WEAK EMBEDDING PROPERTY, INNER FUNCTIONS AND ENTROPY

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To Nikolai Nikolski on occasion of his birthday

ABSTRACT. Following Gorkin, Mortini, and Nikolski, we say that an inner function I in $H^\infty(\mathbb{D})$ has the WEP property if its modulus at a point z is bounded from below by a function of the distance from z to the zero set of I. This is equivalent to a number of properties, and we establish some consequences of this for H^∞/IH^∞.

The bulk of the paper is devoted to *wepable* functions, i.e. those inner functions which can be made WEP after multiplication by a suitable Blaschke product. We prove that a closed subset E of the unit circle is of finite entropy (i.e. is a Beurling–Carleson set) if and only if any singular measure supported on E gives rise to a wepable singular inner function. As a corollary, we see that singular measures which spread their mass too evenly cannot give rise to wepable singular inner functions. Furthermore, we prove that the stronger property of porosity of E is equivalent to a stronger form of wepability (*easy wepability*) for the singular inner functions with support in E. Finally, we find out the critical decay rate of masses of atomic measures (with no restrictions on support) guaranteeing that the corresponding singular inner functions are easily wepable.

1. INTRODUCTION

1.1. **Background.** Let $H^\infty = H^\infty(\mathbb{D})$ be the algebra of bounded analytic functions on the unit disc $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. A function $I \in H^\infty$ is called *inner* if it has radial limits of modulus 1 at almost every point of the unit circle.

Any inner function I factors as $I = BS$ where B is a Blaschke product and S is a *singular* inner function, that is, an inner function without zeros in $\mathbb{D}$.

A Blaschke product B is called an *interpolating* Blaschke product if its zero set $\Lambda = (z_n)_n$ forms an interpolating sequence for H^∞, that is

$H^\infty|\Lambda = \ell^\infty|\Lambda$. Let $\rho(z, w)$ be the pseudohyperbolic distance between the points z and w in the unit disc $\mathbb{D}$ defined as

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|, \qquad z, w \in \mathbb{D}.$$

A celebrated result of Carleson says that this holds if and only if $\inf_{n \neq m} \rho(z_n, z_m) > 0$ and

(1) $$\sup_{z \in \mathbb{D}} \sum_n \frac{(1 - |z_n|^2)(1 - |z|^2)}{|1 - \overline{z_n}z|^2} < \infty.$$

It was also proved by Carleson that (1) is equivalent to the embedding $H^1 \subset L^1(d\mu)$, where H^1 is the standard Hardy space and $d\mu = \sum_n (1 - |z_n|^2)\delta_{z_n}$, δ_{z_n} being the point mass at z_n. In other words, (1) holds if and only if there exists a constant $C > 0$ such that $\sum_n (1 - |z_n|^2)|f(z_n)| \leq C\|f\|_1$, for any function f in the Hardy space H^1 of the analytic functions in $\mathbb{D}$ for which

$$\|f\|_1 = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{it})|dt < \infty.$$

It is well known that a Blaschke product B is an interpolating Blaschke product if and only if there exists a constant $C = C(B) > 0$ such that

(2) $$|B(z)| > C\rho(z, Z(B)),$$

where $Z(B)$ denotes the zero set of B (see the monographs [12], [11, p. 217] or [14]). This fact easily extends to general inner functions.

1.2. **Weak Embedding Property.** In 2008, Gorkin, Mortini, and Nikolski [8] introduced the following new class of inner functions. An inner function I satisfies the *Weak Embedding Property (WEP)*, a weaker version of (2), if for any $\varepsilon > 0$ one has

$$\eta_I(\varepsilon) := \inf\{|I(z)| : \rho(z, Z(I)) > \varepsilon\} > 0.$$

A Blaschke product I with zeros $(z_n)_n$ satisfies the WEP if and only if for every $\varepsilon > 0$,

$$\sup_{z \in \mathbb{D},\, \inf_n \rho(z, z_n) > \varepsilon} \left\{ \sum \frac{(1 - |z_n|^2)(1 - |z|^2)}{|1 - \overline{z_n}z|^2} \right\} < \infty,$$

which is a weakening of the Carleson embedding property (1).

