of $S = \cap_{i=1}^{c} Z(f_i)$ and suppose $codim(Y) = c-1$. This is going to imply a contradiction: By induction we have $codim(\cap_{i=1}^{c-1} Z(f_i)) = c - 1$, so Y is also an irreducible component of $\cap_{i=1}^{c-1} Z(f_i)$. Restricting to the open set U obtained by throwing away all other components, we get the ideal of Y generated by $f_1, \dots f_{c-1}$, which is a radical ideal (by Lemma 1.1.2). Now f_c vanishes on Y, so we have a dependence relation $f_c = \sum_{i=1}^{c-1} a_i f_i$, where the a_i's are regular functions on U, i.e. without any f_i in the denominator. Substituting this sum for f_c in the product $f_1 \cdots f_r$, we see that raising to the $(p-1)$'th power gives us p'th powers in all terms. So the divisor is not splitting and the contradiction is established. □

Remark. — If a component of a splitting divisor is singular in codimension 1, then in fact this singularity cannot be contained in any other component: By

Corollary 2.4.3, in the completion at a general point the singularity looks like two components intersecting in dimension $n-2$. And so by the proposition there cannot be yet another component through here.

2.3.2 Example. — If three lines form a splitting divisor for $\mathbb{P}^2$, then by the proposition they must be in general position (i.e. have no point in common). On the other hand, if they are in general position, we can choose coordinates so that the divisor is $Z(xyz)$, and then obviously it is split.

2.3.3 Lemma. — *Let $Y = Z(f) \subset \mathbb{P}^n$ be a hypersurface of degree $d \leq n+1$. Then Y is split if and only if there exist $n+1-d$ independent hyperplanes $H_d, \ldots, H_n$, no one contained in Y, such that $Y \cap H_d \cap \cdots \cap H_n$ is compatibly split in $H_d \cap \cdots \cap H_n = \mathbb{P}^{d-1}$.*

Proof. — If Y is split, then by Proposition 2.2.7 there exist $H_d, \ldots, H_n$ such that $Y + H_d + \cdots + H_n$ is a splitting divisor, and they are independent by Proposition 2.3.1. Now by Lemma 1.1.2, the splitting is compatible with $H_d \cap \cdots \cap H_n = \mathbb{P}^{d-1}$, and also with $Y \cap H_d \cap \cdots \cap H_n$, thereby giving a compatibly splitting of the latter in the former.

For the reverse implication, suppose we have the independent hyperplanes such that $Y \cap L$ is compatibly split in $L = H_d \cap \cdots \cap H_n = \mathbb{P}^{d-1}$. Choose coordinates such that $H_i = Z(x_i)$. Let us now show that $Y + H_d + \cdots + H_n$ is a splitting divisor, i.e. that $C((f \cdot x_d \cdots x_n)^{p-1}) \neq 0$. Since we get all the needed $x_d \cdots x_n$ factors from the linear factors, all terms of f having such factors are useless for making the critical term. So we find that $C((f \cdot x_d \cdots x_n)^{p-1}) = C((f_{|L})^{p-1})$, where $f_{|L} = f(x_0, \ldots, x_{d-1}, 0, \ldots, 0)$ is the homogeneous polynomial of degree d defining $Y \cap L$ in L. And since $Y \cap L$ is compatible split in L, we have $C((f_{|L})^{p-1}) \neq 0$, and so $Y + H_d + \cdots + H_n$ is a splitting divisor in $\mathbb{P}^n$. □

2.3.4 Corollary. — *Any nondegenerate quadric hypersurface is split.*

Proof. — Cut the quadric with $n-1$ general hyperplanes (i.e. chosen such that the quadric in each dimension is again nondegenerate). Then we arrive at a nondegenerate quadric in $\mathbb{P}^1$ — two distinct points which are split cf. Example 2.2.5. Now apply the above lemma. □

Remark. — The result remains true in characteristic 2, but note that a lot of quadrics then degenerate to double hyperplanes, and thus fail to be varieties in our sense of the word. So the intersecting hyperplanes must be chosen carefully.

2.3.5 Example. — Let $Q+H$ be the union of a conic and a line in $\mathbb{P}^2$. Then by the lemma $Q \cup H$ is split if and only if $Q \cap H$ is compatibly split in $H = \mathbb{P}^1$. By Example 2.2.5, this is the case when H intersects Q in two distinct points. If Q is smooth this means that H is not tangent to Q, and if Q is two lines, we are back at Example 2.3.2.

Remark. — The important thing in the above example is that the intersection of a line tangent to a conic is nonreduced and therefore cannot be split. As the next example shows, nothing prevents two components in a splitting divisor from being tangent:

2.3.6 Example. — The divisor $Z(z) + Z(w) + Z(zw + xy)$ is easily seen to be a splitting divisor in $\mathbb{P}^3$, but the two components through the origin $x = y = z = 0$ have common tangent plane at this point.

