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GROWTH ALTERNATIVE FOR HECKE–KISELMAN MONOIDS

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Abstract: The Gelfand–Kirillov dimension of Hecke–Kiselman algebras defined by oriented graphs is studied. It is shown that the dimension is infinite if and only if the underlying graph contains two cycles connected by an (oriented) path. Moreover, in this case, the Hecke–Kiselman monoid contains a free noncommutative submonoid. The dimension is finite if and only if the monoid algebra satisfies a polynomial identity.

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1. Introduction

Let Θ be an arbitrary finite simple digraph with n vertices $\{1,\dots,n\}$. So, in other words, it is a simple graph in which two vertices can be connected by an oriented edge (an arrow) or an unoriented edge (an edge). In [**10**] the following monoid HK_Θ associated with Θ has been defined, by specifying generators and the set of defining relations:

(i) HK_Θ is generated by idempotents $e_i^2 = e_i$, where $1 \leq i \leq n$;
(ii) if the vertices i, j are not connected in Θ, then $e_ie_j = e_je_i$;
(iii) if i, j are connected by an arrow $i \to j$ in Θ, then $e_ie_je_i = e_je_ie_j = e_ie_j$;
(iv) if i, j are connected by an edge in Θ, then $e_ie_je_i = e_je_ie_j$.

If the graph Θ is unoriented (has no arrows), the monoid HK_Θ is isomorphic to the so-called 0-Hecke monoid $H_0(W)$, where W is the Coxeter group of the graph Θ, see [**8**]. The latter monoid plays an important role in representation theory. In the case Θ is oriented (all edges are arrows) and acyclic, the monoid HK_Θ is finite and it is a homomorphic image of the so-called Kiselman monoid K_n, see [**10**], [**13**].

The monoid HK_Θ has been studied in particular in [**2**], [**3**], [**7**], [**9**]. So far, the emphasis has been concentrated on three problems, which still remain open in full generality: on the description of all graphs Θ, for which the monoid HK_Θ is finite; on the existence of faithful representations of the monoid HK_Θ in the multiplicative semigroup $\mathbb{M}_n(\mathbb{Z})$ of matrices over the ring of integers; and whether HK_Θ is always a $\mathcal{J}$-trivial monoid. The latter means that the Green $\mathcal{J}$-relation on HK_Θ is trivial; in other words: $\mathrm{HK}_\Theta\, s\, \mathrm{HK}_\Theta = \mathrm{HK}_\Theta\, t\, \mathrm{HK}_\Theta$ implies that $s = t$, for $s, t \in \mathrm{HK}_\Theta$. The last problem has been solved in the affirmative in the case Θ is oriented, as well as in the case Θ is unoriented, and also in certain mixed cases (see [**3**]). This problem is essential for the representation theory of such monoids, since representations of $\mathcal{J}$-trivial monoids admit a very satisfactory approach, see [**8**].

The present paper is motivated by a study of the growth of Hecke–Kiselman monoids, or in other words of the Gelfand–Kirillov dimension of the semigroup algebras $A_\Theta := \mathrm{K}[\mathrm{HK}_\Theta]$ over a field K. The main aim is to establish the growth alternative for A_Θ in the case when Θ is an oriented graph. In other words, we show that the growth is either polynomial or exponential. Clearly, the Gelfand–Kirillov dimension of A_Θ is 0 if and only if the monoid HK_Θ is finite. In the case of oriented graphs Θ this means exactly that the graph is acyclic, see [**13**].

Our main result reads as follows.

Theorem 1. *Assume that Θ is a finite oriented simple graph. The following conditions are equivalent.*

(1) *Θ does not contain two different cycles connected by an oriented path of length ≥ 0.*
(2) *A_Θ is a PI-algebra.*
(3) $\mathrm{GKdim}(A_\Theta) < \infty$.
(4) *The monoid HK_Θ does not contain a free submonoid of rank 2.*

Here, if $\Theta = (V(\Theta), E(\Theta))$ is an oriented graph with the set of vertices $V(\Theta)$ and the set of arrows $E(\Theta)$, then we say that two induced subgraphs Θ_1 and Θ_2 of Θ are:

- connected by a path of length 0, if $V(\Theta_1) \cap V(\Theta_2) \neq \emptyset$;
- connected by a path of length 1, if $V(\Theta_1) \cap V(\Theta_2) = \emptyset$ and there exist $w \in V(\Theta_1)$, $v \in V(\Theta_2)$, and an arrow $w \to v$, or $v \to w$, in the graph Θ;
- connected by a path of length $k > 1$, if the above two cases do not occur and if k is the minimal integer such that there exists

a sequence of vertices $w_0, \dots, w_k \in V(\Theta)$ such that $w_0 \in V(\Theta_1)$, $w_k \in V(\Theta_2)$, $w_i \notin V(\Theta_1) \cup V(\Theta_2)$ for $0 < i < k$ and in Θ there exist arrows $w_i \to w_{i+1}$, or there exist arrows $w_{i+1} \to w_i$, for all $i = 0, 1, \dots, k-1$.

We note that the growth alternative result for A_Θ in the case when the graph Θ is unoriented follows from classical results on Coxeter groups and monoids. As mentioned above, in this case $\mathrm{HK}_\Theta = H_0(W)$, where W is the Coxeter group of the graph Θ. The growth of W and $H_0(W)$ is actually the same, which follows from the word property theorem of Tits [**19**], see also [**20**, Theorem 1]. More precisely, the reduced words for the Coxeter group and Coxeter monoid are the same, and two reduced words represent the same element of the Coxeter group if and only if they represent the same element of the Coxeter monoid. The theorem by de la Harpe [**11**] states, however, that the growth of the Coxeter group W is either polynomial, in the case when W is finite or affine (and W is abelian-by-finite in this case), or exponential in other cases. Moreover, it is known that there exists a finite automaton that recognizes the language of normal forms of elements of a Coxeter group [**6**], hence also the language of normal forms of elements of the corresponding monoid $H_0(W)$. In our particular case, by a standard argument (see [**22**, p. 97]) one obtains the following result.

Theorem 2. *Assume that Θ is a finite unoriented simple graph. The following conditions are equivalent.*

(1) *Θ is a disjoint union of Dynkin and extended Dynkin diagrams.*
(2) $\mathrm{GKdim}(A_\Theta) < \infty$.
(3) *The monoid HK_Θ does not contain a free submonoid of rank 2.*

Clearly, Theorems 1 and 2 imply certain conditions concerning the Gelfand–Kirillov dimension of the algebra A_Θ, when Θ is any mixed digraph. However, we do not dwell on those details in this paper.

A similar growth alternative has been known in several other contexts; in particular, it holds in the class of monomial algebras, that provide a rich area of examples of algebras with a particular growth behavior and have been used to answer several questions on the Gelfand–Kirillov dimension of arbitrary algebras. Recall that for an ideal I of the free monoid $\langle X\rangle$ on a set X one defines $\mathrm{K}_0[\langle X\rangle/I] = \mathrm{K}[\langle X\rangle]/\mathrm{K}[I]$. The result obtained independently by Anick [**1**] and Ufnarovskii [**21**] can be stated as follows (see also [**15**, Theorem 24.19]).

Theorem 3. *Assume that I is a finitely generated ideal of the free monoid $\langle X\rangle$ on a finite set X. Then the following conditions are equivalent.*

(1) $\mathrm{GKdim}(\mathrm{K}_0[\langle X\rangle/I]) < \infty$.

(2) $\langle X\rangle/I$ *does not contain a free submonoid of rank* 2.

(3) $\mathrm{K}_0[\langle X\rangle/I]$ *is a PI-algebra.*

Moreover, if these conditions are satisfied, then $\mathrm{GKdim}(\mathrm{K}_0[\langle X\rangle/I])$ *is an integer.*

In certain more general classes of algebras (for example, algebras admitting a finite Gröbner basis, or in the case of automaton algebras) similar results have been obtained by associating to the algebra, or to the set M of its normal words, a certain graph $\Gamma(M)$, and by deriving results on the growth of the algebra in terms of the structure of the graph $\Gamma(M)$, see [**22**].

