# THE ARBOREAL APPROACH TO PAIRS OF INVOLUTIONS IN RANK TWO

JAUME AGUADÉ

## 1. Introduction

We consider representations of the infinite dihedral group $D_\infty$ in $GL_2(\mathbb{Z}_p)$ ($\mathbb{Z}_p$ is the ring of $p$-adic integers for a chosen prime $p$). Each of these representations is given by a pair of involutions $\sigma_1$, $\sigma_2$ up to conjugation. These representations were classified in [1] using some numerical invariants which were introduced in that paper in a completely formal way. Actually, these invariants appeared in a natural way in the computations of the mod $p$ cohomology of the classifying spaces of rank two Kac-Moody groups and some related spaces as discussed in [2], but the proofs in [1] are independent of all the topological machinery in [2]. The interested reader may read section 7 in [1] for a quick overview of the relationship between representations of $D_\infty$ and rank two Kac-Moody groups.

In the present paper we provide a new classification of the representations of $D_\infty$ in $GL_2(\mathbb{Z}_p)$ and new proofs for the classification theorems in [1]. The proofs that we present here are simpler and more illuminating than the proofs in [1]. These new proofs are geometrical, non-computational, and more complete than the rather abstract original proofs. The key idea here is to use the geometry of the tree of $GL_2(\mathbb{Z}_p)$ as studied in the classic book [6]. In this way we obtain geometrical interpretations of the algebraic invariants that we introduced in [1] and all the theory in that paper fits into a much sharper and simpler picture.

## 2. The tree of $GL_2(\mathbb{Z}_p)$

In this section we review the tree of $GL_2(\mathbb{Z}_p)$ as it is described in [6], Ch. II, §1, we fix some notation that will be used along this paper and we state a couple of technical results that we need later.

Let $V$ be a $\mathbb{Q}_p$-vector space of dimension two. On the set of all rank two $\mathbb{Z}_p$-lattices in $V$ we identify $L \sim \alpha L$ for any unit $\alpha \in \mathbb{Q}_p$. Let $X$

The author is partially supported by grants MTM204-06686, SGR2005-00606 and PR2007-0097.

be the quotient set. There is a well defined *distance* in $X$ such that two (equivalence classes of) lattices $L$, $L'$ are at distance $n$ if and only if there is a sublattice $L''$ of $L$ such that $L'' \sim L'$ and $L/L''$ is a cyclic group of order $p^n$. Then, if we join by an edge any two points of $X$ which are at distance one, one can show that the graph $X$ is indeed a *tree* and the distance in $X$ coincides with the graph-theoretic distance.

Let us choose a base point $[L_0] \in X$. Then it is easy to see that the points at distance $n$ from $[L_0]$ are in one-to-one correspondence to points in the projective line $P_1(\mathbb{Z}/p^n\mathbb{Z})$. In particular, points at distance one are in one-to-one correspondence to points in the projective line $P_1(\mathbb{Z}/p\mathbb{Z})$ and this implies that exactly $p+1$ edges meet at each vertex of $X$. By letting $n$ grow to infinity, we see that the set $X_\infty$ of *ends* of $X$ can be identified to $P(L_0) \cong P(V)$. More precisely, this correspondence sends a line $R \in P(V)$ to the end of the path with vertices $[L_n]$

$$L_n = p^n L_0 + (L_0 \cap R).$$

From now one, let us choose a basis in $V$ to identify $V = \mathbb{Q}_p^2$. Then we can take $L_0 = \mathbb{Z}_p^2 \subset \mathbb{Q}_p^2$. For any $\{a, b\} \in P_1(\mathbb{Q}_p)$ we can choose a representative such that $a, b \in \mathbb{Z}_p$ and at least one of the coordinates is a unit. In this case, the path from $[L_0]$ to the end $\{a, b\} \in X_\infty$ is given by the sequence with vertices $\{[L_n^{a,b}],\ 0 \leq n < \infty\}$ with

$$L_n^{a,b} = p^n L_0 + \mathbb{Z}_p(a, b).$$

We leave as an exercise to the reader the proof of the next result:

**Proposition 1.** *Let $\omega$, $\omega'$ be the unique irreducible paths starting at $[L_0]$ and pointing to the (different) points $\{1, 0\}$, $\{a, b\}$ in $X_\infty = P_1(\mathbb{Q}_p)$, respectively. Then $\omega \cap \omega'$ consists of exactly $n$ edges, where $n = \max\{\nu_p(b/a), 0\}$.* □

There is a natural action of $GL(V)$ on $X$ via graph automorphisms, $\Phi : GL(V) \to \mathrm{Aut}(X)$. When we pass to the set of ends $X_\infty$ this action becomes the natural projective action of $GL(V)$ on $P(V)$ and the kernel of $\Phi$ is the center of $GL(V)$. We are interested in the isotropy groups of vertices of $X$:

**Proposition 2.** *If $\phi \in GL_2(\mathbb{Q}_p)$ fixes $[L_0] \in X$, then $\phi$ has the form $\alpha M$ for some $\alpha \in \mathbb{Q}_p^*$ and $M \in GL_2(\mathbb{Z}_p)$.*

*Proof.* $\phi L_0 \subset \alpha L_0$ for some $\alpha \in \mathbb{Q}_p^*$ implies $\phi = \left(\begin{smallmatrix} \alpha u & \alpha u' \\ \alpha v & \alpha v' \end{smallmatrix}\right)$ for $u, v, u', v' \in \mathbb{Z}_p$. $\phi^{-1}\alpha L_0 \subset L_0$ implies that $uv' - vu' \not\equiv 0\,(p)$. □

Hence, if we restrict to elements of $GL(V)$ whose determinant is a unit in $\mathbb{Z}_p$, then the isotropy group of the base point $[L_0]$ is $GL(L_0) =$

$GL_2(\mathbb{Z}_p)$. This fact is used in this paper in the following way: assume we have two matrices $A$, $B \in GL_2(\mathbb{Z}_p)$ which are conjugate in $GL_2(\mathbb{Q}_p)$ by a matrix $P$. Then $P$ induces an automorphism of $X$ and if we are able to show (e. g. by a geometric argument) that this automorphism fixes the vertex $[L_0] \in X$, then $P$ can be chosen in $GL_2(\mathbb{Z}_p)$ and the matrices $A$, $B$ are conjugate in $GL_2(\mathbb{Z}_p)$. This basic idea is the leitmotiv of this paper.