Finite products of interpolating Blaschke products satisfy the WEP with $\eta_I(\varepsilon) \succeq \varepsilon^N$ and in fact, a Blaschke product B is the product of N interpolating Blaschke products if and only if there exists a constant $C = C(B) > 0$ such that $|B(z)| > C\rho(z, Z(B))^N$ for any $z \in \mathbb{D}$ [7].

However there are other inner functions that satisfy the WEP. In [8], an explicit example was presented of a Blaschke product satisfying the WEP which cannot factor into a finite product of interpolating Blaschke products. This example was extended and complemented in [13]. A different class of examples has been given in [2] showing that for every strictly increasing function $\psi : (0,1) \to (0,1)$ there exists a Blaschke product B satisfying the WEP such that $\eta_B(\varepsilon) = o(\psi(\varepsilon))$ as $\varepsilon \to 0$.

1.3. **Operator Theory motivations.** Given an inner function I consider the quotient algebra H^∞/IH^∞. The zeros $Z(I)$ of I in $\mathbb{D}$ are naturally embedded in the maximal ideal space $\mathfrak{M}$ of H^∞/IH^∞. It is proved in [8] that I satisfies the WEP if and only if H^∞/IH^∞ has no corona, that is, $Z(I)$ is dense in $\mathfrak{M}$.

Another condition shown to be equivalent to the WEP in [8] is the norm controlled inversion property which says that for any $\varepsilon > 0$, there exists $m(\varepsilon) > 0$ such that if $f \in H^\infty$, $\|f\|_{H^\infty} = 1$ and $\inf\{|f(z)| : z \in Z(I)\} > \varepsilon > 0$, then f is invertible in H^∞/IH^∞ and $\|1/f\|_{H^\infty/IH^\infty} \leq m(\varepsilon)$.

Consider a vector-valued version of this: for $f := (f_1, \ldots, f_n) \in (H^\infty)^n$, let $\|f\|_{\infty,n}^2 := \sup_{z \in \mathbb{D}} \sum_{j=1}^n |f_j(z)|^2$ and for I inner,

$$\chi_I(f) := \inf\{\|g\|_{\infty,n} : \exists h \in H^\infty : \sum_{j=1}^n g_j f_j + hI \equiv 1\}.$$

This is like a "Corona constant" for the n-tuple f in the quotient space H^∞/IH^∞.

Following Gorkin, Mortini and Nikolski, for $\delta \in (0,1)$, $n \geq 1$, we define

$$c_n(\delta, I) := \sup\left\{\chi_I(f) : \delta^2 \leq \inf_{\lambda \in Z(I)} \sum_{j=1}^n |f_j(\lambda)|^2, \|f\|_{\infty,n} \leq 1\right\},$$

which is a decreasing function of δ, and

$$\delta_n(I) := \inf\{\delta : c_n(\delta, I) < \infty\}.$$

It turns out that these values do not depend on n.

Proposition 1. *For any $n \geq 1$, $\delta_n(I) = \tilde{\delta}(I) := \inf\{\varepsilon : \eta_I(\varepsilon) > 0\}$.*

Another result concerns possible rates of growth of $c_n(\delta, I)$.

Proposition 2. *For every decreasing function $\phi : (0,1) \to (0,\infty)$ there exists a Blaschke product B such that $\delta_n(B) = 0$ and $c_n(\delta, B) \geq \phi(\delta)$, $0 < \delta < 1$, $n \geq 1$.*

Note that by the definition of $\tilde{\delta}$, the function I satisfies the WEP if and only if $\tilde{\delta}(I) = 0$.

Inner functions I satisfying the WEP can also be described in terms of spectral properties of the model operator acting on the Model Space $K_I = H^2/IH^2$, see [8].

1.4. **Wepable functions.** An inner function I is called *wepable* [2] if it can enter as a factor in a WEP inner function, i.e. if there exists J inner such that IJ satisfies the WEP. Clearly, if I is a singular inner function, thus without zeros, it cannot be WEP, but it can be wepable. It is easy to see that only the Blaschke factor in J will help make IJ a WEP function.

