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Joint Approximation in BMO

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A characterization of Farrell sets for the space of harmonic or holomorphic functions whose boundary values are in BMO (VMO) is obtained. The main step of the proof is the construction of certain VMO functions which are related to the sharpness of the John–Nirenberg inequality.

1. INTRODUCTION

Let $\mathscr{A}$ be a space of functions in the upper half space $\mathbb{R}^{d+1}_+ = \{(x, y): x \in \mathbb{R}^d, y > 0\}$ or in the unit disk $\mathbb{D}$ of the complex plane. Let $\mathscr{C}$ denote the set of continuous functions which belong to $\mathscr{A}$. Suppose that there is given a topology τ in $\mathscr{A}$ such that $\mathscr{C}$ is dense in $\mathscr{A}$. A relatively compact subset $F \subset \mathbb{R}^{n+1}_+$ (or $F \subset \mathbb{D}$) is called a Farrell set (for $(\mathscr{A}, \tau)$) if for any $f \in \mathscr{A}$ bounded on F there exists a sequence of continuous functions $\{p_n\}$ in $\mathscr{C}$ tending to f (in the topology τ) and pointwise-boundedly on F, that is, $p_n \to f$ pointwise on F and

$$\|p_n\|_F \to \|f\|_F,$$

where $\|f\|_F$ denotes the supremum of f over F.

This concept was introduced by Rubel, who also raised the question of describing such sets [R]. Actually, Farrell sets have been geometrically described for several spaces of functions, for instance, (a) for the space of all holomorphic functions in the unit disk $\mathbb{D}$ endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$ [R–S], (b) for the space H^∞ of bounded analytic functions in the unit disk with the topology of

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pointwise bounded convergence [S], (c) for the usual Hardy spaces $H^p(\mathbb{D})$, $0<p<\infty$, with the weak and the norm topology [R–S, P, P–S, B–P–S–T], (d) for the space of bounded harmonic functions in the upper half space with the weak-$*$ topology [P–T, B–T]. In cases (b), (c), and (d), Farrell sets are those relatively compact sets $F\subset B$ such that almost any point in $\bar{F}\cap\partial B$ is in the nontangential closure of F. Here B denotes either $\mathbb{D}$ or $\mathbb{R}^{n+1}_{+}$.

Given a point $x\in\mathbb{R}^d$ and $\alpha>0$, let

$$\Gamma(x,\alpha)=\{(y,t)\in\mathbb{R}^{d+1}_{+}:|y-x|\leqslant\alpha t\}$$

be the cone with vertex at x and aperture α. Given a set F in the upper half space $\mathbb{R}^{d+1}_{+}$, the point $x\in\mathbb{R}^d$ is in the nontangential closure of F, if there exists $\alpha>0$ such that x is in the closure of $F\cap\Gamma(x,\alpha)$. The nontangential closure of F is denoted by F_{nt}.

In this paper we study Farrell sets for the space of functions of bounded mean oscillation, obtaining a similar geometric description.

In the Euclidean space $\mathbb{R}^d$, let Q denote any cube with sides parallel to the axis and write $|Q|$ for its Lebesgue measure. A locally integrable function f on $\mathbb{R}^d$ has bounded mean oscillation, $f\in\mathrm{BMO}(\mathbb{R}^d)$, if

$$\|f\|_{*}=\sup_{Q}\frac{1}{|Q|}\int_{Q}|f-f_Q|<\infty,$$

where

$$f_Q=\frac{1}{|Q|}\int_{Q}f=\fint_{Q}f$$

is the mean of f over Q. Every bounded function is in BMO but BMO contains unbounded functions. For instance, $\log|x|$ is in $\mathrm{BMO}(\mathbb{R}^d)$. One denotes by $\mathrm{VMO}(\mathbb{R}^d)$, the space of functions of vanishing mean oscillation, the closure in BMO of the continuous functions with compact support. Equivalently, VMO consists of those $f\in\mathrm{BMO}$ such that the averages $|Q|^{-1}\int_Q|f-f_Q|$ tend to zero uniformly as the side length of Q tends either to zero or to infinity. Again, VMO contains unbounded functions (e.g., $|\log|x||^{1/2}$).

The celebrated duality theorem of Fefferman [F–S] states that $\mathrm{BMO}(\mathbb{R}^d)$ is the dual space of the Hardy space, $H^1(\mathbb{R}^d)$, of functions g in L^1 whose Riesz transforms

$$R_j g(x)=\lim_{\varepsilon\to 0}\int_{\{|y|>\varepsilon\}}\frac{y_j}{|y|^{d+1}}\,g(x-y)\,dy,\qquad j=1,\ldots,d,$$

are in L^1. We set $\|g\|_{H^1}=\|g\|_1+\sum_{j=1}^d \|R_j g\|_1$. Actually, given $f \in \mathrm{BMO}(\mathbb{R}^d)$, the corresponding functional Λ_f is defined on the dense subspace of H^1 of continuous compactly supported functions g with mean 0 by

$$\Lambda_f(g)=\int_{\mathbb{R}^n} f(x)\, g(x)\, dx$$

and extends continuously to H^1.

Given a function f in $\mathrm{BMO}(\mathbb{R}^d)$, we denote by $f(z)$ the value of its harmonic extension at the point $z \in \mathbb{R}_+^{d+1}$, that is,

$$f(z)=(f * P_t)(x)=\int_{\mathbb{R}^d} f(y)\, P_t(x-y)\, dy,$$

where $z=(x, t)$, $t>0$, and

$$P_t(y)=c(d)\,\frac{t}{(\|y\|^2+t^2)^{(d+1)/2}}, \qquad y \in \mathbb{R}^d,$$

is the Poisson kernel. The space of harmonic extensions of functions in $\mathrm{BMO}(\mathbb{R}^d)$ is denoted by $\mathrm{BMO}(\mathbb{R}_+^{d+1})$. If $f \in \mathrm{BMO}(\mathbb{R}^d)$, the functions $f_t(x)=(f * P_t)(x)$, $x \in \mathbb{R}^d$, $t>0$, converge pointwise and weak-$*$ to f as $t \to 0$. So the continuous functions in $\mathrm{BMO}(\mathbb{R}^d)$ are weak-$*$ dense in $\mathrm{BMO}(\mathbb{R}^d)$.

THEOREM 1. *Let F be a relatively compact set in the upper half space $\mathbb{R}_+^{d+1}$. Then the following conditions are equivalent*:

(a) *F is a Farrell set for* $\mathrm{BMO}(\mathbb{R}_+^{d+1})$ *equipped with the weak-$*$ topology.*

(b) *F is a Farrell set for* $\mathrm{VMO}(\mathbb{R}_+^{d+1})$ *equipped with the norm topology.*

(c) *Almost every point of $\bar{F} \cap \mathbb{R}^d$ is the nontangential limit of points of F, that is, $|\bar{F} \cap \mathbb{R}^d \backslash F_{nt}|=0$.*

The space BMOA (VMOA) consists of the holomorphic functions in the unit disk that are Poisson extensions of functions of $\mathrm{BMO}(\partial\mathbb{D})$ ($\mathrm{VMO}(\partial\mathbb{D})$). The problem of finding a geometrical description of Farrell sets for BMOA (VMOA) was proposed in [R–S].

THEOREM 2. *Let F be a relatively closed set of the unit disk. Then the following conditions are equivalent*:

(a) *F is a Farrell set for* BMOA *endowed with the weak-$*$ topology.*

(b) *F is a Farrell set for* VMOA *endowed with the norm topology.*

(c) *Almost every point of $\bar{F} \cap \partial\mathbb{D}$ is a nontangential limit of points of F, that is, $|\bar{F} \cap \partial\mathbb{D} \backslash F_{\mathrm{nt}}|=0$.*

Using duality methods, Pérez-González, Stray, and Trujillo-González have recently proved Theorem 2, except for the implication from (b) to (c). In both Theorems 1 and 2, when the geometric condition (c) is satisfied, our methods are constructive. The main technical step is the construction of VMO functions with certain regularity properties, as stated in Theorems 4 and 5 below.

On the other hand, the necessity of condition (c) in Theorems 1 and 2 is closely related to the following fact. A set $F \subset \mathbb{R}^2_+$ is dominating (for H^∞) if for any bounded analytic function f, one has

$$\sup\{|f(z)|: z \in F\} = \sup\{|f(z)|: z \in \mathbb{R}^2_+\}.$$

Such sets are also described by the condition $|\mathbb{R} \backslash F_{nt}| = 0$ (see [B–S–Z]). An analogous result for Hardy spaces H^p, $0 < p < \infty$, has been obtained by Thomas [T].

We will say that a set $F \subset \mathbb{R}^{d+1}_+$ is dominating for BMO (for VMO) in its closure if for any $u \in \mathrm{BMO}$ ($u \in \mathrm{VMO}$) such that $\sup_F |u| < \infty$, one has $u \in L^\infty(\bar{F})$.

THEOREM 3. *Let F be a relatively compact set in $\mathbb{R}^{d+1}_+$. The following properties are equivalent*:

(a) *F is dominating in its closure for* $\mathrm{BMO}(\mathbb{R}^{d+1}_+)$.