## 3. Pairs of involutions over a field

In this section we classify the irreducible representations of $D_\infty$ in $GL_2(K)$ for any field $K$ of characteristic different from 2. We obtain geometrical interpretations of the results of section 6 of [1] and of the papers [3], [5].

Any non-central involution $\sigma$ in $GL_2(K)$ is conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and so $\sigma$ determines two lines $\sigma^+$, $\sigma^-$ in $K^2$ which correspond to the eigenspaces of eigenvalues $+1$ and $-1$, respectively. Hence, an irreducible representation $\rho = \langle \sigma_1, \sigma_2 \rangle$ of $D_\infty$ in $GL_2(K)$ produces four points $\sigma_1^+$, $\sigma_1^-$, $\sigma_2^+$, $\sigma_2^-$ in the projective line $P_1(K)$, uniquely determined up to an automorphism of $P_1(K)$. Notice that these four points are only subjected to the restrictions $\sigma_1^+ \neq \sigma_1^-$, $\sigma_2^+ \neq \sigma_2^-$ and $\{\sigma_1^+, \sigma_1^-\} \neq \{\sigma_2^+, \sigma_2^-\}$.

Let us define $\Gamma(\rho)$ to be the cross-ratio

$$\Gamma(\rho) = (\sigma_2^+, \sigma_1^+, \sigma_1^-, \sigma_2^-).$$

Recall that given four points $a, b, c, d$ in a projective line, $a, b, c$ distinct, the cross-ratio $(a, b, c, d)$ is defined as the element $x \in K \cup \{\infty\}$ such that there is an element in $PGL_2(K)$ which sends $a, b, c, d$ to $\infty, 0, 1, x$, respectively. The property of $a, b, c$ being distinct can be weakened by defining $(a, a, c, d) = 1$, $(a, b, a, d) = 0$, $(a, b, b, d) = \infty$.

In our case, we always have $\Gamma(\rho) \neq \infty$ and we have $\Gamma(\rho) \neq 0, 1$ if and only if the four points $\sigma_1^+$, $\sigma_1^-$, $\sigma_2^+$, $\sigma_2^-$ are all different. $\Gamma$ provides a surjection from the set of irreducible representations $D_\infty \to GL_2(K)$ into $K$. Also, if $\Gamma \neq 0, 1$ then the cross ratio is the only projective invariant of these four points, and $\Gamma(\rho) = \Gamma(\rho') \neq 0, 1$ implies $\rho \sim \rho'$. For each value of $\Gamma$ in $\{0, 1\}$ there are two non-equivalent representations. Let us introduce a new invariant to distinguish between them:

$$\tau = \begin{cases} 1 & \text{if } \sigma_1^- \in \{\sigma_2^+, \sigma_2^-\} \\ 0 & \text{if } \sigma_1^+ \in \{\sigma_2^+, \sigma_2^-\} \end{cases}$$

Then, the invariants $\Gamma \in K$, $\tau \in \mathbb{Z}/2$ provide a complete classification of the irreducible representations of $D_\infty$ in $GL_2(K)$.

A generic representation $\rho$ is given by the two involutions $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\sigma_2 = M^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) M$ for some matrix $M = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \in GL_2(K)$. Then, it is straightforward to compute the invariants $\Gamma(\rho)$, $\tau(\rho)$ as functions of the entries $x$, $y$, $z$, $t$. We get:

$$\Gamma(\rho) = \frac{xt}{xt - yz}$$

$$\tau(\rho) = \begin{cases} 0 & xz = 0 \\ 1 & xz \neq 0 \end{cases}$$

and so we recover the invariants of section 6 of [1].

## 4. Integral representations: the regular case

From now on we consider representations $\rho$ of $D_\infty$ in $GL_2(\mathbb{Z}_p)$ given by two involutions $\sigma_1, \sigma_2 \in GL_2(\mathbb{Z}_p)$. Up to conjugation, the elements of order two in $GL_2(\mathbb{Z}_p)$ are $\pm I$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Moreover, these last two elements are conjugate if $p$ is odd. In this section we assume that the two involutions $\sigma_1$, $\sigma_2$ are both equivalent (in $GL_2(\mathbb{Z}_p)$) to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We call this the *regular* case (it was called the $\mathrm{Rep}_{1,1}$ case in [1]). For an odd prime $p$, all irreducible representations are regular.

As in the previous section, $\rho$ determines four points $\sigma_1^+$, $\sigma_1^-$, $\sigma_2^+$, $\sigma_2^-$ in $P_2(\mathbb{Q}_p)$. Recall from section 2 that the points of $P_2(\mathbb{Q}_p)$ are identified to the ends of the tree $X$ which was defined there. It makes sense to consider the *geodesic* path joining two ends. By this we mean a $\mathbb{Z}$-indexed sequence of consecutive edges which points to these two ends and does not have any repeated edge. Such a sequence is unique up to translations in the indexing set.