A more general conjecture, coming from [**1**], can be stated as follows: if R is a finitely presented algebra then the growth of R is either subexponential or R contains a free noncommutative subalgebra. This problem, as well as related questions concerning free subalgebras of division rings, have attracted a lot of attention, see [**4**] for example. Most notably, a conjecture formulated independently by Makar-Limanov and Stafford says that a division algebra D either contains a free noncommutative subalgebra over its center or it is a locally PI-algebra.

Another example of a growth alternative result was given by Okniński and Salwa in the context of the semigroup version of the Tits alternative, [**16**]. They proved that a finitely generated subsemigroup S of the linear group $GL_n(\mathrm{K})$ over a field K either contains a free subsemigroup of rank two, or generates an almost-nilpotent subgroup. The second condition yields that the Gelfand–Kirillov dimension of the semigroup algebra $\mathrm{K}[S]$ must be an integer.

2. Definitions and the necessary background

Let A be a finitely generated algebra over a field K and let V be an arbitrary finite dimensional subspace that generates the algebra A. Recall that the Gelfand–Kirillov dimension of A is defined by

$$\mathrm{GKdim}(A) = \limsup(\log_n(d_V(n))),$$

where $d_V(n) = \dim_K(V^0 + V^1 + \cdots + V^n)$ is the growth function associated with V and $V^k = \text{lin}_K\{v_1 \cdots v_k \mid v_i \in V,\ 1 \le i \le k\}$. In the case S is a finitely generated monoid with a generating set Z, the growth function of the semigroup algebra $K[S]$ can be computed by calculating the growth function $d_S(n)$ of S, namely $d_S(n) = |\{w \in S \mid w = y_1 \cdots y_k,$ for some $y_i \in Z,\ k \le n\}|$.

Let X be a finite set and let F denote the set of all words in the alphabet X (including the empty word ϵ, which is identified with the unity 1 of the free algebra $K\langle X\rangle$). For every $x \in X$ and $w \in F$ by $\deg_x w$ we mean the degree of the word w in x. By $|w|$ we denote the length of the word w. The support of the word w, denoted by $\text{supp}(w)$, stands for the set of all $x \in X$ such that $\deg_x w > 0$. By $\text{pref}_n(w)$, $\text{suff}_n(w)$ we denote respectively the prefix and the suffix of the word w of length n.

We say that the word $w = x_1 \cdots x_r \in F$ is a subword of the word $v \in F$, where $x_i \in X$, if $v = v_1x_1 \cdots v_rx_rv_{r+1}$, for some $v_1, \ldots, v_{r+1} \in F$. If the length $|v_1x_1 \cdots v_rx_r|$ is minimal possible then we say that this is the first occurrence of w in v. If $v_2, \ldots, v_r$ are trivial words, then we say that w is a factor of v. By $l_n(w) = x_n$ we denote the n-th element of the set X appearing in the word w.

Assume that the set X is well ordered. Then on the set F there exists an induced degree-lexicographic order $\le$. To every element $x \in K[F]$ one can associate its leading term $\overline{x}$ with respect to $\le$. For every subset $R \subseteq K[F]$ by $\alpha(R)$ we denote the set of all leading terms of elements of R. If I is an ideal of the free algebra $K[F]$ then a word $w \in F$ is called normal (modulo I), if w is not a leading term of any element of I.

Describing the normal words of the algebra of the form $K\langle X\rangle/I$ is related to finding the reduced Gröbner basis of the ideal I. The so-called diamond lemma is often used in this context. We will follow the approach and terminology of [**5**].

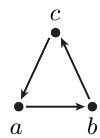
We will also fix notation and terminology used in the context of Hecke–Kiselman monoids. Consider an oriented digraph $\Theta = (V(\Theta), E(\Theta))$. Equalities (i)–(iv) presented in Section 1 are simply called the edge relations. Let w be a word in the free monoid $\langle V(\Theta)\rangle$ on the set $V(\Theta)$. By $[w]$ we denote the equivalence class of the word w with respect to HK_Θ. Also, we define two relations on the monoid $\langle V(\Theta)\rangle$, coming from the graph Θ: for $w, w' \in \langle V(\Theta)\rangle$ we write $w \sim w'$ if $[w] = [w']$, and $w \approx w'$ if $w = xyz$, $w' = xy'z$, where $x, y, y', z \in X$ and $y = y'$ is an edge relation in HK_Θ.

An element $v \in V(\Theta)$ is called a source vertex, if v is the beginning of every edge incident to v in Θ. Dually, one defines a sink vertex.

3. Hecke–Kiselman monoids defined by a cycle

In what follows, Θ will denote an oriented graph. According to Theorem 3 in [**13**] and Lemma 2.6 in [**2**], we know that $\mathrm{GKdim}(A_\Theta) = 0$ if and only if Θ is acyclic. Consequently, we start with the case when Θ is a cycle of length 3.

Example 1. Consider the graph Θ of the form:

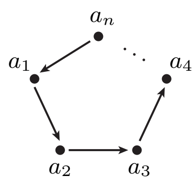


Then $\mathrm{GKdim}(A_\Theta) = 1$.

Proof: The monoid HK_Θ is generated by the idempotents a, b, c subject to the relations: $aba = bab = ab$, $aca = cac = ca$, $bcb = cbc = bc$. It is not hard to see that every word of the free monoid on a, b, c can be rewritten to an equivalent word in HK_Θ that is a finite factor of one of the following infinite words $(abc)^\infty$, $(acb)^\infty$. Thus are no more than 6 words of length n in HK_Θ, for all n. Clearly, this implies that $\mathrm{GKdim}(A_\Theta) = 1$. □

Our next step is to consider the case of a cyclic oriented graph Θ of an arbitrary length.

Example 2. Consider the graph Θ_n with n vertices $a_1, \ldots, a_n$, where $n \geq 4$, that is an oriented cycle:



Then $\mathrm{GKdim}(A_{\Theta_n}) = 1$.

Proof: Let $F = \langle X \rangle$ be the free monoid with the set X of free generators $a_1, \ldots, a_n$. We will proceed by induction on the number of vertices n. The following observation is crucial.

Lemma 4. *Suppose that the cycle Θ_{n-1} consists of $n-1$ vertices $b_1, b_2, \ldots, b_{n-1}$. Consider an epimorphism ϕ from the free monoid $\langle b_1, \ldots, b_{n-1}\rangle$ to the submonoid $\langle a_2, \ldots, a_{n-1}, a_n a_1\rangle$ of F defined in the following way:*

$$\phi(b_i) = \begin{cases} a_{i+1}, & \text{for } 1 \leq i \leq n-2, \\ a_n a_1, & \text{for } i = n-1. \end{cases}$$

Then ϕ induces a homomorphism $\overline{\phi}\colon \mathrm{HK}_{\Theta_{n-1}} \longrightarrow \mathrm{HK}_{\Theta_n}$ given by the formula $\overline{\phi}([w]) = [\phi(w)]$, for every $w \in \langle b_1, \dots, b_{n-1}\rangle$.