(b) *F is dominating in its closure for* $\mathrm{VMO}(\mathbb{R}^{d+1}_+)$.

(c) $|\bar{F} \cap \mathbb{R}^d \backslash F_{nt}| = 0$.

Certainly a similar result holds for the spaces BMOA, VMOA, as well as for the Hardy spaces H^p.

Recall that functions in BMO satisfy the John–Nirenberg inequality; that is, there exist two positive constants C_1, $C_2 > 0$ such that whenever $f \in \mathrm{BMO}$ then for every $\lambda > 0$ and every cube Q, one has

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leqslant C_1 \exp(-C_2 \lambda / \|f\|_*) \, |Q|.$$

So, roughly speaking, the set of points of a given cube where a function in BMO differs from its mean has small Lebesgue measure. The next lemma tells that this is best possible. That is, given $A \subset Q$ such that $|A|/|Q| = \varepsilon$ is sufficiently small, there exists $f \in \mathrm{BMO}$, $\|f\|_* = 1$ such that $|f(x) - f_Q| \geqslant C \log \varepsilon^{-1}$ for any $x \in A$. This was shown by Garnett and Jones in [G–J, p. 379]. We need the following strengthened version.

MAIN LEMMA. *Let A be a measurable set of $\mathbb{R}^d$ and let Q be a cube of $\mathbb{R}^d$. Assume $|A \cap Q| < \varepsilon \, |Q|$, $0 < \varepsilon < 1$. Then there is a function $\varphi \in \mathrm{VMO}$ such that*

(i) *φ is supported on $\frac{3}{2}Q$ and $|\operatorname{supp} \varphi| < C\varepsilon\, |Q|$.*

(ii) $\|\varphi\|_* \leqslant C$.

(iii) $0 \leqslant \varphi \leqslant \log \varepsilon^{-1}$.

(iv) *If $x \in A \cap Q$ then $\varphi(x) = \log \varepsilon^{-1}$.*

(v) $⨍_Q \varphi \leqslant C$.

Here $C = C(d)$ is a constant depending only on the dimension.

If one requires the function φ to be in BMO instead of VMO, then this is Lemma 2.2 of [G–J]. Certainly, minor modifications of the proof in [G–J] should give our Main Lemma. However, in Section 3, for the reader's convenience, we will present a slightly different and shorter proof. Constructions of similar type can be found in [J, U].

Our two next results are the main steps in the proof of Theorems 1, 2, and 3. In both proofs the Main Lemma plays a central role and the construction is quite explicit.

THEOREM 4. *Let $A \subset \mathbb{R}^d$ be a measurable set and let $x_0 \in \bar{A}$ such that*

$$\lim_{\delta \to 0} \frac{|Q(x_0, \delta) \cap A|}{|Q(x_0, \delta)|} = 0,$$

where $Q(x_0, \delta)$ denotes the cube centered at x_0 of side length δ. Then there exists a nonnegative function $g \in \mathrm{VMO}(\mathbb{R}^d)$ such that

(a) $\|g\|_* \leqslant C = C(d)$.

(b) $\lim_{\delta \to 0} \frac{1}{|Q(x_0, \delta)|} \int_{Q(x_0, \delta)} g(y)\, dy = 0$.

(c) $\lim_{y \to x_0,\, y \in A} g(y) = \infty$.

THEOREM 5. *Let $A \subset \mathbb{R}^d$ and let $x_0 \in \bar{A}$ be as in Theorem 4. Then there exists a nonnegative function $f \in \mathrm{VMO}(\mathbb{R}^d)$ such that $f(y) \leqslant 1$ for all $y \in A$, $\|f\|_* \leqslant C = C(d)$ and*

$$\lim_{\delta \to 0} \frac{1}{|Q(x_0, \delta)|} \int_{Q(x_0, \delta)} f(y)\, dy = \infty.$$

2. PRELIMINARY FACTS

In this section we collect some results which will be used throughout the paper. First, we state a criterion for weak-$*$ convergence in BMO.

Lemma 2.1. *Let* $\{f_k\}$ *be a sequence of functions in* $\mathrm{BMO}(\mathbb{R}^d)$ *and* $f \in \mathrm{BMO}(\mathbb{R}^d)$. *Assume that* $f_k \to f$ *in the weak-*$*$ *topology and for some* $z_0 \in \mathbb{R}^{d+1}_+$, $f_k(z_0) \to f(z_0)$. *Then* $\sup_k \|f_k\|_* < \infty$ *and* $f_k(z) \to f(z)$, *for all* $z \in \mathbb{R}^{d+1}_+$. *Moreover,* f_k *tends to* f *uniformly on compact sets of* $\mathbb{R}^{d+1}_+$.

Proof. The conclusion $\sup_k \|f_k\|_* < \infty$ is clear. We only will pay attention to the pointwise convergence. For fixed $t > 0$, the Poisson kernel P_t does not belong to H^1. It is well known that bounded functions with mean 0 that decay sufficiently rapidly at infinity belong to H^1 [G–R, p. 327]. Thus for any $x \in \mathbb{R}^d$, the function $P_t(x-y) - C_d \chi_{B(0,1)}(y)$, where $C_d = |B(0,1)|^{-1}$, is in H^1. Then for each $z \in \mathbb{R}^{d+1}_+$ we have

$$f_k(z) - C_d \int_{B(0,1)} f_k(y)\, dy \to f(z) - C_d \int_{B(0,1)} f(y)\, dy.$$

In particular, taking $z = z_0$, we get

$$\int_{B(0,1)} f_k(y)\, dy \to \int_{B(0,1)} f(y)\, dy.$$

Therefore,

$$f_k(z) \to f_z(z)$$

for any $z \in \mathbb{R}^{d+1}_+$. ∎

Given a cube Q in $\mathbb{R}^d$ we denote by z_Q the point $(x_Q, \ell(Q))$ in the upper-half space, where x_Q is the center of Q and $\ell(Q)$ is its side length. Also, $P_{z_Q}(y) = P(x_Q - y, \ell(Q))$ denotes the Poisson kernel.

Our next auxiliary result says that when one computes the harmonic extension of a BMO function, contributions from “far away” are negligible.

Lemma 2.2. *Given* $\varepsilon > 0$, *there exists* $N = N(\varepsilon, d) > 0$ *such that for any cube* $Q \subset \mathbb{R}^d$ *and any* $f \in \mathrm{BMO}(\mathbb{R}^d)$, $\|f\|_* \leqslant 1$, *satisfying*

$$|f(x)| \leqslant 1 - \varepsilon$$

for all $x \in NQ$, *one has*

$$|f(z_Q)| \leqslant 1.$$

Proof. Observe that for any cube P in $\mathbb{R}^d$, one has

$$|f_{2P} - f_P| \leqslant C \|f\|_*.$$

Hence, estimating the Poisson kernel in dyadic blocks, one gets

$$\begin{aligned}\int_{\mathbb{R}^d \setminus NQ} |f(y)|\, P_{z_Q}(y)\, dy &\leqslant C \sum_{k \geqslant 1} \frac{|f|_{2^k NQ}}{2^k N} \\ &\leqslant \frac{C}{N} \sum_{k \geqslant 1} 2^{-k}(k\, \|f\|_* + |f|_{NQ}) \\ &\leqslant \frac{C}{N}.\end{aligned}$$

Hence

$$\left| f(z_Q) - \int_{NQ} P_{z_Q}(y)\, f(y)\, dy \right| \leqslant \frac{C}{N}$$

and the result follows. ∎

Given a natural number n, let $D(n)$ denote the set of dyadic cubes Q of edge length 2^{-n}; that is, Q is the cartesian product of intervals $[\ell 2^{-n}, (\ell+1)\, 2^{-n}]$ where ℓ is an integer. Related to each $D(n)$ we consider a partition of unity. We associate to each dyadic cube $Q \in D(n)$ a measurable function Ψ_Q supported in $\frac{5}{4} Q$ such that $0 \leqslant \Psi_Q \leqslant 1$, $\Psi_Q(x) = 1$ for all $x \in \frac{3}{4} Q$ and

$$\sum_{Q \in D(n)} \Psi_Q(x) \equiv 1$$

for all $x \in \mathbb{R}^d$. For instance, we could take $\Psi_Q = \chi_Q$ for all $Q \in D(n)$; however, in the proof of Theorem 1, we will require the continuity of the functions Ψ_Q.

The following result is probably well known but, since we have not found it in the literature, a proof is provided.

PROPOSITION 2.3. *Let f be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Let $\varphi(f, n)$ be the function defined by*

$$\varphi(f, n) = \sum_{Q \in D(n)} a_Q \Psi_Q \qquad \textit{where} \quad a_Q = \frac{\int f \Psi_Q}{\int \Psi_Q}.$$

(a) *If $f \in L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, then $\varphi(f, n) \to f$ in $L^p(\mathbb{R}^d)$ as $n \to \infty$.*

(b) *If $f \in \mathrm{BMO}(\mathbb{R}^d)$ then $\varphi(f, n) \to f$ in the weak-$*$ topology as $n \to \infty$. That is, for any $h \in H^1(\mathbb{R}^d)$*

$$\langle \varphi(f, n), h \rangle \to \langle f, h \rangle \qquad \textit{as} \quad n \to \infty.$$

(c) *If $f \in \mathrm{VMO}(\mathbb{R}^d)$ then $\|\varphi(f, n) - f\|_* \to 0$ as $n \to \infty$.*

Remark. In both parts (b) and (c) one easily gets that for any $\eta > 0$ the harmonic extension of $\varphi(f, n)$ tends to the harmonic extension of f uniformly in $\{(x, y) \in \mathbb{R}^{d+1} : y > \eta\}$.