Let $\Sigma_1$, $\Sigma_2$ be the geodesic paths joining $\sigma_1^+$ to $\sigma_1^-$ and $\sigma_2^+$ to $\sigma_2^-$, respectively. Since $\sigma_1$ and $\sigma_2$ are both (integrally) conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and since we saw in section 2 that the geodesic path joining $\{1, 0\}$ to $\{0, 1\}$ passes through the vertex $[L_0]$, we have that also $\Sigma_1$, $\Sigma_2$ pass through $[L_0]$. These two paths will then coincide along some edges on both sides of $[L_0]$ and eventually diverge and never meet again.

This observation allows us to introduce two discrete invariants $\alpha$, $\beta$ of the representation $\rho$:

$$\alpha(\rho) = \#\{\text{edges of } \Sigma_1 \cap \Sigma_2 \text{ between } [L_0] \text{ and } \sigma_1^+\}, \quad 0 \leq \alpha(\rho) \leq \infty$$

$$\beta(\rho) = \#\{\text{edges of } \Sigma_1 \cap \Sigma_2 \text{ between } [L_0] \text{ and } \sigma_1^-\}, \quad 0 \leq \beta(\rho) \leq \infty$$

Conjugation in $GL_2(\mathbb{Z}_p)$ gives an automorphism of $X$ leaving $[L_0]$ fixed. Since $\alpha$ and $\beta$ are defined in a graph-theoretical way, they depend only on the representation class of $\rho$. If the points $\sigma_1^+$, $\sigma_1^-$, $\sigma_2^+$, $\sigma_2^-$ are all different, then $\alpha$ and $\beta$ are both non-negative integers while if they are not all different, then $\alpha$ or $\beta$ may have the value $\infty$.

We can also consider the invariant $\Gamma(\rho)$ which we introduced in the previous section. The computations we made there show that if $\rho$ is integral, then $\Gamma(\rho) \in \mathbb{Z}_p$.

**Theorem 3.** *$\Gamma$, $\alpha$ and $\beta$ are a complete set of invariants for the representations of $D_\infty$ in $GL_2(\mathbb{Z}_p)$ in the regular case.*

*Proof.* We mean that two representations are equivalent if and only if the invariants $\Gamma$, $\alpha$ and $\beta$ are the same for both. This follows easily from the properties of the tree $X$. Assume $\rho$ and $\rho'$ are two irreducible representations with the same invariants. First of all, we have that $\rho$ is conjugate to $\rho'$ in $GL_2(\mathbb{Q}_p)$. This is clear if $\Gamma \neq 0, 1$ since we saw that in this case $\Gamma$ classifies the representations over $\mathbb{Q}_p$. Also, if $\Gamma = 0, 1$ we need a further invariant $\tau$ to classify the representations, but it is clear that the value of $\tau$ can be deduced from the values of $\alpha$, $\beta$.

Hence, there is a matrix $P \in GL_2(\mathbb{Q}_p)$ which conjugates $\rho$ to $\rho'$. Since we can always multiply $P$ by any central matrix, we can assume that $P$ has entries in $\mathbb{Z}_p$. Then $P$ gives an automorphism of the graph $X$ which sends the geodesic paths $\Sigma_1$, $\Sigma_2$ to the corresponding geodesic paths $\Sigma_1'$, $\Sigma_2'$. The equality of the invariants $\alpha$, $\beta$ for both $\rho$ and $\rho'$ implies that this automorphism must fix the vertex $[L_0]$. Hence, by proposition 2, $P$ can be chosen in $GL_2(\mathbb{Z}_p)$. $\square$

Let us finish this section by effectively computing the invariants that we have introduced above. Consider the generic representation $\rho$ given by the two involutions $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\sigma_2 = M^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) M$ for some matrix $M = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_p)$. Then we saw in the previous section that the invariant $\Gamma$ is given by

$$\Gamma(\rho) = \frac{xt}{xt - yz}.$$

The values of $\alpha$ and $\beta$ can be deduced from the computation in proposition 1. If we let $a = \nu_p(z/t)$, $b = \nu_p(y/x)$, then proposition 1 gives $\alpha$ and $\beta$ as functions of $a$ and $b$ in the following way: If both $a, b \geq 0$, then $\alpha = a$, $\beta = b$. If any of $a, b$ is negative, then both $a$ and $b$ are negative and $\alpha = -b$, $\beta = -a$. Then, from the fact that $M$ must be invertible in $GL_a(\mathbb{Z}_p)$ we deduce

$$\alpha(\rho) = \nu_p(xz),$$
$$\beta(\rho) = \nu_p(yt).$$

We have recovered theorem 1 in [1]. From here it is not difficult to see that the range of the invariants $\Gamma$, $\alpha$, $\beta$ is $\mathbb{Z}_p \times \{0, 1, \ldots, \infty\}^2$, subject only to the restriction:

$$\alpha+\beta=\nu_p(\Gamma)+\nu_p(\Gamma-1).$$

Moreover, from this geometric interpretation of the invariants $\alpha$ and $\beta$ one can easily produce a complete list of representatives as in table 1 of [1].