Proof: We only need to prove that $\overline{\phi}$ is well defined. It is sufficient, therefore, to check that for every edge relation $w_1 = w_2$ that is defined by Θ_{n-1} on the free monoid $\langle b_1, \dots, b_{n-1}\rangle$ we have $\phi(w_1) \sim \phi(w_2)$, where $\sim$ is the relation defined on F by Θ_n. This is trivial, if $b_{n-1} \notin \operatorname{supp}(w_1)$. We verify the remaining cases below:

- $\phi(b_{n-1}^2) = a_n a_1 a_n a_1 \sim a_n a_1 a_1 \sim a_n a_1 = \phi(b_{n-1})$;
- for $i \leq n-1$ and $i \notin \{1, n-2\}$, we have $\phi(b_{n-1} b_i) = a_n a_1 a_{i+1} \sim a_n a_{i+1} a_1 \sim a_{i+1} a_n a_1 = \phi(b_i b_{n-1})$;
- $\phi(b_{n-2} b_{n-1} b_{n-2}) = a_{n-1} a_n a_1 a_{n-1} \sim a_{n-1} a_n a_{n-1} a_1 \sim a_{n-1} a_n a_1 = \phi(b_{n-2} b_{n-1})$;
- $\phi(b_{n-1} b_{n-2} b_{n-1}) = a_n a_1 a_{n-1} a_n a_1 \sim a_n a_{n-1} a_1 a_n a_1 \sim a_n a_{n-1} a_n a_1 \sim a_{n-1} a_n a_1 = \phi(b_{n-2} b_{n-1})$;
- $\phi(b_{n-1} b_1 b_{n-1}) = a_n a_1 a_2 a_n a_1 \sim a_n a_1 a_n a_2 a_1 \sim a_n a_1 a_2 a_1 \sim a_n a_1 a_2 = \phi(b_{n-1} b_1)$;
- $\phi(b_1 b_{n-1} b_1) = a_2 a_n a_1 a_2 \sim a_n a_2 a_1 a_2 \sim a_n a_1 a_2 = \phi(b_{n-1} b_1)$. □

We proceed with the proof. The following order

$$a_n < a_1 < \cdots < a_{n-1} \tag{3.1}$$

on the set of generators induces a degree-lexicographic order $\leq$ on F. A word $w \in F$ is called reduced if for every $w' \sim w$ we have $w \leq w'$. Otherwise, we say that w is a reducible word. Of course, for every word $w \in F$ there exists exactly one reduced word $w' \in F$ such that $w \sim w'$, which we call the reduced form of w.

Let Θ_n' be the induced subgraph of Θ_n consisting of the vertices $a_1, \dots, a_{n-1}$, and let F' be the set of all reduced forms of the elements of the monoid $\mathrm{HK}_{\Theta_n'}$ in the free monoid $\langle a_1, \dots, a_{n-1}\rangle$. This set is finite as Θ_n' is acyclic.

Notice that every reduced word w in F belongs to the following set:

$$F' \cup F' a_n F' \cup \bigcup_{r=1}^{\infty} F' a_n w_1 a_n w_2 a_n \cdots w_r a_n F',$$

where $w_j \in F'$, for $1 \leq j \leq r$. We will prove the following five claims regarding the case when a reduced word w is of the form

$$s \cdot a_n w_1 \cdots a_n w_r a_n \cdot t, \tag{3.2}$$

where $w_j \neq 1$ and $s, t \in F'$.

(1) Every w_j must be in one of the following three forms:
   (a) $w_j = e := a_{n-1}a_{n-2}\cdots a_1$,
   (b) $w_j = f := a_{n-2}a_{n-3}\cdots a_1 a_{n-1}$,
   (c) $w_j = a_k a_{k-1}\cdots a_1 y a_{n-1}$, where $k < n-2$ and $y \in \langle a_2, \dots, a_{n-2}\rangle$.
(2) If $w_j = e$, for some j, then $w_1 = w_2 = \cdots = w_r$.
(3) If $w_j = f$, for some j, then $w_j = w_{j+1} = \cdots = w_r$.
(4) If w_j, w_{j+1} are of the form (c) from (1), then $w_{j+1} = a_1 y a_{n-1}$, for some $y \in \langle a_2, \dots, a_{n-2}\rangle$.
(5) If $w_j = f$, for some j, and $r \geq 2$, then $w_2 = f$.

If we prove these statements above we will see that the growth function $d(k)$ of HK_{Θ_n}, that counts, for the given k, the number of reduced words of length less than, or equal to k in F, is linear. If fact, it can be presented as $d_1(k) + d_2(k)$, where:

- $d_1(k)$ is the number of reduced words of length $\leq k$ from the set

$$F' \cup F' a_n \langle e a_n \rangle F' \cup (F' a_n \cup F' a_n F' a_n)\langle f a_n \rangle F',$$

  where $\langle e a_n\rangle$ and $\langle f a_n\rangle$ are the free submonoids of F generated by ea_n, fa_n, respectively;
- $d_2(k)$ is the number of reduced words w of length $\leq k$ of the form (3.2), where $r \geq 1$ and w_j are, for $j \geq 2$, of the form $a_1 y_j a_{n-1}$, where $y_j \in \langle a_2, \dots, a_{n-2}\rangle$.

Both $d_1(k)$ and $d_2(k)$ are linear. Since F' is a finite set, this is clear in the case of $d_1(k)$. To see that $d_2(k)$ is linear observe that the factor $w' = a_n w_2 a_n \cdots a_n w_r$ of the reduced word w belongs to $\langle a_2, \dots, a_{n-1}, a_n a_1\rangle$. Thus, from Lemma 4 we get that there exists a reduced word v of length $\leq k$ in $\langle b_1, \dots, b_{n-2}\rangle$ such that $\phi([v]) = [w']$. Since $w_1 \in F'$ we can see that $d_2(k)$ is bounded by the value $|F'|^3 \cdot d'(k)$, where d' is the growth function of $\mathrm{HK}_{\Theta_{n-1}}$. The induction hypothesis yields that $d'(k)$ is linear, so $d_2(k)$ must be linear as well. This yields that $d(k)$ is linear and $\mathrm{GKdim}(A_{\Theta_n}) = 1$.

The proofs of (1)–(5) rely on the following easy observations.

Remark 5. Let v, w be elements of F such that $a_{i-1} \notin \mathrm{supp}(v)$ and $a_{i+1} \notin \mathrm{supp}(w)$, for any $i \in \{1, \dots, n\}$ (here we write $a_0 = a_n$ and $a_{n+1} = a_1$). The following relations hold:

$$a_i v a_i \sim a_i v, \quad a_i w a_i \sim w a_i.$$

In particular, an element of F that contains a factor of the form $a_i v a_i$ or $a_i w a_i$ is not reduced.

This remark follows easily from the defining relations for the (particular) monoid HK_{Θ_n}, see also [**9**, Lemma 3.4]. The next one is completely obvious.

Remark 6. A word $w \in F$ with a factor $a_i a_j$, where $a_i a_j \sim a_j a_i$ and $a_i > a_j$ (with respect to (3.1)) is not reduced.

We begin with the proof of (1). Consider a reduced word $a_n w a_n$, where $a_n \notin \mathrm{supp}(w)$. First, notice that $\deg_{a_1} w = 1$ and $\deg_{a_{n-1}} w = 1$. Indeed, if $a_1 \notin \mathrm{supp}(w)$, then the word $a_n w a_n$ is not reduced, according to Remark 5. The same argument asserts that if w contains a factor of the form $a_1 w' a_1$, where $\mathrm{supp}(w') \subseteq X \setminus \{a_1, a_n\}$, then again it cannot be reduced. Similarly, one shows that $\deg_{a_{n-1}} w = 1$. Secondly, from Remark 6 it follows immediately that $\mathrm{suff}_1(w) \in \{a_1, a_{n-1}\}$ (otherwise, this element commutes with a_n). Since $|w| > 0$, we consider two cases.

Case 1: Let $\mathrm{suff}_1(w) = a_1$. We prove that $w = e$. We proceed in two steps. First, we prove inductively that $w = ve = v a_{n-1} \cdots a_1$, for some $v \in F'$, and next we show that $v = 1$.