Let us describe some of the results in [2]. Let $dA(z)$ be area measure in the unit disc. An inner function I such that for any $\varepsilon > 0$ one has

$$(3) \qquad \int_{\{z:|I(z)|<\varepsilon\}} \frac{dA(z)}{1-|z|^2} = \infty$$

is not wepable. Moreover there exist singular inner functions I satisfying (3). Hence there exists singular inner functions which are not wepable, answering a question in [8]. Condition (3) is a sort of Blaschke condition and has also appeared in [9]. It was also shown in [2] that condition (3) does not characterise (non)-wepable inner functions.

1.5. **Results about the support of the singular measure.** Given a measurable set $E \subset \mathbb{T} = \partial\mathbb{D}$, let $|E|$ denote its normalised length, $|\mathbb{T}| = 1$. Recall that a closed set $E \subset \mathbb{T}$ with $|E| = 0$ has finite entropy (has finite Carleson characteristic, is a Beurling–Carleson set) if

$$\mathcal{E}(E) := \sum |J_k| \log |J_k|^{-1} < \infty,$$

where $(J_k)_k$ are the connected components of $\mathbb{T} \setminus E$; more precisely, this value is the entropy of the family $(J_k)_k$. A classical result of Carleson says that a closed set $E \subset \mathbb{T}$ is the zero set of an analytic function whose derivatives of any order extend continuously to the closed unit disc if and only if E has zero length and finite entropy [3].

Given an inner function I let $\text{sing}(I)$ denote the set of points of the unit circle where I can not be extended analytically. If $I = BS$ where B is a Blaschke product with zeros $(z_n)_n$ and

$$S_\mu(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)\right), \qquad z \in \mathbb{D},$$

where μ is a positive singular measure, then $\text{sing}(I) = (\overline{\{z_n\}} \cap \mathbb{T}) \cup \text{supp}\,\mu$, where $\text{supp}\,\mu$ denotes the (closed) support of μ.

Theorem 1. *Let E be a closed subset of the unit circle. The following conditions are equivalent:*

(a) Every singular inner function whose singular set is contained in E is wepable;

(b) E has zero length and finite entropy.

The sufficiency of the conditions in (b) is obtained by careful constructions of Blaschke products which are carried out in Section 3.

The necessity of the conditions in (b) is related to estimate (3) and follows from the following result which may be of independent interest. We identify the unit circle with the interval $[0, 2\pi)$ and consider the dyadic arcs $[2\pi \cdot k2^{-n}, 2\pi \cdot (k+1)2^{-n})$, $0 \le k < 2^n$, $n \ge 0$. Those arcs have normalised length equal to 2^{-n}.

Given an arc $J \subset \mathbb{T}$ of center ξ_J, write $z(J) = (1 - \frac{3}{4}|J|)\xi_J$. We also denote $Q(J) := \{re^{i\theta} : e^{i\theta} \in \overline{J}, 1 - |J| \le r < 1\}$ (the Carleson box associated to J) and $T(J) := \{re^{i\theta} : e^{i\theta} \in \overline{J}, 1 - |J| \le r \le 1 - |J|/2\}$ (the top half of the box).

Given a finite measure μ in the unit circle let $P[\mu]$ be its Poisson integral.

Lemma 1. *Let E be a closed subset of the unit circle. The following conditions are equivalent:*

(a) E has zero length and finite entropy;

(b) $\sum |J| < \infty$, where the sum is taken over all dyadic arcs J such that $J \cap E \ne \varnothing$;

(c) For any positive singular measure μ whose support is contained in E and any $C > 0$ one has $\sum |J| < \infty$, where the sum is taken over all dyadic arcs $J \subset \mathbb{T}$ such that $P[\mu](z(J)) \ge C$.

The condition in (c) can be understood as a discrete version of (3) with $I = S_\mu$.

The condition in (b) can be seen as a discrete version of $\int_{\Gamma(E)} \frac{dA(z)}{1-|z|^2} < \infty$, where $\Gamma(E)$ denotes the union of all the Stolz angles with vertex on a point of E. For a related result, see [4, Lemma A.1].