Proof. (a) Observe that by Jensen's inequality,

$$\begin{aligned}\int \Big|\sum_Q a_Q \Psi_Q(x)\Big|^p \, dx &\leqslant \int \Big(\sum_Q |a_Q|^p \, \Psi_Q(x)\Big)\, dx \\ &\leqslant \sum_Q \int |f(x)|^p \, \Psi_Q(x)\, dx \\ &= \int |f(x)|^p \, dx,\end{aligned}$$

that is, $\|\varphi(f, n)\|_p \leqslant \|f\|_p$.

Now the conclusion follows because it holds trivially for continuous function with compact support. Given $\varepsilon > 0$ there is a continuous function g with compact support such that $\|f - g\|_p < \varepsilon$. Therefore,

$$\begin{aligned}\|f - \varphi(f, n)\|_p &\leqslant \|f - g\|_p + \|g - \varphi(g, n)\|_p + \|\varphi(g, n) - \varphi(f, n)\|_p \\ &\leqslant 2\|f - g\|_p + \|g - \varphi(g, n)\|_p < 3\varepsilon\end{aligned}$$

if n is large enough, because g is continuous and has compact support.

(b) Let f be a function in BMO and write

$$M(f, \delta) = \sup_{\ell(Q) \leqslant \delta} \frac{1}{|Q|} \int_Q |f - f_Q|.$$

Let Q and Q' be adjacent dyadic cubes of generation n. Consider a cube P containing $\frac{5}{4} Q$ and $\frac{5}{4} Q'$ such that $\ell(P) \leqslant 3 \cdot 2^{-n}$. Observe that

$$|a_Q - a_{Q'}| \leqslant |a_Q - f_P| + |f_P - a_{Q'}| \leqslant 3 \, \frac{|P|}{|Q|} \frac{1}{|P|} \int_P |f(y) - f_P| \, dy,$$

that is,

$$|a_Q - a_{Q'}| \leqslant 3^{d+1} M(f, 3 \cdot 2^{-n}). \tag{2.1}$$

To simplify notation write $\varphi_n = \varphi(f, n)$. We now show

$$\|f - \varphi_n\|_* \leqslant 5^d M(f, 3 \cdot 2^{-n}). \tag{2.2}$$

We first consider cubes P with $\ell(P)\leqslant 2^{-n}$. Then P only intersects adjacent dyadic cubes of $D(n)$ (at most 2^d) and therefore from (2.1) it follows that for any $x\in P$, one has

$$\begin{aligned}|\varphi_n(x)-(\varphi_n)_P| &\leqslant \max\{|a_Q-a_{Q'}| : Q, Q' \text{ adjacent dyadic cubes of } D(n)\}\\ &\leqslant 3^{d+1}M(f, 3\cdot 2^{-n}).\end{aligned}$$

Then, in this case

$$\frac{1}{|P|}\int_P |(f-\varphi_n)-(f-\varphi_n)_P| \leqslant M(f, 2^{-n})+3^{d+1}M(f, 3\cdot 2^{-n}).$$

When the cube P has $\ell(P)>2^{-n}$ we get

$$\begin{aligned}\int_P |f(x)-\varphi_n(x)|\,dx &\leqslant \int_P \sum_Q |f(x)-a_Q|\,\Psi_Q(x)\,dx\\ &\leqslant \sum_Q \int_P\left(\frac{1}{\int\Psi_Q}\int |f(x)-f(y)|\,\Psi_Q(y)\,dy\right)\Psi_Q(x)\,dx\\ &\leqslant 2M(f, 2\cdot 2^{-n})\sum_{(5/4)Q\cap P\neq\varnothing}\left|\frac{5}{4}Q\right|^2|Q|^{-1}\\ &\leqslant 2^{d+1}\left(\frac{5}{4}\right)^{2d}|P|\,M(f, 2\cdot 2^{-n}),\end{aligned}$$

where we have used the simple fact that for any cube R one has

$$\int_R\int_R |f(x)-f(y)|\,dx\,dy\leqslant |R|\int_R |f-f_R|.$$

Finally

$$\frac{1}{|P|}\int_P |(f-\varphi_n)-(f-\varphi_n)_P|\leqslant\frac{2}{|P|}\int_P |f-\varphi_n|\leqslant 2^{d+2}\left(\frac{5}{4}\right)^{2d}M(f, 2\cdot 2^{-n})$$

and (2.2) is proved.

Therefore, $\|\varphi(f,n)\|_*\leqslant 5^d\|f\|_*$ for all n. Thus we only need to check that $\langle\varphi(f,n), h\rangle\to\langle f, h\rangle$ when h is bounded, has compact support and $\int h=0$ (because this class of functions is dense in H^1). But now, as in part (a), $\varphi(f,n)h\to fh$ in L^1 and

$$\langle\varphi(f,n), h\rangle=\int\varphi(f,n)h\to\int fh=\langle f, h\rangle\qquad\text{as}\quad n\to\infty.$$

(c) When $f \in \mathrm{VMO}$, $M(f, \delta) \to 0$ as $\delta \to 0$ and (2.2) shows that φ_n tends to f in norm. ∎

If f is a complex-valued function and $\rho > 0$, denote by $T_\rho f$ its truncation at height ρ, that is $T_\rho f(x) = f(x)$ if $|f(x)| \leqslant \rho$ and $T_\rho f(x) = \rho\, f(x)/|f(x)|$ if $|f(x)| \geqslant \rho$. Since $T_\rho f = \Phi \circ f$, where Φ satisfies

$$|\Phi(z) - \Phi(w)| \leqslant |z - w|, \qquad z, w \in \mathbb{C},$$

it is clear that

$$\|T_\rho f\|_* \leqslant \|f\|_*.$$

LEMMA 2.4. *Let f be a function in* $\mathrm{BMO}(\mathbb{R}^d)$. *Then*

(a) *$T_\rho f$ converges to f in the weak-$*$ topology, as ρ tends to ∞.*

(b) *If $f \in \mathrm{VMO}(\mathbb{R}^d)$, $T_\rho f$ tends to f in the norm topology, as ρ tends to ∞.*

Proof. Part (a) follows easily because $\|T_\rho f\|_* \leqslant \|f\|_*$ and because for any cube Q one has $T_\rho(f\chi_Q) \to f\chi_Q$ in L^1 as $\rho \to \infty$. To get part (b) one uses the following fact. If $f \in \mathrm{VMO}$, given $\varepsilon > 0$ there are $\delta > 0$ (small) and $R > 1$ (large) such that

$$\frac{1}{|Q|} \int_Q |f - f_Q| < \varepsilon$$

if either $\ell(Q) < \delta$ or $\ell(Q) > R$ or the distance from the center of Q to the origin is bigger than R. ∎

Sundberg established:

THEOREM 2.5 [Su, p. 754]. *Let f be a function in* $\mathrm{BMO}(\mathbb{R}^d)$. *For fixed $C > 0$, $(T_\rho f)(z)$ tends to $f(z)$ uniformly on the set $\{z \in \mathbb{R}^{d+1}_+ : |f(z)| \leqslant C\}$.*

3. PROOFS OF THEOREMS 4 AND 5

This section is devoted to proving Theorems 4 and 5 and to presenting a proof of the Main Lemma. In order to establish these results we will need to construct unbounded functions in BMO (or VMO) of small norm. The next two lemmas provide an easy way to do this.

We say that a Lipschitz function a is adapted to the cube Q if a is supported in $\frac{3}{2}Q$ and $|\mathrm{grad}\, a| \leqslant b\ell(Q)^{-1}$ for some constant b.

LEMMA 3.1 [G–J, Lemma 2.1]. *Let $\{Q_j\}$ be a sequence of cubes in $\mathbb{R}^d$ satisfying the packing condition*

$$\sum_{Q_j \subset Q} |Q_j| \leqslant C_1 |Q| \qquad \textit{for all cubes } Q.$$

Let a_j be adapted to Q_j. Then $\sum_j a_j \in \mathrm{BMO}$ and $\|\sum a_j\|_* \leqslant CbC_1$ where C is a universal constant.

LEMMA 3.2. *With the same conditions as in the above lemma, assume moreover $|\mathrm{grad}\, a_j| \leqslant b(\ell(Q_j))\, \ell(Q_j)^{-1}$ where b is a bounded positive function with $\lim_{t\to 0} b(t) = \lim_{t\to\infty} b(t) = 0$. Then $\sum_j a_j \in \mathrm{VMO}$.*

An easy proof of Lemma 3.1 is given in [G–J, p. 379] and Lemma 3.2 follows from a slight modification of it.