## 5. Integral representations: the irregular case

In this section $p=2$ and we consider representations $\rho$ of $D_\infty$ in $GL_2(\mathbb{Z}_2)$ given by two involutions $\sigma_1,\sigma_2$ which are both conjugate (in $GL_2(\mathbb{Z}_2)$) to $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. We call these representations *irregular*. As in the previous section, let $\Sigma_i$ be the geodesic path in $X$ joining $\sigma_i^+$ and $\sigma_i^-$, $i=1,2$. Let $\Gamma(\rho)\in\mathbb{Q}_2$ be the invariant introduced in section 3. If we write $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma_2=M^{-1}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)M$ for some matrix $M=\left(\begin{smallmatrix}x&y\\z&t\end{smallmatrix}\right)\in GL_2(\mathbb{Z}_2)$ then an easy calculation shows $\Gamma(\rho)=(\Gamma^*(\rho)+2)/4$ for a new invariant

$$\Gamma^*(\rho)=\frac{x^2-y^2-z^2+t^2}{xt-yz}\in\mathbb{Z}_2.$$

Since $p=2$, in the tree $X$ there are now exactly three edges meeting at each vertex. The vertices at distance one from our base point $[L_0]$ are $[L_1^{1,0}]$, $[L_l^{0,1}]$ and $[L_1^{1,1}]=[L_1^{1,-1}]$ where we have used the notation that we introduced in section 2. Notice that

$$L_1^{1,1}=L_1^{1,-1}=\{(a,b)\in L_0\mid a\equiv b\,(2)\}.$$

If we consider now the irreducible paths starting at the base point $[L_0]$ and pointing to $\{1,1\}$ and $\{1,-1\}$ respectively, we easily see that these two paths have only one common edge. Then, the geodesic path in $X$ which joins the ends $\{1,1\}$ and $\{1,-1\}$ contains the vertex $[L_1^{1,1}]$ but does not contain the vertex $[L_0]$. Hence, the geodesic path which joins the ends $\sigma^+$ and $\sigma^-$ for any involution $\sigma$ which is conjugate to $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ does not contain the vertex $[L_0]$ but contains only one vertex at distance one from $[L_0]$.

**Proposition 4.** *For each odd $\lambda\in\mathbb{Z}_2$ there is exactly one irregular representation $\rho$ of $D_\infty$ in $GL_2(\mathbb{Z}_2)$ with $\Gamma^*(\rho)=\lambda$.*

*Proof.* Assume $\rho$ is a representation with $\Gamma(\rho)$ odd and let $\sigma_i^+$, $\sigma_i^-$, $M$, $\Sigma_i$ be as above. The ends of $\Sigma_2$ are $\sigma_2^+=\{t-y,x-z\}$ and $\sigma_2^-=\{t+y,-x-z\}$. Then $\Gamma^*(\rho)\equiv(t+y)+(z+x)\,(2)$ and since $\Gamma^*(\rho)$ is odd one of $t+y$, $z+x$ is a unit and the other is not. Hence, $\Sigma_2$ contains the vertex $[L_1^{t-y,x-z}]$ which is either $[L_1^{1,0}]$ or $[L_1^{0,1}]$. This implies that $\Sigma_2$ cannot intersect $\Sigma_1$. We have proved that if $\Gamma^*(\rho)$ is odd, then the geodesic paths $\Sigma_1$, $\Sigma_2$ are disjoint.

Then, if $\Gamma^*(\rho) = \Gamma^*(\rho')$ is odd, then $\rho$ and $\rho'$ are both irreducible and equivalent in $GL_2(\mathbb{Q}_p)$. Let $P \in GL_2(\mathbb{Q}_p)$ be such that it conjugates $\rho$ into $\rho'$. Without loss of generality we can assume that $P$ has entries in $\mathbb{Z}_p$. Then $P$ gives an automorphism of the graph $X$ which sends the geodesic paths $\Sigma_1$, $\Sigma_2$ to the corresponding geodesic paths $\Sigma'_1$, $\Sigma'_2$. Since $[L_0]$ is the only vertex of $X$ at distance one from both $\Sigma_1$ and $\Sigma_2$, $P$ must fix $[L_0]$. Hence, $P \in GL_2(\mathbb{Z}_2)$ and $\rho$ is conjugate to $\rho'$.

By taking $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ we prove the existence of at least one representation $\rho$ with $\Gamma^*(\rho) = \lambda$. □

Let us assume now that $\Gamma^*(\rho)$ is even. In this case, $x + y \equiv z + t \,(2)$ and since the case $x + y \equiv z + t \equiv 0 \,(2)$ would imply $M \equiv \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ (2) which is absurd, we have that both $x+y$ and $z+t$ are odd. Hence, the geodesic paths $\Sigma_1$, $\Sigma_2$ both contain the vertex $[L_1^{1,1}]$. This means that $\rho$ can be considered as a representation on $L_1^{1,1}$ of *regular* type and, as a representation on $L_1^{1,1}$, it is classified by the two invariants $\alpha$, $\beta$ of section 4. Notice that the values $\alpha = 0$ or $\beta = 0$ are not possible, because in $X$ there are only three edges meeting at each vertex. We observe now that this classification over $L_1^{1,1}$ gives also a classification over $L_0$. More precisely, if $P \in GL_2(\mathbb{Q}_p)$ fixes $L_1^{1,1}$ and conjugates two irregular representations with $\Gamma^*$ even, then one easily sees that $P$ must fix $[L_0]$ and so these two representations are equivalent.