2.3.7 Example. — Let us split $Gr(2,4)$, the Grassmannian of lines in $\mathbb{P}^3$, which is Plücker embedded as $Z(x_0x_5 - x_1x_4 + x_2x_3) \subset \mathbb{P}^5$. First we cut it with $Z(x_5)$, getting $Z(x_2x_3 - x_1x_4) \subset \mathbb{P}^4$, which is the singular Schubert variety $S_{2,4} \subset Gr(2,4)$. Cutting with $Z(x_4)$ we get $Z(x_2x_3) \subset \mathbb{P}^3$, which is the union of the two next Schubert varieties $S_{1,4}$ and $S_{2,3}$. And finally with $Z(x_1)$ and $Z(x_0)$ we get down to the three points, and thus $Z(x_0x_1x_4x_5(x_0x_5 - x_1x_4 + x_2x_3))$ is a splitting divisor for $\mathbb{P}^5$ compatible with $Gr(2,4)$ *and* with all the Schubert varieties. (Note that $S_{1,3}$ is just the intersection of $S_{1,4}$ and $S_{2,3}$, and thus split as well, and that $S_{1,2}$ is one of the three points.)

2.4 Singularities of codimension 1

In this section we investigate singularities. For simplicity, we work only in $\mathbb{A}^n$, and by a splitting divisor we mean a divisor whose projective closure is a splitting divisor for $\mathbb{P}^n$. By Lemma 1.1.1, everything works for $\mathbb{P}^n$ as well.

2.4.1 Example. — For $p > 2$, the criterion easily gives that the irreducible singular plane cubic $f = y^2 - x^2(x - \lambda)$ is split if and only if $\lambda \neq 0$. For $p = 2$, the curve is a cusp for all λ, and is seen not to be split. So nodal cubics are split, the cuspidal one is not.

Remark. — In fact these four plane curves: ordinary elliptic curve, nodal cubic, smooth conic, and a line, are the only split projective curves. See [OM].

A convenient way to distinguish the cusp from the node is looking at the scheme theoretic singular locus:

Definition. — Given a closed subscheme $X = Z(f_1, \ldots, f_s) \subset \mathbb{A}^n$ of codimension c. Let $Sing^1(X) = Sing(X)$ be the scheme defined by the ideal generated by the f_i's together with all the $c \times c$ minor determinants of the Jacobian matrix $||\partial f_i/\partial x_j||$ of partial derivatives of the f_i's. Define recursively $Sing^r(X) = Sing(Sing^{r-1}X))$.

In the example above, we can now characterize the cusp as having $codim_X(Sing^2(X)) = 1$, whereas the node has $Sing^2(X) = \emptyset$. Let us make the convention that $codim_X(\emptyset) = dim X + 1$

In this setup it is easy to generalize the example:

2.4.2 Proposition. — *For a Frobenius split hypersurface $X = Z(f) \subset \mathbb{A}^n$, the codimension of $Sing^2(X)$ in X is at least 2.*

Proof. — Suppose that $S \subset Sing(Sing(X))$ is reduced of dimension $n-2$, and let us show that this prevents X from being split.

Choose coordinates such that the origin P is a smooth point of S and such that in the completion of $\mathcal{O}_{\mathbb{A}^n}$ at P, the ideal of S is (x_1, x_2). Since $S \subset Sing^1(X)$, we have that $\partial f/\partial x_j$ (more precisely their completion) belongs to (x_1, x_2) for all j, and thus all terms of f have degree at least 2 with respect to x_1 and x_2, and we can write

$$f = Ax_1^2 + Bxy + Cx_2^2 + \text{ rest}$$

where $A, B, C \in k[x_3, \ldots, x_n]$, and the rest consists of terms of degree at least 3 with respect to x_1 and x_2.

Also all the 2×2 minor determinants of $||\partial f_i/\partial x_j||$ belong to (x_1, x_2), in particular, for the upper left hand determinant, we get

$$\begin{vmatrix} 2A + \frac{\partial^2 \text{rest}}{\partial^2 x_1} & B + \frac{\partial^2 \text{rest}}{\partial x_1 \partial x_2} \\ & \\ B + \frac{\partial^2 \text{rest}}{\partial x_1 \partial x_2} & 2C + \frac{\partial^2 \text{rest}}{\partial^2 x_2} \end{vmatrix} \in (x_1, x_2),$$

and since all double derivatives of the rest have x_1 or x_2 factors, we conclude that $2A \cdot 2C - B^2 \in (x_1, x_2)$, and indeed $4AC - B^2 = 0$. Now we can write

$Ax_1^2 + Bxy + Cx_2^2$ as $u \cdot (ax_1 + cx_2)^2$ for some polynomials u, a and c, and this last square factor means that all terms of $(Ax_1^2 + Bxy + Cx_2^2)^{p-1}$ have p'th power factors of x_1 or x_2. The same goes for f^{p-1} (since the rest has even higher degree of x_1 and x_2), contradicting that X is split. □

Remark. — This proposition accounts in one go for the three non-split curves drawn in the introduction.