Proof of Main Lemma. It should be observed that (v) follows from (i) and (ii). Without loss of generality we assume that $A \cap Q = \bigcup_j Q_j$ is a countable union of dyadic subcubes of Q, that have pairwise disjoint interiors.

Finite Case. We begin by considering the case when $A \cap Q$ is a finite union of dyadic cubes, that is, $A \cap Q = \bigcup_{j=1}^N Q_j$. Let us proceed with a standard stopping time argument. Consider the family $\{L\}$ of dyadic subcubes of Q. Let $\{L_j^1\}$ denote the finite collection of maximal dyadic cubes satisfying

$$2\varepsilon\, |L| < |L \cap A|.$$

They are called first generation stopping time cubes and have pairwise disjoint interiors. Since L_j^1 is maximal we have $2\varepsilon\, |L_j^1| < |L_j^1 \cap A| \leqslant 2\varepsilon 2^d\, |L_j^1|$. On each L_j^1, repeat the procedure, this time with the condition $|L \cap A| > 2(2\varepsilon 2^d)\, |L|$. In this way a collection of second generation stopping time cubes $\{L_j^2\}$ is obtained. Continuing the process $M(d, \varepsilon)$ times (to be made precise later), we will get a finite number of finite families of dyadic cubes $\{L_j^k\}$, $k = 1, ..., M(d, \varepsilon)$, such that

(a) Each family $\{L_j^k\}$ is maximal according to the rule $|L \cap A| > 2\varepsilon(2 \cdot 2^d)^{k-1}\, |L|$.

(b) For every k, one has $Q \cap A \subset \bigcup_{j=1}^{j(k)} L_j^k$, $j(k) \leqslant N$.

(c) If $k_1 > k_2$ then either $L_j^{k_1}$ and $L_i^{k_2}$ have disjoint interiors or $L_j^{k_1} \subset L_i^{k_2}$.

(d) For any L_j^k one has $2\varepsilon(2\cdot 2^d)^{k-1}\,|L_j^k| < |L_j^k \cap A| \leqslant \varepsilon(2\cdot 2^d)^k\,|L_j^k|$. Thus

$$\sum_{L_i^k \subset L_j^{k-1}} |L_i^k| \leqslant \frac{|L_j^{k-1}|}{2}.$$

To determine the bound $M(d, \varepsilon)$, we decide that the process will terminate whenever we get a cube L_i^k of the collection of cubes Q_j of $Q \cap A$. In this case, one has

$$2\varepsilon(2\cdot 2^d)^{k-1}\,|L^k| < |L^k| = |L^k \cap A| \leqslant \varepsilon(2\cdot 2^d)^k\,|L^k|,$$

that is,

$$2\varepsilon(2\cdot 2^d)^{k-1} < 1 \leqslant \varepsilon(2\cdot 2^d)^k.$$

Therefore we take $M = M(d, \varepsilon)$ equal to the maximum of 1 and the integer part of $((d+1)\log 2)^{-1}\log \varepsilon^{-1}$.

Let a_j^k be a Lipschitz function adapted to the cube L_j^k so that $0 \leqslant a_j^k \leqslant 1$ and $a_j^k \equiv 1$ on L_j^k. Then the continuous function $\sum_{k=1}^{M} \sum_j a_j^k$ is supported on $\frac{3}{2}Q$, takes values between 0 and M, is equal to M on $Q \cap A$, and by Lemma 3.1 has BMO-norm bounded by a universal constant. We can apply Lemma 3.1 because property (d) implies the necessary packing condition.

Finally, define

$$\varphi = \frac{\log \varepsilon^{-1}}{M} \sum_k \sum_j a_j^k$$

so that $\varphi(x) = \log \varepsilon^{-1}$ if $x \in Q \cap A$. Since $M \geqslant C(d)\log \varepsilon^{-1}$, one has $\|\varphi\|_* \leqslant C$, where C is a constant depending on d but independent of ε, Q, and A. So we have proved the lemma when $Q \cap A$ is a finite union of cubes.

General Case. Let $A \cap Q = \bigcup_{j=1}^{\infty} Q_j$. We choose an increasing sequence of integers $O = N_0 < N_1 < N_2 < \cdots < N_k < \cdots$ with the condition

$$\sum_{j=N_{k-1}+1}^{N_k} |Q_j| < \varepsilon^{k^2}\,|Q|, \qquad k = 1, 2, \ldots$$

Set

$$F_k = \bigcup_{j=N_{k-1}+1}^{N_k} Q_j.$$

Now, on each F_k we proceed as in the previous finite case and we get a continuous function φ_k supported in $\frac{3}{2}Q$ and $\|\varphi_k\|_* \leqslant C$. Moreover, $0 \leqslant \varphi_k \leqslant k^2 \log \varepsilon^{-1}$, $\varphi(x) = k^2 \log \varepsilon^{-1}$ if $x \in F_k$ and

$$\fint_Q \varphi_k \leqslant C.$$

Write

$$\tilde{\varphi} = \sum_{k=1}^{\infty} k^{-2} \varphi_k.$$

Note that the nonnegative function $\tilde{\varphi}$ is supported in $\frac{3}{2}Q$, and if $x \in A \cap Q$ then $\tilde{\varphi}(x) \geqslant \log \varepsilon^{-1}$; moreover, $\fint_Q \tilde{\varphi} \leqslant C$ and $\|\tilde{\varphi}\|_* \leqslant \sum_k k^{-2} \|\varphi_k\|_* \leqslant C$. The last task is to check that $\tilde{\varphi}$ belongs to VMO. Fix $\varepsilon > 0$ and choose N so that $\sum_{k>N} k^{-2} < \varepsilon(2C)^{-1}$. Therefore

$$\begin{aligned} \fint_Q |\tilde{\varphi} - \tilde{\varphi}_Q| &\leqslant \sum_{k=1}^{N} \fint_Q k^{-2} |\varphi_k - (\varphi_k)_Q| + \sum_{k>N} k^{-2} \|\varphi_k\|_* \\ &< \sum_{k=1}^{N} \fint |\varphi_k - (\varphi_k)_Q| + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

if $|Q|$ is either sufficiently small or sufficiently large, because the last sum involves a finite number of VMO functions.

Finally, we take

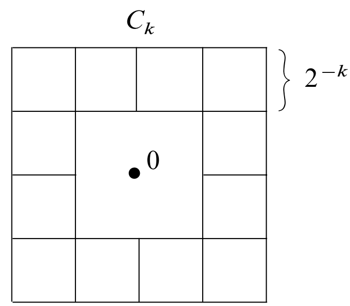
$$\varphi(x) = \min(\tilde{\varphi}(x), \log \varepsilon^{-1}). \quad \blacksquare$$

We start by proving Theorem 4. Theorem 5 will follow from a modification of that proof.

Proof of Theorem 4 *and Theorem* 5. One may assume $x_0 = 0$. Consider the Whitney type decomposition of $\mathbb{R}^d \backslash \{0\}$. That is, $\mathbb{R}^d \backslash \{0\}$ may be written as a "disjoint" union of dyadic cubes

$$\mathbb{R}^d \backslash \{0\} = \bigcup_{i=-\infty}^{\infty} \bigcup_{j=1}^{k(d)} Q_{i,j},$$

where $k(d) = 4^d - 2^d = 2^d(2^d - 1)$ and $Q_{i,j}$ is a dyadic cube of $\mathbb{R}^d$, with $\ell(Q_{i,j}) = 2^i$ and contained in $Q(2^{i+2}) \backslash Q(2^{i+1})$. From now on, $Q(\delta)$ will denote the cube of side length δ centered at the origin.



For each integer $k \geqslant 1$, we denote by C_k the "annuli" of size 2^{-k}; that is,

$$C_k = \bigcup_{j=1}^{k(d)} Q_{k,\,j}\,.$$

Since the set A has null density at 0 we have

$$|Q_{k,\,j} \cap A| < \varepsilon_k \, |Q_{k,\,j}|$$

where $\{\varepsilon_k\}$ is a nonincreasing sequence tending to zero. Next we apply the Main Lemma to each $Q_{k,\,j}$ and we get the corresponding function $\varphi_{k,\,j}$. Write

$$\varphi_k = \sum_{j=1}^{k(d)} \varphi_{k,\,j}\,.$$

If we defined $g = \sum_{k=1}^{\infty} \varphi_k$, we would get the properties in Theorem 4, but maybe g is not in VMO. Since we want g to be in VMO, we will multiply the functions φ_k by suitable constants and then we will add them.

The above sequence $\{\varepsilon_k\}$ can tend to zero very slowly. Let $\{\eta_k\}$ be a strictly decreasing sequence (to be determined later) of positive real numbers tending to zero very quickly. For $m \geqslant 1$ let $I_m = (\eta_{m+1}, \eta_m]$ and define the nonnegative functions

$$\Psi_0 = \sum_{\varepsilon_k \in (\eta_1,\, \infty)} \varphi_k,$$

$$\Psi_m = \sum_{\varepsilon_k \in I_m} \varphi_k.$$

We note that if $x \in A \cap (\bigcup_{\varepsilon_k \in I_m} C_k)$ one has $\Psi_m(x) \geqslant |\log \eta_m|$ because since $\varepsilon_k \in I_m$, $\varphi_k(x) = |\log \varepsilon_k| \geqslant |\log \eta_m|$. Notice also that Ψ_m have BMO norm uniformly bounded, $\|\Psi_m\|_* \leqslant C \max \|\varphi_{k,\,j}\|_* \leqslant C$, because the functions $\{\varphi_{k,j}\}$, involved in the definition of Ψ_m, have almost disjoint supports [B–V, Lemma 1.7.3]. Clearly, $\Psi_m \in \mathrm{VMO}$, and $⨍_{Q(\delta)} \Psi_m \leqslant C$.