1.6. **Results about regularity of singular measures.** Positive singular measures can fairly distribute their mass. For instance, there exist singular probability measures μ on the unit circle such that

$$\sup\left\{\left|\frac{\mu(J)}{\mu(J')} - 1\right| + \left|\frac{\mu(J) - \mu(J')}{|J|}\right|\right\} \to 0 \quad \text{as } |J| \to 0,$$

where the supremum is taken over any pair of adjacent arcs $J, J' \subset \mathbb{T}$ of the same length (see [1]). As a consequence of Theorem 1 we will prove that positive singular measures μ such that S_μ is wepable cannot

distribute their mass as evenly. Actually a Dini type condition governs the growth of the density of such measures.

Corollary 1. (a) *Let μ be a positive singular measure on the unit circle and consider $w(t) = \sup \mu(J)$, where the supremum is taken over all arcs $J \subset \mathbb{T}$ with $|J| = t$. Assume that*

$$\sum_{n \geq 1} \frac{2^{-n}}{w(2^{-n})} = \infty.$$

Then S_μ satisfies condition (3) and, hence, it is not wepable.
   (b) *Let $w : [0,1] \to [0,\infty)$ be a nondecreasing function with $w(0) = 0$ such that $w(2t) < 2w(t)$ for any $t > 0$. Assume that*

$$\sum_{n \geq 1} \frac{2^{-n}}{w(2^{-n})} < \infty.$$

Then there exists a positive singular measure μ in the unit circle satisfying $\mu(J) < w(|J|)$ for any arc $J \subset \mathbb{T}$, such that its support has zero length and finite entropy, and hence S_μ is wepable.

Definition 1. *A closed subset E of the unit circle is called* porous *if there exists a constant $C > 0$ such that for any arc $J \subset \mathbb{T}$, there exists a subarc $J' \subset J \setminus E$ with $|J'| > C|J|$.*

The porosity condition (the Kotochigov condition, the (K) condition) appears naturally in the free interpolation problems for different classes of analytic functions smooth up to the boundary, see [5].
   The next auxiliary result is a scale invariant version of Lemma 1.

Lemma 2. *Let E be a closed subset of the unit circle. The following conditions are equivalent:*
   (a) *E is porous;*
   (b) *There exists a constant $C > 0$ such that for any dyadic arc J one has*

$$\sum_{I \in E(J)} |I| \leq C|J|,$$

where $E(J)$ is the family of the dyadic arcs $I \subset J$ such that $I \cap E \neq \varnothing$;
   (c) *There exists a constant $C > 0$ such that for any finite positive measure μ with support contained in E, any number $A > 0$ and any dyadic arc $J \subset \mathbb{T}$, one has*

$$\sum_{I \in E(A)} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2} \leq \frac{C}{A} P[\mu](z(J)),$$

where $E(A)$ is the family of the dyadic arcs I such that $P[\mu](z(I)) \geq A$.

Let S be a wepable inner function. It may happen that any Blaschke product B such that BS satisfies the WEP must have some of its zeros $(z_n)_n$ located at points where $|S|$ is close to 1; more precisely, $\limsup_{n\to\infty} |S(z_n)| = 1$. (Theorems 1 and 2 together prove the existence of such S).

Definition 2. *An inner function S will be called* easily wepable *if there exists a constant $m < 1$ and a Blaschke product B such that SB satisfies the WEP and $Z(B) \subset \{z \in \mathbb{D} : |S(z)| < m\}$.*

Theorem 2. *Let E be a closed subset of the unit circle. The following conditions are equivalent:*

(a) Every singular inner function whose singular set is contained in E is easily wepable;

(b) The set E is porous.

In [10] it was proved that a closed set E of the unit circle is porous if and only if for any singular inner function S whose singular set is contained in E and any $a \in \mathbb{D} \setminus \{0\}$ the inner function $(S-a)/(1-\bar{a}S)$ is a finite product of interpolating Blaschke products.

Now we describe the critical decay rate of masses of atomic measures (with no restrictions on support) guaranteeing that the corresponding singular inner functions are easily wepable.