We begin with the induction. Let $w = w' a_1$, for some $w' \in F'$. Since $|w| \geq 2$, it follows that $w' \neq 1$. Again, from Remark 6 it follows that $\mathrm{suff}_1(w')$ cannot be an element of X that commutes with a_1 and clearly $\mathrm{suff}_1(w') \notin \{a_1, a_n\}$. Consequently, $a_n w a_n = a_n w'' a_2 a_1 a_n$, for some $w'' \in F'$ and the first step of induction follows. Assume now that $w = w'' a_k a_{k-1} \cdots a_1$ for some $1 < k < n-1$. Of course, $w'' \neq 1$, as otherwise $a_{n-1} \notin \mathrm{supp}(a_n w a_n)$, a contradiction. We want to show that $\mathrm{suff}_1(w'') = a_{k+1}$. As before, using Remark 6, we see that $\mathrm{suff}_1(w'') \neq a_q$, where $q > k+1$. Also, it is clear that $\mathrm{suff}_1(w'') \notin \{a_{k-1}, a_k\}$. If $\mathrm{suff}_1(w'') = a_l$, for $l < k-1$, then the word $a_n w a_n$ has a factor $a_l z a_l$, where $a_{l-1} \notin \mathrm{supp}(z)$, so it is not reduced by Remark 5. This shows that $q = k+1$, completing the induction step. We have shown that $w = ve$, for some $v \in F'$. However, if we had $v \neq 1$ and $\mathrm{suff}_1(v) = a_i$, for $1 \leq i \leq n-3$ (otherwise v would contain a reducible factor) then w would contain a factor $a_i z' a_i$ with $a_{i-1} \notin \mathrm{supp}(z')$, which would make it reducible. Therefore $v = 1$, and it follows that $w = e$.

Case 2: Let $\mathrm{suff}_1(w) = a_{n-1}$. Since $\deg_{a_1} w = \deg_{a_{n-1}} w = 1$, there exist $x, y \in F'$ such that $a_n w a_n = a_n x a_1 y a_{n-1} a_n$ and $\mathrm{supp}(x), \mathrm{supp}(y) \subseteq X \setminus \{a_1, a_{n-1}, a_n\}$. Applying arguments similar to those used in the previous case, we see that $x a_1$ is of one of the forms: $a_k a_{k-1} \cdots a_2 a_1$, where $1 \leq k \leq n-2$. If $k = n-2$ then clearly $w = f$ as the element $a_{n-2} \cdots a_2 a_1 a_s$ is reducible, for all $s < n-1$, again by Remark 5. If $1 \leq k < n-2$, then the element w is exactly of the form (c) listed in (1).

Statement (1) is verified. We proceed to prove (2). Take a reduced word w of the form (3.2). If $r = 1$ there is nothing to prove. Assume that $r \geq 2$. If $k \neq n-1$ then the word ea_na_k contains a factor of the form $a_k \cdots a_1 a_n a_k$ which does not contain a_{k+1}. Thus, none of these words may be reduced, due to Remark 5. Similarly, no words of the form $a_k a_n e$, where $k \neq 1$ may be reduced. In view of (1), this proves the assertion of (2). Similarly, one shows that if fa_na_k is reduced then $k = n-2$. Therefore, if $w_j = f$, for some j, then $w_{j+1} = \cdots = w_r = f$, so that (3) holds.

The statement of (4) is also easy to check. Each reduced word of the form $a_n w_j a_n w_{j+1} a_n$, as defined in this case, contains a factor $a_{n-1}a_na_k$, for $k < n-2$. But unless $k = 1$, we have $a_{n-1}a_na_k \sim a_ka_{n-1}a_n$. Moreover $a_ka_{n-1}a_n < a_{n-1}a_na_k$ in the deg-lex order on F, defined by (3.1). Since $a_n w_j a_n w_{j+1} a_n$ is reduced, such case cannot occur and w_{j+1} must be of the form $a_1 y a_{n-1}$, where $y \in \langle a_2, \ldots, a_{n-2} \rangle$, as claimed.

Assume that $w_j = f$, for some j, and choose minimal such j. Suppose $j \geq 3$. It follows from (1)–(3) that each of $w_1, \ldots, w_{j-1}$ is of the form (c). Hence, by (4) we must have $w_i = a_1 y_i a_{n-1}$, for $i = 2, \ldots, j-1$, where $y_i \in \langle a_2, \ldots, a_{n-2} \rangle$. Suppose $y_{j-1} = xa_s$, for some s and some $x \in F$. Then the factor $a_s a_{n-1} a_n a_{n-2} \cdots a_s$ of $a_n w_{j-1} a_n w_j a_n$ does not contain a_{s-1}, a contradiction (because it must be reduced). It follows that y_{j-1} is the empty word and $w_{j-1} = a_1 a_{n-1}$. Since w_{j-2} must be of the form (c), we see that $a_{n-1}a_na_1a_{n-1}$ is a factor of $w_{j-2}a_nw_{j-1}$. Since this is not a reduced word, we get a contradiction. It follows that $j \leq 2$. Hence (5) is proved. □

4. Free submonoids of HK_Θ

In this section we present a family of examples of graphs Θ, for which the monoid HK_Θ contains a free submonoid of rank 2. We begin with the case of two cycles of length three connected by a path of length 0.

Example 3. Let Θ be a graph with vertices a, b, c, d and Ψ be a graph with vertices a, b, c, e, f of the following form:

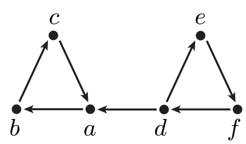
Then the submonoid $\langle [acb], [adb] \rangle$ of HK_Θ and the submonoid $\langle [acb], [afe] \rangle$ of HK_Ψ are both free of rank 2.

Proof: It is clear that no word of the submonoid $\langle acb, adb\rangle$ of the free monoid $F = \langle a, b, c, d\rangle$ can be rewritten in F in any way using the edge relations defining the monoid HK_Θ except by the idempotent relations of type $x = xx$, for $x \in \{a, b, c, d\}$. The same argument holds for the submonoid $\langle acb, afe\rangle$ of the free monoid $F' = \langle a, b, c, e, f\rangle$. □

We will now consider the case of two disjoint cycles connected by a path of a nonzero length. The following definition will be useful.

Definition 7. Let Θ be a simple oriented digraph and let $F = \langle V(\Theta)\rangle$ a free monoid generated by the set of vertices of Θ. Suppose that a cycle Θ' of length n of the form $x_1 \to x_2 \to \cdots \to x_n \to x_1$ is a subgraph of Θ. We will say that the word $w \in F$ is Θ'-free if the maximal subword v of w with $\mathrm{supp}(v) \subseteq V(\Theta')$ is a factor of the infinite word $(x_n x_{n-1} \cdots x_1)^\infty$.

Example 4. Let Θ be a graph of the form:



Then the submonoid $\langle [acb], [fed]\rangle$ of HK_Θ is free.

Proof: Let $F = \langle X\rangle$ be the free monoid on the set $X = \{a, b, c, d, e, f\}$ and let $H = \langle acb, fed\rangle$ be a submonoid of F. We define a subset W of F as follows: an element w belongs to W if and only if the following three conditions are satisfied:

(1) $\mathrm{supp}(w) = X$;

(2) the first appearance of a in w is earlier then the ones of b and c, and the first appearance of f in w is earlier then the ones of d and e;

(3) every word in W is both $\{a, b, c\}$-free and $\{d, e, f\}$-free.

The outline of the proof is as follows. First, we prove that W is closed under equivalence classes of F under the relation $\sim$ defined for Θ in Section 2. Then, to each element $v \in W$ we assign an infinite sequence N_v of nonnegative integers $(i_1, j_1, i_2, j_2, \dots)$ such that $N_v = N_{v'}$ if $v \sim v'$. It is clear that H is a subset of W and it will be straightforward to verify that $N_w \neq N_{w'}$, for any two different members w, w' of H. From this we will easily deduce that $\langle [acb], [fed]\rangle$ is a free submonoid of rank 2 in HK_Θ.

Take any $w \in W$ and $w' \in F$ such that $w' \sim w$. We claim that $w' \in W$. Clearly, it is enough to confirm this in the case when w and w'

satisfy $w \approx w'$, as the relation $\sim$ is the transitive closure of $\approx$ defined in Section 2. Thus, we may assume that $w = u_1 t u_2$ and $w' = u_1 t' u_2$, for some $u_1, u_2 \in F$ and an edge relation $t = t'$ in HK_Θ. Since $w \in F$, we only need to consider the following cases:

(a) $\{t, t'\} \in \{x, xx\}$, $x \in X$;
(b) $\{t, t'\} \in \{xy, yx\}$, where $xy \approx yx$, and $x, y \in X$;
(c) $\{t, t'\} \subseteq \{da, ada, dad\}$.