To determine the values of $\alpha$, $\beta$ out of $x, y, z, t$ we notice that on the standard basis of $L_1^{1,1}$ the representation $\rho$ is given by the two involutions $\bar{\sigma}_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\bar{\sigma}_2 = Q^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) Q$ with $Q = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)^{-1} M \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Then we obtain, for this induced representation $\bar{\rho}$:

$$\begin{aligned} \alpha(\bar{\rho}) + 2 &= \nu_2((x + y + z + t)(x + y - z - t)) \\ \beta(\bar{\rho}) + 2 &= \nu_2((x - y + z - t)(x - y - z + t)). \end{aligned}$$

Then, if for an irregular representation $\rho$ with $\Gamma^*(\rho) \equiv 0 \,(2)$ we define $\alpha(\rho)$ and $\beta(\rho)$ by the above formulas, we have proved:

**Proposition 5.** *The invariants $\Gamma^*$, $\alpha$, $\beta$ are a complete set of invariants for the irregular representations $\rho$ with $\Gamma^*(\rho)$ even. For any $\lambda \in \mathbb{Z}_2$, $\alpha = 1, 2, \ldots, \infty$, $\beta = 1, 2, \ldots, \infty$ such that $\alpha+\beta = \nu_2(\lambda)+\nu_2(\lambda-1)$, there is exactly one irregular representation $\rho$ with $\Gamma^*(\rho) = 4\lambda - 2$, $\alpha(\rho) = \alpha$, $\beta(\rho) = \beta$.* □

In the paper [1] it remained open the problem of finding a complete set of representatives for the representations as in the preceding proposition. The present approach provides an easy way to solve that problem: It is enough to take the representatives for the representations of regular type (excluding the cases $\alpha = 0$ and $\beta = 0$) and to traanslate them from $[L_1^{1,1}]$ to $[L_0]$. We leave these computations to the reader.

The result in proposition 4 appears already in [1] (cf. proposition 5), but the classification given by proposition 5 looks quite different from the classification which we obtained in [1]. In that paper, to classify the representations of irregular type we used the invariant $\Gamma^*$ plus three other invariants. Let

$$\begin{aligned}
\epsilon &= x - y + z - t \\
\bar{\epsilon} &= x + y + z + t \\
\bar{\bar{\epsilon}} &= x + y - z - t \\
\bar{\bar{\bar{\epsilon}}} &= x - y - z + t \\
a &= x^2 - y^2 + z^2 - t^2 \\
b &= xz - yt \\
\delta &= \min\{\nu_2(a), \nu_2(b)\}\,.
\end{aligned}$$

Then, we proved in [1] that $\Gamma^*$, $\nu_2(\epsilon)$, $\nu_2(\bar{\epsilon})$, $\delta$ are a complete set of invariants for the representations of irregular type. Hence, it follows that it should be possible to compute $\{\Gamma^*, \alpha, \beta\}$ out from $\{\Gamma^*, \nu_2(\epsilon), \nu_2(\bar{\epsilon}), \delta\}$ and viceversa. The remainder of this section is devoted to these computations. Notice that if we show that $\{\Gamma^*, \alpha, \beta\}$ are determined by $\{\Gamma^*, \nu_2(\epsilon), \nu_2(\bar{\epsilon}), \delta\}$ and viceversa, then proposition 5 provides an independent (and simpler) proof of theorem 3 in [1] and, in the same way, theorem 3 in [1] provides an independent (and more computational) proof of proposition 5 in this paper. Moreover, these computations will clarify the appearance of the somewhat strange invariant $\delta$ in [1] and will show that this invariant is really necessary for the classification of representations of irregular type. Finally, let us mention that $\delta$ appears in the mod 2 cohomology of the classifying spaces of rank two Kac-Moody groups (see [2]) and so the computation of $\delta$ as a function of $\{\Gamma^*, \alpha, \beta\}$ is necessary to complete the picture presented in [1] and [2].

First of all, $\Gamma^*$ odd implies $\nu_2(\epsilon) = \nu_2(\bar{\epsilon}) = \delta = 0$ and there is nothing to prove. Assume $\Gamma^*$ is even, and so are $\epsilon = x - y + z - t$, $\bar{\epsilon}$, $\bar{\bar{\epsilon}}$ and $\bar{\bar{\bar{\epsilon}}}$. The following identities are straightforward ($\Delta = xt - yz$):

$$\begin{aligned}
\alpha + 2 &= \nu_2(\bar{\epsilon}\,\bar{\bar{\epsilon}}) \\
\beta + 2 &= \nu_2(\epsilon\,\bar{\bar{\bar{\epsilon}}}) \\
\bar{\epsilon}\,\bar{\bar{\bar{\epsilon}}} &= \Delta(\Gamma^* + 2) \\
\epsilon\,\bar{\bar{\epsilon}} &= \Delta(\Gamma^* - 2)
\end{aligned}$$

Assume first $\Gamma^* \neq \pm 2$ so that all terms in the equations above are finite. We have $\Gamma^* \equiv \pm 2\,(8)$.

If $\Gamma^* \equiv -2\,(8)$, then $\nu_2(\epsilon)+\nu_2(\bar{\bar{\epsilon}}) = 2$ and since both $\epsilon$, $\bar{\bar{\epsilon}}$ are even we must have $\nu_2(\epsilon) = \nu_2(\bar{\bar{\epsilon}}) = 1$. Then, $\nu_2(\bar{\epsilon}) = \alpha + 1$ and $\nu_2(\bar{\bar{\bar{\epsilon}}}) = \beta + 1$.

If $\Gamma^* \equiv 2\,(8)$ then the same argument proves that $\nu_2(\bar{\epsilon}) = \nu_2(\bar{\bar{\bar{\epsilon}}}) = 1$, while $\nu_2(\epsilon) = \beta + 1$ and $\nu_2(\bar{\bar{\epsilon}}) = \alpha + 1$.