Theorem 3. *Let $(b_s)_{s\geq 1}$ be a non-increasing summable sequence of positive numbers. The following conditions are equivalent:*

(a) Every atomic singular inner function with point masses $(b_s)_{s\geq 1}$ is easily wepable;

(b)

$$
(4) \qquad\qquad b_s \asymp \sum_{k\geq s} b_k, \qquad s \geq 1.
$$

Note that given any decreasing sequence of masses, they can give rise to a measure μ with easily wepable singular function S_μ, simply by locating the masses at points of the form $\exp(i2^{-n})$, for instance, and applying Theorem 2. It would be interesting to have a similar statement to Theorem 3 with "wepable" instead of "easily wepable"; in particular to know whether any condition weaker than (4) can imply automatic wepability, and what rate of decrease of the (b_s) guarantees that there always exists some choice of location of the point masses with produces a non-wepable S_μ. In particular, the construction in the proof of [2, Proposition 6] shows that there exists a non-wepable atomic singular inner function as soon as the point masses decay no more rapidly than $1/(n(\log n)^2)$, $n \to \infty$.

1.7. **Organization of the paper.** In Section 2, we prove Lemmas 1 and 2, and therefore the necessity part of Theorem 1. In Section 3, we prove the remaining part of Theorem 1 and Corollary 1. In Section 4, we give the proof of Theorem 3. In Section 5, we give the proof of Theorem 2, which deals with a situation where the entropy of the singular set is very well controlled. Finally, the proofs of Propositions 1 and 2, which are quite independent from the rest, appear in Section 6.

The letter C will denote a constant whose value may change from line to line.

We denote by $\mathcal{D} = \cup_{n \geq 0} \mathcal{D}_n$ the family of the dyadic arcs, with $\mathcal{D}_n = \{J \subset \mathcal{D} : |J| = 2^{-n}\}$. Note that Card $\mathcal{D}_n = 2^n$.

Given an arc $J \subset \mathbb{T}$ of center ξ and length $|J|$ and $M > 0$ let MJ be the arc of the unit circle of center ξ and length $M|J|$.

Let $z, w \in \mathbb{D}$. Later on, we use the following standard estimates:

$$(5) \qquad \frac{1}{2} \cdot \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2} \leq \log\left|\frac{1 - \bar{w}z}{z - w}\right|,$$

$$(6) \qquad \log\left|\frac{1 - \bar{w}z}{z - w}\right| \leq C(\delta)\frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2}, \qquad \rho(z, w) \geq \delta > 0.$$

2. Proofs of Lemmas 1 and 2

Proof of Lemma 1. (a)$\Leftrightarrow$(b) Let $\mathbb{T} \setminus E$ be the disjoint union of the arcs I_k, $k \geq 1$. Suppose first that $|E| = 0$. Then

$$\sum_{J \in \mathcal{D},\, J \cap E \neq \varnothing} |J| = \sum_{k \geq 1} \sum_{J \in \mathcal{D},\, J \cap E \neq \varnothing,\, J \cap I_k \neq \varnothing} |J \cap I_k|$$

$$= \sum_{k \geq 1} \sum_{n \geq 0} \sum_{J \in \mathcal{D}_n,\, J \cap E \neq \varnothing,\, J \cap I_k \neq \varnothing} |J \cap I_k|$$

$$\asymp \sum_{k \geq 1} \left(\sum_{0 \leq n \leq \log(1/|I_k|)} |I_k| + \sum_{n > \log(1/|I_k|)} 2^{-n} \right) \asymp \mathcal{E}(E).$$

Next, if $|E| > 0$, then

$$\sum_{J \in \mathcal{D},\, J \cap E \neq \varnothing} |J| = \infty.$$

(b)$\Rightarrow$(c) Arguing as above,

$$\sum_{J\in\mathcal{D},\,P[\mu](z(J))\geq C}|J| = \sum_{k\geq 1}\sum_{n\geq 0}\sum_{J\in\mathcal{D}_n,\,P[\mu](z(J))\geq C,\,J\cap I_k\neq\varnothing}|J\cap I_k|$$

$$\lesssim \sum_{k\geq 1}\Big(\sum_{0\leq n\leq 2\log(1/|I_k|)}|I_k| + \sum_{n>2\log(1/|I_k|)}2^{-n/2}\Big) \asymp \mathcal{E}(E).$$