Indeed, the case $\{t, t'\} \subseteq \{xy, xyx, yxy\}$, where $xy \neq da$ is excluded according to condition (3) in the definition of W. Now it is basically clear that w' satisfies conditions (1)–(3) and thus it belongs to W.

We will now introduce a preparatory notation in order to associate a sequence N_w to every $w \in W$, as mentioned in the outline of the proof. From (1)–(3) it follows that w can be represented in two forms. First

$$w = u_1 a_1 u_2 c_1 u_3 b_1 u_4 a_2 u_5 c_2 u_6 b_2 \cdots, \tag{4.1}$$

where a_{i_1}, b_{i_2}, c_{i_3}, u_{i_4} are trivial for sufficiently large i_1, i_2, i_3, i_4 and $u_i \in \langle d, e, f\rangle$, $a_i \in \langle a, d, e, f\rangle$, $b_i \in \langle b, d, e, f\rangle$, $c_i \in \langle c, d, e, f\rangle$. Moreover, the first and last letter of every a_i, b_i, c_i is equal to a, b, c, respectively.

The second form of w is as follows:

$$w = v_1 f_1 v_2 e_1 v_3 d_1 v_4 f_2 v_5 e_2 v_6 d_2 \cdots, \tag{4.2}$$

where again d_{j_1}, e_{j_2}, f_{j_3}, v_{j_4} are trivial for sufficiently large j_1, j_2, j_3, j_4 and $v_j \in \langle a, b, c\rangle$, $d_j \in \langle a, b, c, d\rangle$, $e_j \in \langle a, b, c, e\rangle$, $f_j \in \langle a, b, c, f\rangle$. Moreover, the first and last letter of d_j, e_j, f_j is equal to d, e, f, respectively. It is clear that if any member of F (not necessarily from W) is of the form (4.1) or (4.2) then this form is uniquely determined.

We also need certain numerical invariants of elements of W to proceed. For each factor v of $w \in F$ we define positive integers $s_w(v)$, $t_w(v)$. The first one is the position of the first letter of v in w, in other words: $l_{s_w(v)}(w) = l_1(v)$, according to the notation adopted in Section 2. The number $t_w(v)$ denotes the position of the last letter of v in w, which means that $l_{t_w(v)}(w) = l_{|v|}(v)$.

The sequence N_w is defined by the recurrence in which the consecutive elements $i_1, j_1, i_2, j_2, \ldots$ are defined. If at any step one of the numbers i_l or j_l turns out to be zero, for $l > 1$, then the recurrence terminates and all succeeding elements of N_w are, by definition, equal to zero.

The first step depends on which of the two numbers $s_w(d_1)$ and $t_w(a_1)$ is greater.

Case 1: Let $s_w(d_1) < t_w(a_1)$. Then we put $i_1 = 0$ and $j_1 = r$, where $r > 0$ is the largest integer such that $s_w(d_r) < t_w(a_1)$.

Case 2: Let $s_w(d_1) > t_w(a_1)$. Let $k > 0$ be the largest integer such that $t_w(a_k) < s_w(d_1)$. Then:

- If the word $(acb)^k$ is not a subword of w, then we put $i_1 = k - 1$, and we define j_1 as the maximal s such that $(fed)^s$ is a subword of w. Moreover, we put $i_2 = 0$, which (according to the above rules) terminates the entire recurrence.
- If the word $(acb)^k$ is a subword of w, then we put $i_1 = k$. Moreover, we also define j_1. If a_{i_1+1} is not trivial, then $j_1 = r$, where r is the largest number such that $s_w(d_r) < t_w(a_{i_1+1})$. In the opposite case j_1 is defined as a maximal m such that $(fed)^m$ is a subword of w.

Note that conditions (1)–(3) of the definition of W assert that $j_1 > 0$, no matter the case considered. Thus, if the recurrence has not already terminated, we define the steps for $l > 0$ in the following manner.

The defining step for i_{l+1} (this assumes that in the previous step we have established $j_l > 0$). If d_{j_l+1} is trivial, then $i_{l+1} = r - \sum_{m=1}^{l} i_m$, where r is the greatest integer such that $(acb)^r$ is a subword of w. If $d_{j_l+1} \neq 1$, then let k be the maximal integer such that $t_w(a_k) < s_w(d_{j_l+1})$. Then if $(acb)^k$ is not a subword of w, we put $i_{l+1} = k - \sum_{m=1}^{l} i_m - 1$ and $j_{l+1} = 0$. If, however, $(acb)^k$ is a subword w, then we put $i_{l+1} = k - \sum_{m=1}^{l} i_m > 0$.

The defining step for j_{l+1} (assuming that $i_{l+1} \neq 0$). If $a_{i_{l+1}+1}$ is not trivial, then j_{l+1} is defined as $r - \sum_{m=1}^{l} j_m$, where r is the greatest integer such that $s_w(d_r) < t_w(a_{i_{l+1}+1})$. In the opposite case, j_{l+1} is defined as $p - \sum_{m=1}^{l} j_m$, where p is the maximal integer such that $(fed)^p$ is a subword of w.

We will now show that the condition $w \approx w' \in W$ implies that the sequences $N_w = (i_1, j_1, \dots)$ and $N_{w'} = (i'_1, j'_1, \dots)$ are equal. In other words, we assume that $w = utv$ and $w' = ut'v$, where $v, w \in F$ and t, t' are as in one of the cases (a), (b), (c) listed above. Write w' in the forms (4.1) and (4.2):

$$w' = u'_1 a'_1 u'_2 c'_1 u'_3 b'_1 u'_4 a'_2 u'_5 \cdots , \tag{4.3}$$

$$w' = v'_1 f'_1 v'_2 e'_1 v'_3 d'_1 v'_4 f'_2 v'_5 \cdots , \tag{4.4}$$

with the appropriate conditions on a'_i, b'_i, c'_i, d'_i, e'_i, f'_i, u'_i, v'_i. We will confirm three observations:

(i) If in the form (4.1) of w we have $a_i \neq 1$ ($b_i \neq 1$, $c_i \neq 1$ respectively), then $a'_i \neq 1$ ($b'_i \neq 1$, $c'_i \neq 1$ respectively) in (4.3).
(ii) If in the form (4.2) of w have $d_i \neq 1$ ($e_i \neq 1$, $f_i \neq 1$ respectively), then $d'_i \neq 1$ ($e'_i \neq 1$, $f_i \neq 1$ respectively) in (4.4).
(iii) $s_w(d_i) < t_w(a_j) \Leftrightarrow s_{w'}(d'_i) < t_{w'}(a'_j)$, for all i, j.

It clearly follows from (i)–(iii) that the entire recurrence construction of N_w can be applied to w' leading to $N_w = N_{w'}$, as desired.

Consider the set $L(w)$ of all nontrivial factors $\{a_i, b_i, c_i, u_i\}$ of (4.1), for $i = 1, 2, \dots$ To simplify notation we assume that if z is the n-th element of $L(w)$ (by reading w from the left), then z' is the n-th element of the corresponding set of factors $L(w')$ of (4.3). By the symmetry of (i)–(iii) we may assume that $t < t'$ in the deg-lex order on F.

Assume that $\{t, t'\}$ is as in (a), namely $t = x$, for $x \in X$. Since t is in the support of some $z \in L(w)$, then $t' = xx$ is a factor of $z' \in L(w')$. Moreover $y = y'$, for all $z \neq y \in L(w)$. Thus (i) follows. By the dual argument applied to the factor set of (4.2) we verify (ii). And clearly, (iii) is satisfied. Indeed, if $s_w(d_i)$, $t_w(a_j)$ are less than or equal to $s_w(t)$, then their order cannot change when we pass to w'. In the opposite case both numbers increase by one, while we pass to w'.