Consider a sequence of positive real numbers $\{\alpha_j\}$ such that

(i) $\sum_{j=1}^{\infty} \alpha_j < +\infty$.

(ii) $\lim_{j\to\infty} \alpha_j \log \eta_j^{-1} = +\infty$.

(iii) The sequence $d_{j+1} = \alpha_{j+1} \log \eta_{j+1}^{-1} - \alpha_j \log \eta_j^{-1}$ is decreasing and tends to zero.

For instance, one could take $\eta_j = 2^{-2^j}$ and $\alpha_j = 2^{-j}\sqrt{j}$ to get the required properties. The above condition (iii) will only be used in proving Theorem 5.

Finally we define

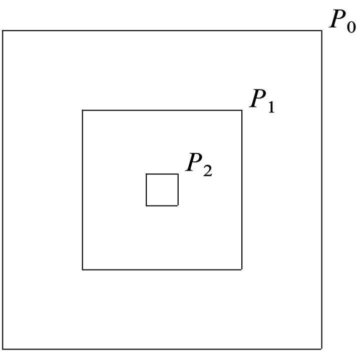
$$g = \sum_m \alpha_m \Psi_m .$$

Clearly, the function g satisfies the properties (a), (b), and (c) in Theorem 4. The only remaining task is to check that $g \in \mathrm{VMO}$. Fix $\varepsilon > 0$ and choose N so that $\sum_{m>N} \alpha_m < \varepsilon (2C)^{-1}$. Thus

$$\begin{aligned} ⨍_Q |g - g_Q| &\leqslant \sum_{m=1}^{N} \alpha_m ⨍_Q |\Psi_m - (\Psi_m)_Q| + \sum_{m>N} \alpha_m \|\Psi_m\|_* \\ &< \sum_{m=1}^{N} ⨍_Q |\Psi_m - (\Psi_m)_Q| + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

if $|Q|$ is either sufficiently small or sufficiently large. Consequently, Theorem 4 is proved.

To prove Theorem 5 we choose a sequence of cubes centered at 0 and tending to 0:



P_0 P_1 P_2

Consider the cubes

$$P_0 = \overline{\bigcup_{k=1}^{\infty} C_k}$$

$$P_m = \overline{\bigcup_{\varepsilon_k \leqslant \eta_m} C_k} \qquad \text{if} \quad m \geqslant 1.$$

Obviously, $\{P_m\}$ satisfies the packing condition mentioned in Lemma 3.1. We assign to every cube P_m a Lipschitz function h_m supported on $\frac{3}{2} P_m$ such that $0 \leqslant h_m(x) \leqslant d_m$, h_m is equal to d_m on P_m and $|\text{grad } h_m| \leqslant 4 d_m \ell(P_m)^{-1}$. Define the unbounded function

$$h = \sum_m h_m.$$

By Lemma 3.2, $h \in \text{VMO}$ and $\|h\|_* \leqslant C$. By construction

$$\lim_{\delta \to 0} \fint_{Q(\delta)} h = +\infty$$

because $h(x) \geqslant \alpha_m \log \eta_m^{-1}$ if $x \in P_m$. But we would like the function to be bounded on the set A. The function g from Theorem 4 is unbounded on A. One could consider a function $h - g$ that satisfies $\|h-g\|_* \leqslant 2C$, $h - g \in$ VMO, and $\lim_{\delta \to 0} \fint_{Q(\delta)} (h-g) = +\infty$, but still we could not assure that $h - g$ is bounded on A.

However, if for all integer $m \geqslant 1$ we had the estimates

$$\alpha_m \log \eta_m^{-1} \leqslant g(x) \leqslant \alpha_{m+1} \log \eta_{m+1}^{-1} \qquad \text{for any} \quad x \in A \cap (P_m \backslash P_{m+1}), \tag{3.1}$$

then we would finish the proof because

$$|h(x) - g(x)| \leqslant d_m, \qquad x \in A \cap (P_m \backslash P_{m+1}).$$

We need to modify the construction of the function g in Theorem 4. Recall that the composition of a BMO (VMO) function g with a real Lipschitz function T is again a BMO (VMO) function and

$$\|T \circ g\|_* \leqslant \|T\|_{\text{Lip}(1)} \|g\|_*.$$

By truncation we can assume $0 \leqslant \Psi_m \leqslant \log \eta_m^{-1}$ and also if $x \in A \cap (P_m \backslash P_{m+1})$ then $\Psi_m(x) = \log \eta_m^{-1}$. Let

$$T_m(t) = \begin{cases} t & \text{if} \quad t < \alpha_m \log \eta_m^{-1} \\ \alpha_m \log \eta_m^{-1} & \text{otherwise.} \end{cases}$$

Write

$$g_1 = \alpha_1 \Psi_1$$

and if $m \geqslant 2$

$$g_m = T_m(\alpha_m \Psi_m + g_{m-1}).$$

Since T_m is Lipschitz, one has

$$\|g_m\|_* \leqslant \alpha_m \|\Psi_m\|_* + \|g_{m-1}\|_* \leqslant \sum_{i=1}^{m} \alpha_i \|\Psi_i\|_* \leqslant C.$$

Observe that if $x \in P_m \backslash P_{m+1}$ then $g_j(x) = g_{m+1}(x)$ for all $j \geqslant m+1$. So we may take a limit and define

$$g(x) = \lim_{m \to \infty} g_m(x).$$

Moreover, $\{g_m\}$ tends to g in L^1 because $0 \leqslant g_m \leqslant \sum_j \alpha_j \Psi_j \in L^1$. Clearly, g satisfies (3.1) and $\lim_{\delta \to 0} ⨍_{Q(\delta)} g = 0$. Now, we have to see that $\|g\|_* \leqslant C$ and $g \in \mathrm{VMO}$. It turns out that $\sup_m \|g_m\|_* < \infty$ and the functions g_m are uniformly in VMO, that is, given $\varepsilon > 0$ there are $\delta > 0$ and $R > 1$, independent of m, such that if $|Q| < \delta$ or $|Q| > R$, then

$$⨍_Q |g_m - (g_m)_Q| < \varepsilon.$$

Take $(C_1)_Q = \alpha_1 (\Psi_1)_Q$ and if $m \geqslant 2$,

$$(C_m)_Q = T_m(\alpha_m (\Psi_m)_Q + (C_{m-1})_Q).$$

Thus

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q |g_m - (C_m)_Q| \\
&\quad = \frac{1}{|Q|} \int_Q |T_m(\alpha_m \Psi_m + g_{m-1}) - T_m(\alpha_m (\Psi_m)_Q + (C_{m-1})_Q)| \\
&\quad \leqslant \frac{\alpha_m}{|Q|} \int_Q |\Psi_m - (\Psi_m)_Q| + \frac{1}{|Q|} \int_Q |g_{m-1} - (C_{m-1})_Q| \\
&\quad \leqslant \frac{1}{|Q|} \sum_{j=1}^{m} \alpha_j \int_Q |\Psi_j - (\Psi_j)_Q|.
\end{aligned}$$

Therefore

$$\|g_m\|_* \leqslant \sum_{j=1}^{m} \alpha_j \|\Psi_j\|_* \leqslant C \sum_{j=1}^{m} |\alpha_j| \leqslant C.$$

Moreover, given $\varepsilon > 0$ choose an integer N such that $\sum_{j>N} \alpha_j < \varepsilon/2C$. Then

$$\frac{1}{|Q|} \int_Q |g_m - (C_m)_Q| \leqslant \sum_{j=1}^{N} \frac{1}{|Q|} \int_Q |\Psi_j - (\Psi_j)_Q| + \frac{\varepsilon}{2} < \varepsilon$$

if $|Q|$ is either small enough or large enough. So the functions g_m are uniformly in VMO.

On the other hand, $g_Q = \lim_{m\to\infty} (g_m)_Q$ since $g_m \to g$ in L^1. Consequently

$$\begin{aligned}\frac{1}{|Q|} \int_Q |g - g_Q| &= \frac{1}{|Q|} \int_Q \lim_{m\to\infty} |g_m - (g_m)_Q| \\ &\leqslant \varliminf_{m\to\infty} \frac{1}{|Q|} \int_{|Q|} |g_m - (g_m)_Q| < \varepsilon\end{aligned}$$

if either $|Q| < \delta$ or $|Q| > R$.