(c)$\Rightarrow$(a) If $|E| > 0$, then we can just take $\mu_0 = K\chi_E\,dm$ with K to be chosen later on. By the Lebesgue density theorem, for a subset E_1 of E, $|E_1| \geq |E|/2$ and for some $\delta > 0$ we have

$$\frac{|E\cap J|}{|J|} \geq \frac{1}{2}$$

for every arc J such that $J\cap E_1 \neq \varnothing$, $|J| \leq \delta$. Hence, for $K \geq K(C)$ we obtain

$$\sum_{J\in\mathcal{D},\,P[\mu_0](z(J))\geq C}|J| \geq \sum_{J\in\mathcal{D},\,|J|\leq\delta,\,J\cap E_1\neq\varnothing}|J| = \infty.$$

Next, we can replace μ_0 by a Cantor type singular measure μ_1 while keeping the sum

$$\sum_{J\in\mathcal{D},\,P[\mu_1](z(J))\geq C}|J|$$

infinite.

Now, suppose that (c) holds and $|E| = 0$, $\mathcal{E}(E) = \infty$, so that (a) and, hence, (b) do not hold. Let $\mathbb{T}\setminus E$ be the disjoint union of the arcs $I_k = (a_k, b_k)$, $k \geq 1$, and take

$$\mu = K\sum_{k\geq 1}|I_k|(\delta_{a_k} + \delta_{b_k})$$

with K to be chosen later on. Given $J \in \mathcal{D}$, if $J\cap E \neq \varnothing$, then $\mu(J) \geq K|J|$, and for $K \geq K(C)$ we have $P[\mu](z(J)) \geq C$. Therefore,

$$\sum_{J\in\mathcal{D},\,P[\mu](z(J))\geq C}|J| \geq \sum_{J\in\mathcal{D},\,J\cap E\neq\varnothing}|J| = \infty.$$

$\square$

Proof of Lemma 2. (a)$\Rightarrow$(b) If E is porous, then there exists $a \in \mathbb{N}$ such that for every $n \geq 0$, $J \in \mathcal{D}_n$, and for every $m \geq 0$, the set $J\cap E$ is covered by 2^{m-s} arcs $I \in \mathcal{D}_{n+m}$, $sa \leq m < (s+1)a$. Fix $J \in \mathcal{D}$,

$|J| = 2^{-n}$. Then

$$\sum_{I \in \mathcal{D}, I \subset J, I \cap E \neq \varnothing} |I| \leq \sum_{m \geq n} \sum_{I \in \mathcal{D}_m, I \subset J, I \cap E \neq \varnothing} |I|$$

$$\leq \sum_{s \geq 0} \sum_{sa \leq m < (s+1)a} 2^{m-s} 2^{-n-m} = \sum_{s \geq 0} a 2^{-n-s} = 2a|J|.$$

(b)$\Rightarrow$(a) If E is not porous, then for every $N \geq 1$ there exist $n \geq 0$, $J \in \mathcal{D}_n$ such that if $I \in \mathcal{D}_{n+N}$, $I \subset J$, then $I \cap E \neq \varnothing$. Then

$$\sum_{I \in \mathcal{D}, I \subset J, I \cap E \neq \varnothing} |I| \geq \sum_{n \leq s \leq n+N} \sum_{I \in \mathcal{D}_s, I \subset J, I \cap E \neq \varnothing} |I|$$

$$= \sum_{n \leq s \leq n+N} |J| = (N+1)|J|.$$

(c)$\Rightarrow$(a) As above, if E is not porous, then for every $N \geq 1$ we can find $J = [2\pi \cdot k 2^{-n}, 2\pi \cdot (k+1) 2^{-n}) \in \mathcal{D}_n$ and points $x_s \in E \cap [2\pi \cdot (k2^N + s) 2^{-n-N}, 2\pi \cdot (k2^N + s + 1) 2^{-n-N})$, $0 \leq s < 2^N$. Set

$$\mu = 10 \cdot 2^{-n-N} A \sum_{0 \leq s < 2^N} \delta_{x_s}.$$

Then

$$P[\mu](z(I)) \geq A, \qquad I \in \mathcal{D}_m, \, n \leq m \leq n+N, \, I \subset J,$$

and

$$\sum_{I \in \mathcal{D}, \, P[\mu](z(I)) \geq A} \frac{|I||J|}{|1 - \overline{z(I)} z(J)|^2} \gtrsim \sum_{n \leq m \leq n+N} 2^{m-n} \cdot \frac{2^{-m} 2^{-n}}{2^{-2n}} = N + 1.$$

For large N this contradicts to (c), because

$$P[\mu](z(J)) \leq CA.$$

To complete the proof of our lemma, we need an auxiliary statement.