If $\{t, t'\}$ is as in (b), we have $t = xy$, where $x \in \{a, b, c\}$ and $y \in \{d, e, f\}$, excluding the case $xy = ad$. To verify (i) we need to consider two possibilities. First: t is a factor, but not a suffix of x_i, for some i. Second: there exist i, k such that x is a suffix of $x_i \in L(w)$ and y is a prefix of u_k. In both cases $t' = yx$ is a factor of $x'_i \in L(w')$, while it is possible that u'_k is trivial. However, all elements of $L(w)$ except x_i and possibly u_k, remain unchanged when we pass to w'. So (i) must hold. By a dual argument one can verify (ii). Again, (iii) easily follows. If $x, y \notin \{a, d\}$ then the numbers $s_w(d_i)$, $t_w(a_j)$ remain unchanged while passing to w'. And if either $x \in \{a, d\}$, or $y \in \{a, d\}$, only one of these numbers can change by 1, since $xy \neq ad$.

In case (c), when $t = da$ or ada we will only prove (i) and (iii). Again, (ii) follows from dual arguments. We can see that t must be a factor of some factor $u_k a_i u_{k+1}$ of w, that appears in (4.1). Since $u_k, u_{k+1} \in \langle d, e, f \rangle$ we have two possible cases:

- $u_k a_i = (u\, \underbrace{d)(a}_{t}\, v)$, for some $u, v \in F$.
- t is a factor of a_i.

Consider the first case. Here, we have $t = da$, $u_k = ud$, $a_i = av$, for some $u, v \in F$ and the following three configurations are possible:

- $u'_k a'_i = (u)(\underbrace{ada}_{t'} v)$ and $x' = x$, for $x \in L(w)$, where $x \neq a_i, u_k$.
- $u'_k a'_i = (u\,\underbrace{d)(ad}_{t'}\,v)$ and $x' = x$, for $x \in L(w)$, where $x \neq a_i, u_k$ (when $v \neq 1$).
- $u'_k a'_i u'_{k+1} = (u\,\underbrace{d)(a)(d}_{t'}\,u_{k+1})$ and $x' = x$, for $x \in L(w)$, where $x \neq a_i, u_k, u_{k+1}$ (when $v = 1$).

In the second case, when t is a factor of a_i we have yet another five possible configurations:

- If $a_i = utv$, where $u, v \in F$ and $u, v \neq 1$, then $a'_i = ut'v$ and $x' = x$, for $x \in L(w)$, $x \neq a_i$.
- If $a_i = tv$, where $v \in F$, $v \neq 1$, then t must be equal to ada and we have $u'_k a'_i = (u_k\,\underbrace{d)(ad}_{t'}\,v)$ and $x' = x$, for $x \in L(w)$, $x \neq a_i, u_k$.
- If $a_i = ut$, where $u \in F$, $u \neq 1$, then we either have:
  - $a'_i = u\,\underbrace{ada}_{t'}$ and $x' = x$ for $x \in L(w)$, $x \neq a_i$, or
  - $a'_i u'_{k+1} = (u\,\underbrace{da)(d}_{t'}\,u_k)$ and $x' = x$, for $x \in L(w)$, $x \neq a_i, u_{k+1}$.
- If $a_i = t$, then $t = ada$ and we have $u'_k a'_i u'_{k+1} = (u_k\,\underbrace{d)(a)(d}_{t'}\,u_{k+1})$, and $x' = x$, for $x \in L(w)$, $x \neq a_i, u_k, u_{k+1}$.

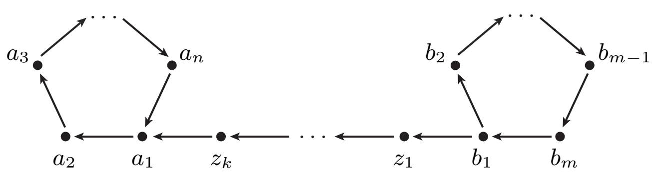
This gives a total of eight configurations (in both cases) and it is clear that (i) holds in each of them. Let us verify (iii). From the definition of W it follows that the elements a in the factors t and t' of w, w' appear in the same respective factors a_i and a'_i of (4.1) and (4.3). The same holds for elements d in the factors of (4.2), (4.4). Consequently, for all pairs (n, m) from the set $N := \{(s_w(d_p), t_w(a_q)),\ p, q = 1, 2, \ldots\}$ such that $n, m < \min\{t_w(a_i), s_w(d_j)\}$ and $n, m > \max\{t_w(a_i), s_w(d_j)\}$ the inequality between of n and m does not change while we pass to w'. Going though the configurations listed above we can easily verify (iii) for the remaining pairs of N.

As a result, we have shown that for two elements $w \approx w'$ from W we have $N_w = N_{w'}$. Thus the equality between these sequences occur also in the case when $w \sim w'$. Now we apply this fact to see that $\langle [acb], [fed] \rangle$ is a free submonoid of HK_Θ.

Let $w = (acb)^{m_1}(fed)^{n_1}(acb)^{m_2}(fed)^{n_2}\cdots$ belong to the monoid H for some nonnegative integers $\overline{w} = (m_1, n_1, m_2, n_2, \ldots)$. Here we assume that either we have $m_1 > 0$, $n_1 > 0$, or $m_1 = 0$, $n_1, m_2 > 0$ and that apart from these first few elements, the succeeding members of $\overline{w}$ are positive, up to some point, after which all of them become zeros. By carefully applying the definition of N_w, it is easy to that $N_w = \overline{w}$. Moreover, it is clear that if $w, w' \in \langle acb, fed\rangle$ and $w \sim w'$, then $\widetilde{w} = (acb)(fed)w \sim (acb)(fed)w' = \widetilde{w'}$ and $\widetilde{w}, \widetilde{w'} \in W$. Hence $N_{\widetilde{w}} = N_{\widetilde{w'}}$, which easily implies that $N_w = N_{w'}$. Hence, the images of all elements of the monoid $\langle acb, fed\rangle$ under the natural homomorphism $F \longrightarrow \mathrm{HK}_\Theta$ are pairwise distinct. The assertion of Example 4 follows. □

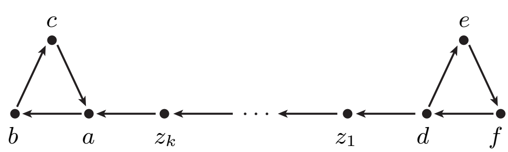
We continue with a generalization that is crucial for the proof of Theorem 1.

Example 5. Consider $\Theta_{n,k,m}$ to be a graph with $n+k+m$ vertices that consists of: a cycle Φ_n of length $n \geq 3$ of the form $a_1 \to a_2 \to \cdots \to a_n \to a_1$, a cycle Ψ_m of length $m \geq 3$ of the form $b_1 \to b_2 \to \cdots \to b_{m-1} \to b_m \to b_1$, and a path of length $k+1$ from b_1 to a_1, for $k \geq 0$. For $k = 0$ this means that we have $b_1 \to a_1$, and for $k > 0$ the path is of the form $b_1 \to z_1 \to \cdots \to z_k \to a_1$ as in the picture below:



Then the submonoid $\langle [a_1a_n\cdots a_2], [b_mb_{m-1}\cdots b_1z_1\cdots z_k]\rangle$ of the monoid $\mathrm{HK}_{\Theta n,k,m}$ is free.

Proof: We will first consider the case when $\Theta_{3,k,3} := \Theta_k$ is a graph of $6+k$ vertices and consists of two cycles $\{a, b, c\}$, $\{d, e, f\}$ joined by a path of length $k+1$, for $k \geq 0$, as shown in the picture (for $k > 0$):



Let us see that HK_{Θ_k} has a free submonoid of rank 2 generated by $[acb]$ and $[fedz_1\cdots z_k]$.

For each $k \geq 0$ consider the set $X_k = \{a, b, c, d, e, f, z_1, \ldots, z_k\}$ (when $k = 0$ this is just $\{a, b, c, d, e, f\}$). Let $F_k = \langle X_k\rangle$ be a free monoid on the set X_k. By $\sim_k$, $\approx_k$ we denote two relations on F_k defined in Section 2 for Θ_k. We proceed by induction on k. The first step has already been verified in Example 4. Let us then assume that the assertion holds for Θ_k, where $k \geq 0$.