Finally, define

$$f = |h - g|;$$

then f satisfies the conclusions of Theorem 5. ∎

4. FARRELL AND DOMINATING SETS

This section is devoted to the proofs of Theorems 1, 2, and 3. Let F be a relatively compact set in the upper half space. Given $\alpha > 0$, let $F_{\mathrm{nt}}(\alpha)$ be the set of points $x \in \mathbb{R}^d$ that are in the closure of $F \cap \Gamma(x, \alpha)$. Thus

$$F_{\mathrm{nt}} = \bigcup_{\alpha>0} F_{\mathrm{nt}}(\alpha).$$

However, for any $\alpha > 0$, the set $F_{\mathrm{nt}} \backslash F_{\mathrm{nt}}(\alpha)$ has zero measure (e.g., [T]). So, except for a set of zero measure, the nontangential closure may be defined using cones of a fixed aperture.

For fixed $k > 1$, given a point z in the upper half space $\mathbb{R}^{d+1}_+$, let

$$T(z) = T_k(z) = \{x \in \mathbb{R}^d : |x - z| < k z_{d+1}\} \tag{4.1}$$

be its scope over $\mathbb{R}^d$. Given a set $F \subset \mathbb{R}^{d+1}_+$ and $\delta > 0$, let F_δ denote the open set

$$F_\delta = \bigcup_{z \in F,\, z_{d+1} < \delta} T(z).$$

Thus, except for a set of measure zero, F_{nt} is the intersection of F_δ; that is,

$$\left| F_{nt} \Big\backslash \bigcap_{\delta > 0} F_\delta \right| = 0.$$

So $|\bar{F} \cap \mathbb{R}^d \backslash F_{nt}| > 0$ if and only if there exists $\eta > 0$ such that

$$|\bar{F} \cap \mathbb{R}^d \backslash F_\eta| > 0.$$

Two proofs of Theorem 3 are presented. The first, using Theorem 5, is shorter, while the second is based on Corollary 4.1 below, which is a consequence of the easier Theorem 4. We will also use this corollary in the proof of Theorem 2.

First Proof of Theorem 3. Part (b) follows trivially from (a). Since Poisson extensions of BMO functions have nontangential limits at almost every point, one gets (a) from (c). So one only has to show that (c) is necessary.

Assume $|\bar{F} \cap \mathbb{R}^d \backslash F_{\mathrm{nt}}| > 0$; that is, $|\bar{F} \cap \mathbb{R}^d \backslash F_\eta| > 0$ for some $\eta > 0$. Let x be a point of density of $\bar{F} \cap \mathbb{R}^d \backslash F_\eta$. Observe that $x \in \bar{F}_\eta$ and apply Theorem 5 with the set $A = F_\eta$. So one obtains a nonnegative function $g \in \mathrm{VMO}(\mathbb{R}^d)$ such that $g(y) \leqslant 1$ at every point $y \in F_\eta$ and

$$\lim_{\delta \to 0} \frac{1}{|Q(x, \delta)|} \int_{Q(x, \delta)} g(y)\, dy = \infty. \tag{4.2}$$

Then, if k in (4.1) is sufficiently large, Lemma 2.2 gives $\sup\{g(z) \colon z \in F, z_{n+1} < \eta\} \leqslant 2$. Consequently

$$\sup\{g(z) \colon z \in F\} < \infty.$$

On the other hand, since $g \in \mathrm{VMO}$, one has

$$\frac{1}{|Q(x, \delta)|} \int_{Q(x, \delta)} |g(y) - g_{Q(x, \delta)}|\, dy \to 0 \qquad \text{as} \quad \delta \to 0.$$

Now, the Chebyshev inequality gives that for any $\varepsilon > 0$ one has

$$\frac{1}{|Q(x, \delta)|} |\{ y \in Q(x, \delta) : |g(y) - g_{Q(x, \delta)}| > \varepsilon\}| \to 0 \qquad \text{as} \quad \delta \to 0.$$

This, together with (4.2), implies that for any $M>0$ one gets

$$\lim_{\delta\to 0}\frac{|\{y\in Q(x,\delta): g(y)>M\}|}{|Q(x,\delta)|}=1.$$

Since x is a point of density of $\bar{F}\cap\mathbb{R}^d$ one deduces that

$$\lim_{\delta\to 0}\frac{|\{y\in Q(x,\delta)\cap\bar{F}: g(y)>M\}|}{|Q(x,\delta)|}=1.$$

This shows that $g\notin L^\infty(\bar{F})$ and finishes the proof. ∎

We will need the following auxiliary result.

COROLLARY 4.1. *Let F be a relatively compact set of the upper half space $\mathbb{R}^{d+1}_+$. Assume $|\bar{F}\cap\mathbb{R}^d\backslash F_{\rm nt}|>0$. Then, for almost every point $x\in\bar{F}\cap\mathbb{R}^d\backslash F_{nt}$ and for any $N>0$, there exists a nonnegative function $g=g(x,N)\in$ VMO$(\mathbb{R}^d)$, $\|g\|_*\leqslant 1$, satisfying*

(a) $\inf_{z\in F} g(z)\geqslant N$.

(b) $\frac{1}{|Q(x,\delta)|}\int_{Q(x,\delta)} g\to 0$ *as* $\delta\to 0$.

(c) $\inf\{g(z): z\in F, |z-x|<\delta\}\to\infty$ *as* $\delta\to 0$.

Proof of Corollary 4.1. Since $|\bar{F}\cap\mathbb{R}^d\backslash F_{nt}|>0$ there exists $\eta>0$ such that $|\bar{F}\cap\mathbb{R}^d\backslash F_\eta|>0$. Let $x\in\mathbb{R}^d$ be a density point of this set. In particular $x\in\bar{F}_\eta$, and

$$\lim_{\delta\to 0}\frac{|Q(x,\delta)\cap\bar{F}|}{|Q(x,\delta)|}=1,$$

and

$$\lim_{\delta\to 0}\frac{|Q(x,\delta)\cap F_\eta|}{|Q(x,\delta)|}=0.$$

Applying Theorem 4 one obtains a function $f\geqslant 0$, $f\in$ VMO, $\|f\|_*\leqslant\frac{1}{2}$, satisfying $\inf\{f(y): y\in F_\eta, |y-x|<\delta\}\to\infty$ as $\delta\to 0$ and

$$\lim_{\delta\to 0}\frac{1}{|Q(x,\delta)|}\int_{Q(x,\delta)} f(y)\,dy=0.$$

We claim that, given $\delta_0>0$ and $N>0$, one has a nonnegative function $h\in$ VMO$(\mathbb{R}^d)$ such that $\|h\|_*\leqslant\frac{1}{2}$, $h(y)\geqslant 2N$ for all $y\in\mathbb{R}^d\backslash Q_{\delta_0}(x)$, and $h\equiv 0$ on a neighbourhood of x. Easily, if δ_0 is sufficiently small, one deduces $h(z)\geqslant N$ if $z\in\mathbb{R}^{d+1}_+$, $z_{n+1}\geqslant\eta$.

Finally, the function $g = f + h$ satisfies the conditions of Corollary 4.1. Recall that given $z \in \mathbb{R}^{d+1}_{+}$ one has

$$\left| g(z) - \frac{1}{|T(z)|} \int_{T(z)} g(y)\, dy \right| \leqslant C \, \|g\|_{*}.$$

Thus, $g(z) \geqslant N$ at any point $z \in F$, $z_{n+1} \leqslant \eta$.

Observe that the function H defined by $H(y) = |\log |x - y||^{1/2}$ if $|x - y| \leqslant 1$ and $H(y) = 0$ otherwise belongs to VMO, $\|H\|_{*} \leqslant C(d)$. Truncating the function H, one gets the function h in the Claim. ∎

Second Proof of Theorem 3. Assume (b) holds. A standard application of the Open Mapping Theorem shows that there exists a constant $C > 0$ such that for any $u \in \mathrm{VMO}(\mathbb{R}^d) \cap L^\infty(F)$, one has

$$\|u\|_{L^\infty(\bar{F})} \leqslant C(\|u\|_{*} + \|u\|_{L^\infty(F)}). \tag{4.3}$$

Assume $|\bar{F} \cap \mathbb{R}^d \backslash F_{\mathrm{nt}}| > 0$. Let $x \in \mathbb{R}^d \cap \bar{F} \backslash F_{\mathrm{nt}}$ be a point given by Corollary 4.1, let $N > 0$ be a large number to be fixed later, and let g_N be the corresponding function and $g = g_N N^{-1/2}$. Define $f = \exp(-g)$. Thus, f is in VMO with $\|f\|_{*} \leqslant N^{-1/2}$, $0 \leqslant f \leqslant 1$, $\sup\{f(z) \colon z \in F\} \leqslant \exp(-N^{1/2})$, and $\lim_{\delta \to 0} f_{Q(x,\delta)} = 1$. Therefore,

$$\lim_{\delta \to 0} \frac{|\{y \in Q(x, \delta) : f(y) \geqslant \frac{1}{2}\}|}{|Q(x, \delta)|} = 1.$$

Since x is a density point of $\bar{F} \cap \mathbb{R}^d$ we deduce

$$\lim_{\delta \to 0} \frac{|\{y \in Q(x, \delta) \cap \bar{F} : f(y) \geqslant \frac{1}{2}\}|}{|Q(x, \delta)|} = 1.$$