Lemma 3. *Let u be a function positive and harmonic on the unit disc, let $A > 0$, and let $\mathcal{G}$ be a subfamily of $\mathcal{D}$ such that the arclength ds on $L = \cup_{J \in \mathcal{G}} \partial T(J)$ is a Carleson measure,*

$$\sup_{z \in \mathbb{D}} \sum_{J \in \mathcal{G}} \int_{\partial T(J)} \frac{1 - |z|^2}{|1 - \bar{w}z|^2} \, ds(w) \leq B.$$

Assume that $u \geq A$ on L. Then for every $J \subset \mathcal{D}$ we have

$$\sum_{I \in \mathcal{G}} \frac{|I||J|}{|1 - \overline{z(I)} z(J)|^2} \leq \frac{CB}{A} u(z(J))$$

for some absolute constant C.

Proof. Consider the function

$$h(z) = \sum_{J \in \mathcal{G}} \int_{\partial T(J)} \log\left|\frac{1 - \bar{w}z}{z - w}\right| \frac{ds(w)}{1 - |w|}.$$

It is harmonic on $\mathbb{D} \setminus L$ and by (6),

$$CBu(z) \geq Ah(z)$$

for $z \in \mathbb{T} \cup L$. By the maximum principle,

$$CBu(z(J)) \geq Ah(z(J)),$$

and hence, by (5),

$$C_1 Bu(z(J)) \geq A \sum_{I \in \mathcal{G}} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2}.$$

$\square$

It remains to prove the implication (a)$\Rightarrow$(c). Set $u = P[\mu]$. Arguing as in the part (a)$\Rightarrow$(b) we obtain that

$$\sum_{I \in \mathcal{D}, I \subset J, 3I \cap E \neq \varnothing} |I| \leq C|J|, \qquad J \subset \mathcal{D}.$$

Hence, the arclength ds on $\cup_{I \in \mathcal{D}, 3I \cap E \neq \varnothing} \partial T(I)$ is a Carleson measure. Fix $A > 0$ and denote by $\mathcal{G}$ the family of all $I \in \mathcal{D}$ such that $3I \cap E \neq \varnothing$ and $u(z(I)) \geq A$. Fix $J \in \mathcal{D}$. Applying Lemma 3, we obtain that

$$u(z(J)) \geq CA \sum_{I \in \mathcal{G}} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2}.$$

Now we need only to estimate

$$\sum_{I \in \mathcal{D}, 3I \cap E = \varnothing, u(z(I)) \geq A} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2}.$$

We set

$$\mathcal{H} = \{I \in \mathcal{D} : 3I \cap E = \varnothing, u(z(I)) \geq A\}$$

and

$$\mathcal{H}_k = \{I \in \mathcal{H} : 2^{k-1}A \leq u(z(I)) < 2^k A\}, \qquad k \geq 1,$$

so that $\mathcal{H} = \cup_{k \geq 1} \mathcal{H}_k$. If $I \in \mathcal{H}$, then an easy estimate of the Poisson integral shows that

$$\frac{u(z(L))}{u(z(I))} \asymp \frac{|L|}{|I|}, \qquad L \in \mathcal{D}, L \subset I.$$

Hence, every arc $I \in \mathcal{H}_k$, $k \geq 1$, contains at most C subarcs $I' \in \mathcal{H}_k$. Therefore, the arclength ds on $\cup_{I \in \mathcal{H}_k} \partial T(I)$ is a Carleson measure with

Carleson constant uniformly bounded in $k \geq 1$. Lemma 3 gives now that

$$2^{-k} u(z(J)) \geq CA \sum_{I \in \mathcal{H}_k} \frac{|I||J|}{|1 - \overline{z(I)} z(J)|^2}.$$

Summing up in $k \geq 1$ we complete the proof.   □

3. Proof of Theorem 1 and Corollary 1

The proof of the sufficiency of Theorem 1 uses the following auxiliary result.