Consider the subset W_{k+1} of F_{k+1} that consists of all words that are both $\{a, b, c\}$-free and $\{d, e, f\}$-free. As in the previous example it is not difficult to show that W_{k+1} is closed under equivalence classes of $\sim_{k+1}$ in F_{k+1}. If $w = rst$ and $v = rs't$, for $w \in W_{k+1}$, $v \in F_{k+1}$, and $s = s'$ is an edge relation in $\mathrm{HK}_{\Theta_{k+1}}$ then the order of appearance of elements from sets $\{a, b, c\}$ and $\{d, e, f\}$ cannot change while we pass from w to v. The possible edge relations are listed below:

(a) $\{s, s'\} = \{x, xx\}$, where $x \in X_{k+1}$.

(b) $\{s, s'\} = \{xy, yx \mid x \in \{a, b, c, z_1, \ldots, z_k\},\ y \in \{d, e, f, z_1, \ldots, z_k\}\}$ except for the cases when $x = y$ and when we have an arrow $x \to y$ in Θ_{k+1}.

(c) $\{s, s'\} = \{xz_{k+1}, z_{k+1}x \mid x \in X_{k+1} \setminus \{a, z_k\}\}$.

(d) $\{s, s'\} \subseteq \{xy, xyx, yxy \mid x = z_i, y = z_{i+1} \text{ for } 1 \leq i \leq k-1\}$ and $\{s, s'\} \subseteq \{dz_1, dz_1d, z_1dz_1\}$.

(e) $\{s, s'\} \subseteq \{z_kz_{k+1}, z_kz_{k+1}z_k, z_{k+1}z_kz_{k+1}\}$.

(f) $\{s, s'\} \subseteq \{z_{k+1}a, z_{k+1}az_{k+1}, az_{k+1}a\}$.

Let $\overleftarrow{w} \in F_{k+1}$ arise from the word $w \in F_{k+1}$ by applying all possible commutation edge relations (that is, of types (b), (c) above) of $\mathrm{HK}_{\Theta_{k+1}}$ in order to move all elements a, b, c as far to the left as possible. This is obviously well defined. We also define an element $\overleftarrow{w}_{z_{k+1}}$ in F_k, for every $w \in F_{k+1}$, as the maximal subword v of $\overleftarrow{w}$ such that $\deg_{z_{k+1}} v = 0$. To simplify notation put $z := z_{k+1}$. Consider a map $\phi_{k+1} \colon [W_{k+1}] \longrightarrow \mathrm{HK}_{\Theta_n}$, that associates to the class $[w]$ of an element $w \in W_{k+1}$ the class of the element $\overleftarrow{w}_z$. We will show that ϕ_{k+1} is a function. This clearly amounts to showing that

$$w \approx_{k+1} v \Rightarrow \overleftarrow{w}_z \sim_k \overleftarrow{v}_z. \tag{4.5}$$

Consider the cases (a)–(f) listed above. The implication in (4.5) is obvious in cases (b) and (d), as we have $\overleftarrow{w} = \overleftarrow{v}$ and thus $\overleftarrow{w}_z = \overleftarrow{v}_z$. Also in case (c) it is clear that $\overleftarrow{w}_z = \overleftarrow{v}_z$. The remaining cases require more detailed explanation.

In case (a) we have $w = rxt$ and $v = rxxt$, for some $x \in X_{k+1}$. When $x = z$ it is clear that $\overleftarrow{w}_z = \overleftarrow{v}_z$. If $x \neq z$ then $\overleftarrow{w} = r'xt'$ and $\overleftarrow{v} = r'xxt'$, for some $r', t' \in F_{k+1}$. Thus $\overleftarrow{w}_z \approx_k \overleftarrow{v}_z$.

In case (e) the following three subcases arise:

- Let $w = rz_kzt$, $v = rzz_kzt$. Then clearly $\overleftarrow{w} \approx_{k+1} \overleftarrow{v}$ with the edge relation $z_kz = zz_kz$, hence $\overleftarrow{w}_z = \overleftarrow{v}_z$.
- Let $w = rz_kzt$, $v = rz_kzz_kt$. Then $\overleftarrow{w} = r'z_kza^nt'$, $\overleftarrow{v} = r'z_kza^nz_kt'$, for some $n \geq 0$ and $r', t' \in F_{k+1}$. Since we have $z_ka^n \sim_k z_ka^nz_k$ then $\overleftarrow{w}_z \sim_k \overleftarrow{v}_z$.
- Finally, if $w = rzz_kzt$ and $v = rz_kzz_kt$, then we proceed analogously as in the previous subcase.

In case (f) we also consider three subcases:

- Let $w = rzat$, $v = rzazt$. Then $\overleftarrow{w} = \overleftarrow{r}zaa^nt'$ and $\overleftarrow{v} = \overleftarrow{r}zaa^nzt'$, with $\overleftarrow{t} = a^nt'$, for some $t' \in F_{k+1}$ and $n \geq 0$. Hence $\overleftarrow{w}_z \sim_k \overleftarrow{v}_z$.
- Let $w = rzat$, $v = razat$. Then $\overleftarrow{w}_z = \overleftarrow{r}_z a \overleftarrow{t}_z$ and $\overleftarrow{v}_z = (\overleftarrow{ra})_z a \overleftarrow{t}_z$. Notice that $(\overleftarrow{ra})_z = paq$ and $\overleftarrow{r} = pq$, for some $p \in F_{k+1}$, $q \in \langle d, e, f, z_1, \ldots, z_k\rangle$. If $\deg_{z_k} q = 0$, then $aq \sim_k qa$ and thus $\overleftarrow{w}_z \sim_k \overleftarrow{v}_z$. If, however, $\deg_{z_k} q > 0$ then let $q = t_1z_k^{i_1}t_2z_k^{i_2} \cdots z_k^{i_l}t_{l+1}$, for some positive integers i_j, l, and $t_i \in \langle d, e, f, z_1, \ldots, z_{k-1}\rangle$. We only need to show that $aqa \sim_k qa$. This is clear, since for $i > 0$ and for any $x \in \langle d, e, f, z_1, \ldots, z_{k-1}\rangle$ we have $z_k^ixa \sim_k z_kax \sim_k az_kax \sim_k az_kxa \sim_k az_k^ixa$. Thus $\overleftarrow{w}_z \sim_k \overleftarrow{v}_z$.
- Finally, if $w = rzazt$, $v = razat$ we proceed analogously as in the previous subcase.

We have proved (4.5). Thus the assignment $w \mapsto \overleftarrow{w}_z$ determines a function ϕ_{k+1} from the subset $[W_{k+1}]$ of all classes of words in W_{k+1} with respect to $\mathrm{HK}_{\Theta_{k+1}}$ to the monoid HK_{Θ_k}. Clearly, the class of any element of $\langle acb, fedz_1 \cdots z_kz_{k+1}\rangle$ in $\mathrm{HK}_{\Theta_{k+1}}$ is mapped to a unique class of an element of the submonoid $\langle acb, fedz_1 \cdots z_k\rangle$, which is free in HK_{Θ_k} by the induction hypothesis. Thus, also the submonoid $\langle [acb], [fedz_1 \cdots z_kz_{k+1}]\rangle$ of $\mathrm{HK}_{\Theta_{k+1}}$ must be free.

We are now ready to prove the general statement of this example. Assume that the assertion is satisfied for some $\Theta_{n,k,m}$, with $n, m \geq 3$, $k \geq 1$. Let $F = \langle V(\Theta_{n,k,m})\rangle$ and $F' = \langle V(\Theta_{n+1,k,m})\rangle$ be free monoids. Consider the set W of all words in F' that are both Φ_{n+1}-free and Ψ_m-free. As in the case of graphs Θ_k one can easily argue that W is closed with respect to equivalence classes of $\sim_{n+1,k,m}$. The remaining argument is similar to the one used in the special case $n = m = 3$. This time we only need to assign to every word $w \in W$ a maximal subword v such that $\deg_{a_{n+1}} w = 0$. This yields a function from the subset of the monoid $\mathrm{HK}_{\Theta_{n+1,k,m}}$ to the monoid $\mathrm{HK}_{\Theta_{n,k,m}}$ such that the classes of elements of the monoid $\langle a_1a_{n+1}a_n \cdots a_2, b_mb_{m-1} \cdots b_1z_1 \cdots z_k\rangle$ are mapped to unique classes of the monoid $\langle a_1a_n \cdots a_2, b_mb_{m-1} \cdots b_1z_1 \cdots z_k\rangle$, which is free by the induction hypothesis. The assertion for $\Theta_{n+1,k,m}$ follows.