Continuing the notation of the proof, if P is not in $Sing^2(X)$, then the upper left hand determinant is non-vanishing in P, and the three polynomials A, B and C must therefore have constant terms A_0, B_0, C_0, respectively, also satisfying $4A_0C_0 - B_0^2 \neq 0$. The leading term of f is then $A_0x_1^2 + B_0x_1x_2 + C_0x_2^2$ which factorizes into two independent linear forms f_g and f_h that generate (x_1, x_2), the completion of the ideal of S. This in turn means that in the completion at P, the polynomial f factors into two power series, and so the singularity is analytically reducible at P. This is essentially [H], Ch. I, Example 5.6.3.

These arguments can in fact be reversed, and we may restate the proposition as

2.4.3 Corollary. — *If a Frobenius split hypersurface $Z(f) \subset \mathbb{A}^n$ has a singularity of codimension 1, then it is analytically reducible in almost all points of the singularity, that is, in the completion at such a point there is a factorization of f into two nonconstant power series.*

Since codimension 1 singularities of split hypersurfaces locally look like the intersection of two components, we expect the singularity to be compatibly split. This is indeed the case:

2.4.4 Proposition. — *Let $X = Z(f) \subset \mathbb{A}^n$ be a splitting divisor, and let $Y \subset \mathbb{A}^n$ be a variety of codimension 2. Then the splitting is compatible with Y if and only if Y is contained in $Sing(X)$.*

Proof. — Let I be the ideal of Y, and choose coordinates such that, completed in the smooth point P, we have I generated by x_1 and x_2.

If Y is in the singular locus, all terms of f must have degree ≥ 2 with respect to x_1 and x_2, which ensures that all terms of of f^{p-1} have a factor

x_1^p, x_2^p or $x_1^{p-1}x_2^{p-1}$, which in turn means that $f^{p-1}\widehat{I} \subset \widehat{I}^{[p]}$. By the standard argument 1.1.3, this implies $f^{p-1}I \subset I^{[p]}$, and the compatibility is established.

For the reverse, remark that all the above implications are in fact biimplications. □

If a splitting divisor has a singularity of lower dimension, it is in general not compatible with it:

2.4.5 Example. — In the splitting divisor $X = Z((xy - z^2)(z-1)) \subset \mathbb{A}^3$, which is a quadric cone and a plane, the vertex of the cone is a singularity of codimension 2, but is not compatibly split. This is because it is not in $Sing^2(X)$. But if we move the plane from $Z(z-1)$ to $Z(z)$, so that it passes through the vertex, then the vertex becomes the singular locus of $Sing^1(X)$, and now it *is* compatibly split.

2.4.6 Example. *Whitney's Umbrella.* — For $p > 2$, Whitney's Umbrella $X = Z(y^2 - t^2x^2 - x^3)$ is a splitting divisor for $\mathbb{A}^3$. It is a family of singular cubics parametrized by t: The general members are nodal and the special member $t = 0$ is a cusp. It is singular along the t-axis — note how analytically it looks like the intersection of two planes, except for the cusp-like point $(0,0,0)$ which is in $Sing^2(X)$. The splitting is compatible with the t-axis, and also with the point $(0,0,0)$, even though it is not in the singular locus of the t-axis. What is important is that it *is* in $Sing^2(X)$.

A natural next step in the study of Frobenius splitting of hypersurfaces and the geometry of splitting divisors is to treat the questions of compatibility of varieties of higher codimensions: Which varieties are split by a splitting divisor?

I believe Proposition 2.4.2 generalizes to the following:

2.4.7 Conjecture. — *For a Frobenius split hypersurface $X = Z(f) \subset \mathbb{A}^n$, the codimension of $Sing^c(X)$ in X is at least c.*

Remark. — Since the intersection of $c+1$ components of a scheme X is always contained in $Sing^c(X)$, the conjecture implies Proposition 2.3.1 as a special case.

A natural generalization of Proposition 2.4.4 is the following, suggested by Example 2.4.6 and many other examples:

2.4.8 Conjecture. — *Let $Z(f) \subset \mathbb{A}^n$ be a splitting divisor and let S be a variety of codimension c in $Z(f)$.*

(i) *If $S \subset Sing^c(Z(f))$, then the splitting is compatible with S.*

(ii) *If the splitting is compatible with S, then $S \subset Sing^c(Z(f))$.*

The conditions (i) and (ii) could of course be true independently of each other. I believe in them both.

Another line of continuation of the ideas developed in this dissertation is to try to generalize some of the results to complete intersections in a general smooth variety. The very local nature of compatibility may make this project possible.

I hope the future will give me opportunity to take up these questions.

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