This shows that for $\Gamma^* \neq \pm 2$, the invariants $\{\Gamma^*, \nu_2(\epsilon), \nu_2(\bar{\epsilon})\}$ are a function of $\{\Gamma^*, \alpha, \beta\}$ and viceversa. In particular, the invariant $\delta$ is redundant in this case. It is easy to see that the previous argument is also valid for $\Gamma = \pm 2$ in the sense that it allows us to compute $\nu_2(\epsilon)$ and $\nu_2(\bar{\epsilon})$ out of $\Gamma^*$, $\alpha$, $\beta$. We summarize this in the following table:

| | $\nu_2(\epsilon)$ | $\nu_2(\bar{\epsilon})$ | $\nu_2(\bar{\bar{\epsilon}})$ | $\nu_2(\bar{\bar{\bar{\epsilon}}})$ |
|---|---|---|---|---|
| $\Gamma^* \equiv 1\,(2)$ | 0 | 0 | 0 | 0 |
| $\Gamma^* \equiv -2\,(8)$ | 1 | $\alpha + 1$ | 1 | $\beta + 1$ |
| $\Gamma^* \equiv 2\,(8)$ | $\beta + 1$ | 1 | $\alpha + 1$ | 1 |

However, we notice that for $\Gamma^* = \pm 2$ some values in this table are $\infty$ and so we cannot compute $\alpha$ or $\beta$ out from $\{\Gamma^*, \nu_2(\epsilon), \nu_2(\bar{\epsilon})\}$ in this particular case. It is then that the invariant $\delta$ is needed.

Notice that $\epsilon = 0$ implies

$$a = x^2 - y^2 + z^2 - t^2 = 2(yt - xz) = 2b.$$

Then,

$$\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}} = 2(x^2 - y^2 + z^2 - t^2) = 4(yt - xz).$$

Hence,

$$\delta = \min\{\nu_2(a), \nu_2(b)\} = \nu_2(b) = \nu_2(\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}}) - 2.$$

This allows us to compute $\alpha$, $\beta$ out from $\nu_2(\epsilon)$, $\nu_2(\bar{\epsilon})$, $\delta$ if $\Gamma^* = 2$. The case $\Gamma^* = -2$ is similar and we can summarize the computation of $\alpha$, $\beta$ out from $\nu_2(\epsilon)$, $\nu_2(\bar{\epsilon})$, $\delta$ for $\Gamma^* = \pm 2$ in the following table:

| | $\alpha$ | $\beta$ |
|---|---|---|
| $\Gamma^* = 2$, $\nu_2(\epsilon) \neq \infty$ | $\infty$ | $\nu_2(\beta) - 1$ |
| $\Gamma^* = -2$, $\nu_2(\bar{\epsilon}) \neq \infty$ | $\nu_2(\bar{\epsilon}) - 1$ | $\infty$ |
| $\Gamma^* = 2$, $\nu_2(\epsilon) = \infty$ | $\delta$ | $\infty$ |
| $\Gamma^* = -2$, $\nu_2(\bar{\epsilon}) = \infty$ | $\infty$ | $\delta$ |

To complete the picture we must show how the invariant $\delta$ can be computed as a function of $\Gamma^*$, $\alpha$, $\beta$. This is the trickiest part in all this analysis. Notice that

$$\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}} = a + 2b, \quad \epsilon\bar{\epsilon} = a - 2b$$

and, according to our previous computations, we have

$$\nu_2(\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}}) = \begin{cases} \beta + 2 & \text{if } \Gamma^* \equiv -2\,(8) \\ \alpha + 2 & \text{if } \Gamma^* \equiv 2\,(8) \end{cases} \qquad \nu_2(\epsilon\bar{\epsilon}) = \begin{cases} \alpha + 2 & \text{if } \Gamma^* \equiv -2\,(8) \\ \beta + 2 & \text{if } \Gamma^* \equiv 2\,(8) \end{cases}$$

Hence, if we assume $\alpha \neq \beta$, then we see that $\nu_2(a+2b) \neq \nu_2(a-2b)$ and so $\nu_2(a) = \nu_2(2b)$. Therefore $\nu_2(b) < \nu_2(a)$ and $\delta = \min\{\nu_2(a), \nu_2(b)\} = \nu_2(b)$. Then, it follows that

$$\begin{aligned} \nu_2(b) + 2 &= \nu_2((a + 2b) - (a - 2b)) \\ &= \min\{\nu_2(a + 2b), \nu_2(a - 2b)\} \\ &= \min\{\alpha + 2, \beta + 2\} \end{aligned}$$

and we have $\delta = \min\{\alpha, \beta\}$. It is easy to see that this argument holds also for $\Gamma^* = \pm 2$.

However, this argument fails if $\alpha = \beta$ and indeed in this case $\delta$ has a more complex form as a function of $\Gamma^*$, $\alpha$, $\beta$. Assume now that $\alpha = \beta$ and notice that the quotient

$$\frac{\bar{\epsilon}\bar{\bar{\bar{\epsilon}}}}{\epsilon\bar{\bar{\epsilon}}} = \frac{\Gamma^* + 2}{\Gamma^* - 2}$$

is known since it is a function of $\Gamma^*$. Let us define a 2-adic *unit* $u$ in the following way:

$$u = \begin{cases} 2^{-2\alpha}\dfrac{\Gamma^* + 2}{\Gamma^* - 2} & \text{if } \Gamma^* \equiv -2\,(8) \\[2ex] 2^{-2\alpha}\dfrac{\Gamma^* - 2}{\Gamma^* + 2} & \text{if } \Gamma^* \equiv 2\,(8) \end{cases}$$

Then,

$$\begin{aligned} a = \frac{\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}} + \epsilon\bar{\epsilon}}{2} &= 2^{\alpha+1}(2^{-\alpha-2}\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}} + 2^{-\alpha-2}\epsilon\bar{\epsilon}), \\ 2b = \frac{\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}} - \epsilon\bar{\epsilon}}{2} &= 2^{\alpha+1}(2^{-\alpha-2}\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}} - 2^{-\alpha-2}\epsilon\bar{\epsilon}). \end{aligned}$$