By symmetric arguments we can verify the freeness property of the appropriate submonoid of $\mathrm{HK}_{\Theta_{n,k,m+1}}$. Since we already know that the assertion follows for all graphs $\Theta_{3,k+1,3}$, with $k \geq 0$, the general assertion follows. □

Here is a remark. One might try to use an approach based on the diamond lemma in order to find the normal form of all elements of HK_Θ. However, this typically leads to infinite Gröbner bases. For instance, in Example 4 it is possible to introduce the following system S of reductions in $\mathrm{K}\langle a,b,c,d,e,f\rangle$ with respect to the natural deg-lex order (see [**5**] for terminology and the necessary background):

(1) (aa,a),	(2) (bb,b),	(3) (cc,c),	(4) (dd,d),
(5) (ee,e),	(6) (ff,f),	(7) (aba,ab),	(8) (bab,ab),
(9) (aca,ca),	(10) (cac,ca),	(11) (bcb,bc),	(12) (cbc,bc),
(13) (dad,da),	(14) (ada,da),	(15) (ded,de),	(16) (ede,de),
(17) (dfd,fd),	(18) (fdf,fd),	(19) (efe,ef),	(20) (fef,ef),
(21) (ea,ae),	(22) (fa,af),	(23) (db,bd),	(24) (eb,be),
(25) (fb,bf),	(26) (dc,cd),	(27) (ec,ce),	(28) (fc,cf),

(29) $(bwdab,wdab)$, (30) $(cwdac,cwda)$,

$$\text{for } w \in \{1,f,ef\}\cdot\langle def\rangle \cup \{1,e,fe\}\cdot\langle dfe\rangle,$$

(31) $(edawe,dawe)$, (32) $(fdawf,fdaw)$,

$$\text{for } w \in \langle cba\rangle\cdot\{1,c,cb\} \cup \langle bca\rangle\cdot\{1,b,bc\},$$

(33) $(awda,wda)$,

$$\text{for } w \in \{1,c\}\cdot\{1,f,ef\}\cdot\langle def\rangle \cup \{1,c\}\cdot\{1,e,fe\}\cdot\langle dfe\rangle,$$

(34) $(dawd,daw)$,

$$\text{for } w \in \langle bca\rangle\cdot\{1,b,bc\}\cdot\{1,e\} \cup \langle cba\rangle\cdot\{1,c,cb\}\cdot\{1,e\}.$$

Then one can show, though through a quite involved and detailed case-by-case investigation, that all ambiguities that arise from the system S (more than 150) are in fact resolvable. This of course implies that different elements of the monoid $\langle acb, fed\rangle$ belong to different equivalence classes of $\sim$. This approach, however, seems inapplicable to the general case of graphs $\Theta_{n,k,m}$, considered in Example 5.

5. The proof of Theorem 1

With all previous examples studied, we are ready to prove the main result of this paper.

Proof of Theorem 1: Let $F = \langle V(\Theta)\rangle$. The implication (1) ⇒ (2) is proved by induction on $r := |V(\Theta)|$. If $|V(\Theta)| \leq 2$ then Θ is acyclic and obviously the monoid HK_Θ is finite, hence A_Θ is a PI-algebra. We proceed to the inductive step. If Θ_1 is a connected component of Θ and $\Theta_2 = \Theta \setminus \Theta_1 \neq \emptyset$, then HK_Θ is a direct product of HK_{Θ_1} and HK_{Θ_2}, so that $A_\Theta = A_{\Theta_1} \otimes A_{\Theta_2}$. Therefore, A_Θ is a PI-algebra by Theorem 6.1.1 in [**17**], because so are the algebras A_{Θ_i}, $i = 1, 2$, by the induction hypothesis. Thus, we may assume that Θ is a connected graph.

Suppose that Θ has a source vertex. Denote it by a. We know that $awa = aw$, for each $w \in \mathrm{HK}_\Theta$, see Remark 5. Let Θ' arise from Θ by removing the vertex a and all arrows that originate from a. We can assume that $A_{\Theta'} \subseteq A_\Theta$ and therefore:

$$A_\Theta = A_{\Theta'} + A_{\Theta'} a A_{\Theta'} = A_{\Theta'} + \underbrace{a(A_{\Theta'} a A_{\Theta'})}_{aA_{\Theta'}} + (1-a) A_{\Theta'} a A_{\Theta'}.$$

The summand $I := (1-a)A_{\Theta'} a A_{\Theta'}$ is clearly an ideal in A_Θ and $I^2 = 0$. Let $\overline{x}$ be the image of $x \in A_\Theta$ under the natural map $A_\Theta \longrightarrow A_\Theta / I$. Then $\overline{A_\Theta} = \overline{A_{\Theta'}} + \overline{aA_{\Theta'}}$, which is a finitely generated right $\overline{A_{\Theta'}}$-module. By the inductive hypothesis we may assume that $\overline{A_{\Theta'}}$ (in fact isomorphic to $A_{\Theta'}$) is a PI-algebra, so from [**14**, Corollary 13.4.9], it follows that $\overline{A_\Theta}$ is PI as well. This, of course, implies that A_Θ is PI. The same argument works in the case when a is a sink vertex.

Therefore, we may assume that Θ neither contains sink nor source vertices. Then, by condition (1), it must be a cycle. Thus, by Example 2 we have that A_Θ is a PI-algebra, see [**18**], which yields the first implication.

The implication (2) $\Rightarrow$ (3) is well known, see [**12**, Corollary 10.7]. The implication (3) $\Rightarrow$ (4) is clear. So, we are left with proving (4) $\Rightarrow$ (1). Assume, contradictory to (1), that the graph Θ contains two cycles that are connected by an oriented path. Namely, we have two different sets of vertices $\mathcal{A} = \{a_1, \dots, a_n\}$ and $\mathcal{B} = \{b_1, \dots, b_m\}$ of Θ such that the induced subgraphs Θ_1, Θ_2 of Θ with the respective vertex sets $\mathcal{A}$, $\mathcal{B}$ are cycles of the form $a_1 \to a_2 \to \cdots \to a_n \to a_1$ and $b_1 \to b_2 \to \cdots \to b_m \to b_1$. If $\mathcal{A} \cap \mathcal{B} \neq \emptyset$, then assume that this intersection contains an element c. We can assume that $c = a_1 = b_1$ (perhaps we need to renumber a_i, b_j). By the same arguments as in Example 3 we argue that all elements of the submonoid of F generated by $a = a_n a_{n-1} \cdots a_1$ and $b = b_m b_{m-1} \cdots b_1$ are essentially non-rewritable in HK_Θ, except for the edge relations of the form $x = xx$. Thus $\langle [a], [b] \rangle$ is a free submonoid in HK_Θ, which contradicts (3). If we are in the case when $\mathcal{A} \cap \mathcal{B} = \emptyset$, we know that Θ contains a subgraph of the form $\Theta_{n,k,m}$ introduced in Example 5, and thus it is clear that the free submonoid of $\mathrm{HK}_{\Theta_{n,k,m}}$ is a submonoid of HK_Θ. This again contradicts (4) and the proof is complete. $\square$

We conjecture that GKdim(A_Θ) always is an integer if it is finite. More generally, one might ask whether A_Θ is an automaton algebra in the sense of Ufnarovskii [**22**] for any Hecke–Kiselman monoid HK_Θ and whether Theorem 1 can be generalized to the entire class of such monoids.

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