So $\|f\|_{L^\infty(\bar{F})} \geqslant \frac{1}{2}$. But, for N large enough, the function f contradicts (4.3). ∎

Proof of Theorem 1. We first show that condition (c) is necessary. We will proceed as in the first proof of Theorem 3. Assume $|\bar{F} \cap \mathbb{R}^d \backslash F_{\mathrm{nt}}| > 0$, that is, $|\bar{F} \cap \mathbb{R}^d \backslash F_\eta| > 0$ for some $\eta > 0$. Let x be a density point of $\bar{F} \cap \mathbb{R}^d \backslash F_\eta$. Observe that $x \in \bar{F}_\eta$ and apply Theorem 5 with the set $A = F_\eta$. So one obtains a nonnegative function $g \in \mathrm{VMO}(\mathbb{R}^d)$, $g(y) \leqslant 1$ for all $y \in F_\eta$, and

$$\lim_{\delta \to 0} g_{Q(x,\delta)} = \infty. \tag{4.4}$$

Then, if the parameter k in (4.1) is sufficiently large, Lemma 2.2 gives that $\sup\{g(z): z \in F, z_{n+1} < \eta\} \leqslant 2$. Consequently

$$\sup\{g(z): z \in F\} < \infty.$$

Now, if F is a Farrell set for BMO (or VMO) there is a sequence of continuous functions $\{P_n\}$ tending to g in the weak-$*$ topology, $P_n(z) \to g(z)$ for any $z \in F$ and $\|P_n\|_F \to \|g\|_F$. Thus, for some absolute constant C and for all n, $|P_n(z)| \leqslant C$, for any $z \in F$, $\|P_n\|_* \leqslant C$ and, by Lemma 2.1, we have

$$P_n(z) \to g(z) \qquad \text{for any} \quad z \in \mathbb{R}^{d+1}_+. \tag{4.5}$$

By continuity, $|P_n(y)| \leqslant C$ at every point $y \in \bar{F} \cap \mathbb{R}^d$. Next, using that x is a density point of $\bar{F} \cap \mathbb{R}^d$ and $\sup_n \|P_n\|_* \leqslant C$ we get

$$|(P_n)_{Q(x,\delta)}| \leqslant 2C$$

if δ is small. Then, from the estimate $|P_n(z) - (P_n)_{T(z)}| \leqslant C \|P_n\|_*$ we deduce that $|P_n(z)| \leqslant 4C$ for all n where $z = (x, t)$ and $0 < t < \delta$. This contradicts (4.5) because from (4.4) the values $g(z)$ are unbounded when $z = (x, t)$, and t tends to 0.

Conversely, assume $|\bar{F} \cap \mathbb{R}^d \backslash F_{nt}| = 0$ and let us show that F is a Farrell set for BMO (VMO). So, given $f \in \text{BMO}$ ($f \in \text{VMO}$), $\|f\|_* = 1$, $|f(z)| \leqslant 1$ for any $z \in F$, one has to find continuous functions P_k tending to f in the weak-$*$ (norm) topology, pointwise in F, and satisfying

$$|P_k(z)| \leqslant 1 \qquad \text{if} \quad z \in F. \tag{4.6}$$

Observe that, by Lemma 2.4 and Theorem 2.5, one can assume that f is bounded. We now claim that it is sufficient to have condition (4.6) for points $x \in \bar{F} \cap \mathbb{R}^d$, that is, given $\varepsilon > 0$ it is enough to find continuous functions Φ_k tending to f in the weak-$*$ (norm) topology, pointwise in F, $\|\Phi_k\|_\infty \leqslant C \|f\|_\infty$, where C is a universal constant, and satisfying

$$|\Phi_k(x)| < 1 + \varepsilon, \qquad \text{for} \quad x \in \bar{F} \cap \mathbb{R}^d. \tag{4.7}$$

To establish this claim, observe that Lemma 2.1 gives that Φ_k tend to f uniformly on compact sets of $\mathbb{R}^{d+1}_+$. Hence, there exists $\eta_k > 0$, $\eta_k \to 0$ as $k \to \infty$, such that

$$|\Phi_k(z)| \leqslant 1 + \varepsilon \qquad \text{for any} \quad z \in F \quad \text{such that } z_{d+1} \geqslant \eta_k.$$

Also since Φ_k are continuous, condition (4.7) gives that there exists a neighborhood U of $\bar{F} \cap \mathbb{R}^d$ in $\{(x, t): x \in \mathbb{R}^d, t \geqslant 0\}$ such that $|\Phi_k(z)| \leqslant 1+\varepsilon$ for $z \in U$. So there is $\delta_k > 0$ such that

$$|\Phi_k(z)| \leqslant 1+\varepsilon \qquad \text{for any} \quad z \in F \quad \text{with } z_{d+1} \leqslant \delta_k.$$

Consequently, $|\Phi_k(z)| \leqslant 1+\varepsilon$ for the points $z \in F$ satisfying either $z_{d+1} \leqslant \delta_k$ or $z_{d+1} \geqslant \eta_k$. Considering a subsequence of Φ_k one may assume that $\eta_k > \delta_k > \eta_{k+1}$ for any $k = 1, 2, \ldots$. Now, one can take

$$P_N = \frac{1}{N} \sum_{k=N+1}^{2N} \Phi_k.$$

It is clear that P_N tend to f in the weak-$*$ (norm) topology and pointwise in F. Also, if $z \in F$ there is at most one k such that $\delta_k < z_{d+1} < \eta_k$ or $\eta_{k+1} < z_{d+1} < \delta_k$. Hence for points $z \in F$, one has

$$|P_N(z)| \leqslant \frac{N-1}{N}(1+\varepsilon) + \frac{\|\Phi_k\|_\infty}{N} \leqslant \frac{(N-1)(1+\varepsilon) + C\,\|f\|_\infty}{N}$$

and this would finish the proof. Therefore, one only has to find the functions Φ_k mentioned in the previous claim.

Since $|f(z)| \leqslant 1$ for $z \in F$ and $|\bar{F} \cap \mathbb{R}^d \setminus F_{nt}| = 0$, Fatou's Theorem gives that $|f(x)| \leqslant 1$ at almost every point $x \in \bar{F} \cap \mathbb{R}^d$. Given $k = 1, 2, \ldots$ let $D(k)$ denote the collection of dyadic cubes in $\mathbb{R}^d$ of length side 2^{-k}. Proposition 2.3 asserts that the functions

$$\varphi_k = \sum_{Q \in D(k)} a_Q \Psi_Q$$

tend to f in the weak-$*$ topology and if $f \in \mathrm{VMO}$ tend to f in norm. Now, we require the continuity of the functions Ψ_Q and therefore the functions φ_k will be continuous. However, the natural choice $\Phi_k = \varphi_k$ does not work becuase φ_k may not be bounded by $1+\varepsilon$ on $\bar{F}$. The same trouble would appear if we took $\Phi_k(x) = f(x, k^{-1})$, the restriction of f at level k^{-1}.

Denote by $\mathscr{B} = \mathscr{B}(k)$ the subcollection of those cubes Q in $D(k)$ satisfying that $\bar{F} \cap \frac{5}{4}Q \neq \emptyset$ and $\mathscr{A} = \mathscr{A}(k, \varepsilon)$ those cubes in $\mathscr{B}$ such that $|a_Q| > 1+\varepsilon$. If $x \in \bar{F} \setminus \bigcup_{\mathscr{A}} \frac{5}{4}Q$ then $|\varphi_k(x)| \leqslant \sum_Q |a_Q|\, \Psi_Q(x) \leqslant 1+\varepsilon$. Thus, we should modify φ_k on the points $x \in \bar{F} \cap (\bigcup_{\mathscr{A}} \frac{5}{4}Q)$. We next claim that

$$\sum_{Q \in \mathscr{A}} |Q| \to 0 \qquad \text{as} \quad k \to \infty. \tag{4.8}$$

Note first that $\sum_{Q\in\mathscr{B}} \Psi_Q \to \chi_{\bar{F}}$ in $L^1(\mathbb{R}^d)$ as $k\to\infty$, because $\bar{F}$ is a compact set, and so $\sum_{Q\in\mathscr{B}} a_Q \Psi_Q$ tend to $f\chi_{\bar{F}}$ in $L^1(\mathbb{R}^d)$. Fix a cube $Q\in\mathscr{A}$. For all $x\in(\frac{3}{4}Q)\backslash\bar{F}$, one has

$$\left|\sum_{Q\in\mathscr{B}} a_Q\Psi_Q(x)-f(x)\,\chi_{\bar{F}}(x)\right|=|a_Q|>1+\varepsilon$$

and for almost every $x\in(\frac{3}{4}Q)\cap\bar{F}$

$$\left|\sum_{Q\in\mathscr{B}} a_Q\Psi_Q(x)-f(x)\,\chi_{\bar{F}}(x)\right|\geqslant|a_Q|-|f(x)|>1+\varepsilon-1=\varepsilon.$$

Consequently,

$$\left|\bigcup_{\mathscr{A}}\tfrac{3}{4}Q\right|\leqslant\left|\left\{x\in\mathbb{R}^d:\left|\sum_{\mathscr{B}} a_Q\Psi_Q(x)-f(x)\,\chi_{\bar{F}}(x)\right|>\varepsilon\right\}\right|\xrightarrow[k\to\infty]{}0$$

and then we get (4.8).