Assume $\Gamma^* \equiv -2\,(8)$. Then $\epsilon\bar{\bar{\epsilon}}/4$ and $\bar{\epsilon}\bar{\bar{\epsilon}}/2^{\alpha+2}$ are units (cf. the table above). Hence

$$\begin{aligned} \nu_2(2^{-\alpha-2}\bar{\bar{\epsilon}}\bar{\bar{\bar{\epsilon}}} \pm 2^{-\alpha-2}\epsilon\bar{\epsilon}) &= \nu_2(2^{-2\alpha-4}\bar{\epsilon}\bar{\bar{\epsilon}}^2\bar{\bar{\bar{\epsilon}}} \pm 2^{-2\alpha-4}\epsilon\bar{\epsilon}^2\bar{\bar{\epsilon}}) \\ &= \nu_2(2^{-4}u\epsilon\bar{\bar{\epsilon}}^3 \pm 2^{-2\alpha-4}\epsilon\bar{\epsilon}^2\bar{\bar{\epsilon}}) \\ &= \nu_2(2^{-2}u\bar{\bar{\epsilon}}^2 \pm 2^{-2\alpha-2}\bar{\epsilon}^2) \\ &= \nu_2(u(\bar{\bar{\epsilon}}/2)^2 \pm (\bar{\epsilon}/2^{\alpha+1})^2) \end{aligned}$$

where $\bar{\bar{\epsilon}}/2$ and $\bar{\epsilon}/2^{2\alpha+2}$ are units. Then

$$u(\bar{\bar{\epsilon}}/2)^2 \pm (\bar{\epsilon}/2^{\alpha+1})^2 \equiv u \pm 1\,(4)$$

and this proves that

$$\delta = \min\{\nu_2(a), \nu_2(b)\} = \begin{cases} \alpha+1 & \text{if } u+1 \equiv 0\,(4) \\ \alpha+2 & \text{if } u+1 \equiv 2\,(4). \end{cases}$$

For $\Gamma^* \equiv 2\,(8)$ we obtain the same results by this same method. This finishes the computation of $\delta(\rho)$ as a function of $\Gamma^*(\rho)$, $\alpha(\rho)$ and $\beta(\rho)$ and we have now an explicit translation between the classification of representations of $D_\infty$ in [1] and the classification of representations of $D_\infty$ in the present paper.

## 6. Integral representations: the mixed case

To complete the classification of all representations of $D_\infty$ in $GL_2(\mathbb{Z}_2)$ it remains to consider the case when one of the involutions is conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ while the other is conjugate to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We call this the *mixed* case.

Like in the previous case, if we have the two involutions $\sigma_1 \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\sigma_2 \sim \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we can consider the geodesic paths $\Sigma_1$, $\Sigma_2$ in the tree $X$ joining $\sigma_1^+$ to $\sigma_1^-$ and $\sigma_2^+$ to $\sigma_2^-$, respectively. We know that $\Sigma_1$ contains the distinguished vertex $[L_0]$ while $\Sigma_2$ does not, but $\Sigma_2$ contains a vertex at distance one from $[L_0]$. If $\Sigma_1$ and $\Sigma_2$ are disjoint, then $[L_0]$ can be characterized geometrically as the only vertex in $\Sigma_1$ which is at distance one from $\Sigma_2$. If $\Sigma_1$ and $\Sigma_2$ have some common edges, then to geometrically determine the vertex $[L_0]$ we need an extra piece of information, namely we need to know if $[L_0]$ lies between $\Sigma_1 \cap \Sigma_2$ and $\sigma_1^+$ or between $\Sigma_1 \cap \Sigma_2$ and $\sigma_1^-$. Let us define the *intersection type* of a representation as *null* if $\Sigma_1 \cap \Sigma_2 = \emptyset$, *positive* if $[L_0]$ lies between $\Sigma_1 \cap \Sigma_2$ and $\sigma_1^+$, and *negative* if $[L_0]$ lies between $\Sigma_1 \cap \Sigma_2$ and $\sigma_1^-$. Then, an argument as in the previous sections shows that two representations are equivalent if and only if they are equivalent over $\mathbb{Q}_2$ and have the same intersection type. If we want to translate this into a classification through invariants, we just need to perform a few computations. Let $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\sigma_2 = M^{-1} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) M$ for some matrix $M = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_2)$. In this case it is straightforward to see that we can write $\Gamma(\rho) = (1 + \Gamma^{**}(\rho))/2$ for a new invariant

$$\Gamma^{**}(\rho) = \frac{zt - xy}{xt - yz}.$$

We also define a new invariant $\gamma(\rho) = yt \mod 2$. We have:

(1) If $\Gamma^{**}(\rho) = \Gamma^{**}(\rho') \neq \pm 1$, then $\rho$ and $\rho'$ are equivalent over $\mathbb{Q}_2$ by the results of section 3. $\Gamma^{**}(\rho) = \pm 1$ if and only if $(x+z)(x-z)(y+t)(y-t) = 0$. If this happens, then an easy analysis shows that $\gamma(\rho) = yt \equiv x + z \mod 2$ coincides with the invariant $\tau$ defined in section 3. Hence, $\Gamma^{**}(\rho)$ and $\gamma(\rho)$ determine the representation $\rho$ up to conjugation in $GL_2(\mathbb{Q}_2)$.

(2) We have $\sigma_2^+ = \{t - y, x - z\}$, $\sigma_2^- = \{t + y, -x - z\}$. Hence, the vertex in $\Sigma_2$ at distance one from $[L_0]$ is $[L_1^{t+y,x+z}]$ and $\Sigma_1$ and $\Sigma_2$ are disjoint if and only if this vertex is different from $[L_1^{1,0}]$ and $[L_1^{1,1}]$. It is easy to see that this is equivalent to $\Gamma^{**} \equiv 0\,(2)$.

(3) If $\Gamma^{**} \equiv 1\,(2)$ then the intersection type is either positive or negative and it is easy to see that it is positive if $\gamma = 1$ and negative if $\gamma = 0$.

Hence we have a new proof of theorem 2 in [1]:

**Theorem 6.** $\Gamma^{**}$ *and* $\gamma$ *are a complete set of invariants for the representations of* $D_\infty$ *in* $GL_2(\mathbb{Z}_2)$ *in the mixed case.* □

## 7. Computability

In this final section we want to show that the classification of representations of $D_\infty$ given in [1] and in the present paper is effective in the sense that one could easily write a computer program —in some suitable computer algebra language, like Magma ([4])— such that given a prime number $p$ and four $2 \times 2$ matrices of order 2, $\sigma_1$, $\sigma_2$, $\sigma_1'$, $\sigma_2'$, the program produces a true/false output depending if there is a matrix in $GL_2(\mathbb{Z}_p)$ which conjugates $\sigma_i$ to $\sigma_i'$ for $i = 1, 2$ or not. In this section we could exhibit such a program but it is much shorter to give just some few hints to convince the reader that this program may be written.

First of all, the program should determine for each of the matrices $\sigma_1$, $\sigma_2$, $\sigma_1'$, $\sigma_2'$ if they are *central* (i.e. equal to $\pm I$), *regular* (i.e. conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$) or, in the case $p = 2$, *irregular* (i.e. conjugate to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). This is trivial to do. In the case of $p = 2$ we must notice that $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq \pm I$ is regular if and only if $b$ and $c$ are even.

Next, the program should solve the trivial case in which one of the involutions is central. This will reduce the problem to the case when we know that none of the four matrices $\sigma_1$, $\sigma_2$, $\sigma_1'$, $\sigma_2'$ is central and $\sigma_i$ is conjugate to $\sigma_i'$ for $i = 1, 2$. In this situation, to decide if the two representations of $D_\infty$ are the same we just need to check if the values of the invariants that we have introduced in the preceding sections are the same for the two representations. These invariants are explicitly given as functions of $x$, $y$, $z$, $t$, where $M = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ is a matrix in $GL_2(\mathbb{Z}_p)$ such that

> $M = Q^{-1}P$ with $Q, P \in GL_2(\mathbb{Z}_p)$ such that $Q^{-1}\sigma_2 Q$ and $P^{-1}\sigma_1 P$ are in *standard form.*

Here standard form means the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ for $p \neq 2$ and either the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p = 2$.

Hence, to effectively compute these invariants we need to solve the following problem:

> Given a $2 \times 2$ matrix $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_p)$ with $\sigma^2 = I$ and $\sigma \neq \pm I$, find $M = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ with entries in $\mathbb{Z}_p$ such that $M^{-1}\sigma M$ is in standard form and $\det(M) \not\equiv 0\,(p)$.

To solve this problem we can proceed in the following way. Let $v_1, v_2 \in \mathbb{Z}_p^2$ be eigenvectors of $\sigma$ with eigenvalues $+1$ and $-1$ respectively and let us choose them such that they are not divisible by $p$. If $\sigma$ is conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ then we take $M$ as the matrix formed by the coordinates of $v_1, v_2$. If $p = 2$ and $\sigma$ is conjugate to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ then we take $M$ as the matrix determined by the coordinates of $(v_1 \pm v_2)/2$. In either case, is is easy to see that $\det(M) \not\equiv 0\,(p)$.

Is is a straightforward exercise to translate all this analysis into a Magma language set of instructions.

## 8. Acknowledgements

It is worthwhile to explain here how was the genesis of this paper. The author was fortunate enough to have Jean-Pierre Serre in the audience of a talk on the work in [1] that he gave at a meeting of the Tunisian Mathematical Society. Serre immediately suggested that the abstract numerical invariants of [1] should have a natural geometrical interpretation in terms of the tree of $GL_2(\mathbb{Z}_p)$ and he outlined how this interpretation should look like. Hence, the present paper can be viewed as a writing out of Serre's ideas together with some results needed to understand the relationship between the classification in [1] and the classification that we obtain here. The author is in deep gratitude to J.-P. Serre for sharing his ideas with him and would also like to thank S. Zarati for his invitation to Tunis where this work began. The hospitality of PIMS in Vancouver is also gratefully acknowledged.

## References

[1] J. Aguadé, C. Broto, L. Saumell, *Rank two integral representations of the infinite dihedral group.* Comm. Alg. 35 (2007), 1539–1551.

[2] J. Aguadé, C. Broto, N. Kitchloo, L. Saumell, *Cohomology of classifying spaces of central quotients of rank two Kac-Moody groups.* J. Math. Kyoto Univ. 45 (2005), no. 3, 449–488.

[3] S.D. Berman, K. Buzási, *Representations of the infinite dihedral group.* Publ. Math. Debrecen 28 (1981), no. 1-2, 173–187.

[4] W. Bosma, J. Cannon, C. Playoust. *The Magma algebra system. I. The user language.* J. Symbolic Comput., 24 (1997), 235–265, 1997

[5] D.Ž. Đokovič, *Pairs of Involutions in the General Linear Group.* J. Algebra 100 (1986), 214–223.

[6] J.-P. Serre, *Arbres, amalgames, $SL_2$.* Astérisque 46 (1977).

Dep. de Matemàtiques, Universitat Autònoma de Barcelona. 08193 Cerdanyola del Vallès. Spain

*E-mail address*: jaume.aguade@uab.cat