We first assume that functions φ_k are nonnegative. For each k we will construct a nonnegative continuous function g_k supported in $\bigcup_{Q\in\mathscr{A}}2Q$ with $\|g_k\|_*\leqslant C$, where C is an universal constant, such that $\varphi_k(x)-g_k(x)\leqslant 1+\varepsilon$ for all $x\in\bar{F}\cap\mathbb{R}^d$. In the BMO setting, $g_k\to 0$ in the weak-$*$ topology as $k\to\infty$. Then $\Phi_k=\max(\varphi_k-g_k,0)$ will satisfy our claim. Notice that $\Phi_k\to f$ in the weak-$*$ topology because $\varphi_k-\Phi_k$ tend to 0. When $f\in\mathrm{VMO}$ we also will get $g_k\to 0$ in BMO and then $\Phi_k=\max(\varphi_k-g_k,0)$ tend to f in BMO (using now that φ_k are uniformly in VMO).

To construct the functions g_k first we point out that there exist constants $C_1,\ C_2>0$ such that for any $Q\in D(k)$ one has

$$|\tfrac{5}{4}Q\cap\bar{F}|\leqslant C_1\exp(-C_2\,|a_Q|)\,|\tfrac{5}{4}Q|. \tag{4.9}$$

To see this one may assume that $|a_Q|$ is large. Remember that a_Q is close to $f_{(5/4)Q}$; that is, $|a_Q-f_{5/4Q}|\leqslant C\,\|f\|_*$. Then (4.9) follows from the John–Nirenberg Theorem applied to $\lambda=|a_Q|-1$ because $|f(x)|\leqslant 1$ at almost every $x\in\bar{F}\cap\mathbb{R}^d$. Moreover, if $f\in\mathrm{VMO}$ one may take $C_2=C_2(k)\to\infty$ as $k\to\infty$.

Fix a cube $Q\in\mathscr{A}$. Now, we apply the proof of the Main Lemma (finite case, because $\frac{5}{4}Q\cap\bar{F}$ is compact) and we obtain a nonnegative continuous function $g=g_Q$ satisfying

$$g \equiv 0 \qquad \text{on } (2Q)^c$$

$$g \equiv a_Q \qquad \text{on } \tfrac{5}{4}Q \cap \bar{F},$$

$$⨍_Q g \leqslant C,$$

$$\|g\|_\infty \leqslant C \|f\|_\infty,$$

$$\|g\|_* \leqslant m(C_2),$$

where C is a constant independent of k and $m(C_2) \to 0$ as $C_2 \to \infty$. In particular, if $f \in \mathrm{VMO}$, $m(C_2) \to 0$ as $k \to \infty$. Define

$$g_k = \sum_{Q \in \mathscr{A}} g_Q.$$

Again, $\|g_k\|_* \leqslant C \max_{Q \in \mathscr{A}} \|g_Q\|_* \leqslant Cm(C_2)$. From (4.8) and the estimate $\|g_k\|_\infty \leqslant C \|f\|_\infty$ we have $g_k \to 0$ in L^1. Thus, $g_k \to 0$ in the weak-$*$ topology (or in norm if $f \in \mathrm{VMO}$).

In the general setting, we write $\varphi_k = \max(\varphi_k, 0) - \max(-\varphi_k, 0) = \varphi_k^+ - \varphi_k^-$. We apply the above construction to φ_k^+ and φ_k^- separately. We get nonnegative continuous functions g_k and h_k with the required properties. Finally, take

$$\Phi_k = \max(\varphi_k^+ - g_k, 0) - \max(\varphi_k^- h_k, 0)$$

and the proof is completed. ∎

Proof of Theorem 2. We first show that condition (c) is necessary. Assume $|\bar{F} \cap \partial\mathbb{D} \backslash F_{\mathrm{nt}}| > 0$ and let $\xi \in \partial\mathbb{D}$ be the point and $g = g_N \geqslant 0$ the function given by Corollary 4.1. Consider

$$H = \exp(-g - i\tilde{g}),$$

where $\tilde{g}$ denotes the conjugate function of g with $\tilde{g}(0) = 0$. Then $H \in \mathrm{VMOA}$, $\|H\|_* \leqslant 1$ and

$$|H(z)| \leqslant e^{-N}, \qquad z \in F$$

$$\lim_{r \to 1} |H(r\xi)| = 1.$$

Since $H \in \mathrm{VMOA}$, given $\varepsilon > 0$ there exists a small arc $I \subset \partial\mathbb{D}$ centered at ξ such that

$$|\{\eta \in I : |H(\eta)| < \tfrac{3}{4}\}| < \varepsilon\, |I|.$$

Assume F is a Farrell set for BMOA or VMOA; then there exist analytic polynomials P_n, $\|P_n\|_* \leqslant C$, converging pointwise to H in $\mathbb{D}$ and such that

$$\sup_{z \in F} |P_n(z)| \leqslant e^{-N}.$$

We now use an argument from [B–P–S–T]. Since $(\|P_n\|_2 \leqslant \sqrt{2\pi}\, \|P_n\|_* + |P_n(0)|)$, the norms $\|P_n\|_2$ are uniformly bounded and, passing to a subsequence if necessary, one can assume $P_N \to H$ weakly in L^2. Then, if $E = \bar{F} \cap I$, one has

$$\int P_n \bar{H} \chi_E \to \int_E |H|^2 \geqslant \frac{9}{16} \frac{|E|}{2},$$

while

$$\left| \int P_n \bar{H} \chi_E \right| \leqslant e^{-N}\, |E|,$$

which gives a contradiction because $|E| > 0$.

Conversely, assume $|\bar{F} \cap \partial\mathbb{D} \backslash F_{nt}| = 0$ and let us show that F is a Farrell set for BMOA (VMOA). Let f be a function in BMOA and assume $\sup\{|f(z)|: z \in F\} \leqslant 1$. Then, C. Sundberg [Su] constructed $f_N \in H^\infty$, $\|f_N\|_* \leqslant C\, \|f\|_*$ such that for any $C > 0$, $f_N \to f$ uniformly on the set $\{z \in \mathbb{D} : |f(z)| \leqslant C\}$. Since Farrell sets for H^∞ are characterized by condition (c) [S], there exists a sequence $\{P_k^{(N)}: k = 1, 2, ...\}$ of analytic polynomials with $\sup\{|P_k^{(N)}(z)|: z \in F\} \leqslant 1$, tending to f_N in the weak-$*$ topology of H^∞ as $k \to \infty$. This gives convergence in the weak-$*$ topology of BMOA and a diagonal process finishes the proof.

Now assume that $f \in$ VMOA, $\sup\{|f(z)|: z \in F\} \leqslant 1$, and $\varepsilon > 0$. Theorem 1 provides a continuous function φ in the unit circle such that

$$\|\varphi - f\|_* < \varepsilon$$

$$\sup\{|\varphi(z)|: z \in F\} \leqslant 1 + \varepsilon.$$

Here $\varphi(z)$ denotes the value of the harmonic extension of φ at the point $z \in \mathbb{D}$. We want to approximate f by functions of the form $\varphi - h$, where $h \in \mathscr{C}(\bar{\mathbb{D}})$ satisfies

$$\bar{\partial} h = \bar{\partial} \varphi \qquad \text{on } \mathbb{D} \tag{4.10}$$

$$\sup\{|h(\xi)|: \xi \in \partial\mathbb{D}\} \leqslant C\varepsilon. \tag{4.11}$$

Assume one can find such a function h. Observe that (4.10) and (4.11) show that h is harmonic in $\mathbb{D}$ and bounded by $C\varepsilon$. So $\varphi - h$ is a holomorphic function in the unit disk and continuous up to the boundary. So the estimates

$$\|\varphi - h - f\|_* \leqslant (1+C)\varepsilon$$

$$\sup\{|\varphi(z) - h(z)| : z \in F\} \leqslant 1 + (1+C)\varepsilon$$

will finish the proof.

To find the function h, observe that the Fefferman–Stein decomposition (see [G, p. 252]) provides

$$\varphi - f = u + \tilde{v}, \qquad \text{on } \mathbb{T},$$

where u, v are continuous functions on the unit circle and

$$\|u\|_\infty + \|v\|_\infty \leqslant C\varepsilon.$$

Then, the function $h = \varphi - f + i(v + i\tilde{v})$ satisfies (4.10) because $\bar{\partial} h = \bar{\partial}(\varphi - f) = \bar{\partial}\varphi$ and (4.11) because on $\mathbb{T}$, $h = u + iv$. ∎

As Pérez-González pointed out to us, one can also prove that a Farrell set for BMOA must be Farrell for VMOA. Actually if F is a Farrell set for BMOA and $f \in$ VMOA, one can find polynomials P_n tending to f in the weak-$*$ topology of BMOA. Since $f \in$ VMOA, the polynomials P_n also tend to f in the weak topology of VMOA. Then, there exist convex lineal combinations of P_n which tend to f in norm.

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