Lemma 4. *Let E be a closed subset of the unit circle of zero length and finite entropy. Let $\mathcal{G} = \{J_n\}$ be the family of maximal dyadic subarcs $J_n \subset \mathbb{T}$ such that $2J_n \subset \mathbb{T} \setminus E$. Then*

(a) *The interiors of J_n are pairwise disjoint and $\mathbb{T} \setminus E = \cup J_n$.*

(b) *We have $\sum_n |J_n| \log |J_n|^{-1} < \infty$.*

(c) *If J_n and J_m are in the same connected component of $\mathbb{T} \setminus E$ and $\overline{J_n} \cap \overline{J_m} \neq \varnothing$, then $4|J_n| \geq |J_m| \geq |J_n|/4$.*

(d) *Let J be a connected component of $\mathbb{T} \setminus E$ and consider the subfamily $\mathcal{G}(J) = \{L_n\}$ of the arcs $L \in \mathcal{G}$ with $L \subset J$ ordered so that $|L_{n+1}| \leq |L_n|$ for any n. Then $|L_1| \geq |J|/8$, $|L_k|/4 \leq |L_{k+1}| \leq |L_k|$ and $|L_{k+4}| \leq |L_k|/2$ for any $k \geq 1$.*

Proof. The maximality gives that the interiors of J_n are pairwise disjoint. It is also clear that $\mathbb{T} \setminus E = \cup J_n$. Let us now prove (c). Assume that $|J_n| \geq |J_m|$. Since $2J_n \subset \mathbb{T} \setminus E$, every dyadic arc J adjacent to J_n of length $|J| = |J_n|/4$ satisfies $2J \subset \mathbb{T} \setminus E$. Since $\overline{J_n} \cap \overline{J_m} \neq \varnothing$ we can assume that $J \cap J_m \neq \varnothing$. By maximality, $J \subset J_m$. Hence $|J_m| \geq |J| = |J_n|/4$ and (c) is proved. Let us now prove (d). The maximality gives that $|L_1| \geq |J|/8$ and (c) gives that $|L_k|/4 \leq |L_{k+1}|$. Assume that $|L_{k+4}| > |L_k|/2$. Then $|L_{k+i}| = |L_k|$, $0 \leq i \leq 4$. Then there would exist $i \in \{0, 1, 2, 3, 4\}$ such that the dyadic arc L containing L_{k+i} of length $|L| = 2|L_{k+i}|$ satisfies $2L \subset \mathbb{T} \setminus E$. This would contradict the maximality of L_{k+i} and (d) is proved. Finally, let us prove (b). Take a connected component J of $\mathbb{T} \setminus E$. The estimates in (d) give that

$$\sum_{L \in \mathcal{G}(J)} |L| \log |L|^{-1} \leq C|J| \log |J|^{-1},$$

and (b) follows because E has finite entropy.   □

Proof of Theorem 1. The necessity part follows from Lemma 1 and property (3).

Let us now prove the sufficiency part. Let μ be a positive singular measure whose support E has zero length and finite entropy. Let $\mathcal{G}$ be the family given by Lemma 4 of maximal dyadic arcs of the unit circle whose double is contained in $\mathbb{T} \setminus E$. Let $\mathcal{F}$ be the family of dyadic arcs of $\mathbb{T}$ which are not contained in any arc of $\mathcal{G}$. Then

$$(7) \qquad \sum_{J \in \mathcal{F}} |J| < \infty.$$

Indeed, for every $J \in \mathcal{F}$, $2J$ is the union of four dyadic intervals of the same length such that at least one of them intersects E. Since E has zero length and finite entropy, Lemma 1 gives that

$$\sum_{J \in \mathcal{F}} |J| \leq \sum_{J \in \mathcal{F}} |2J| \lesssim \sum_{J \in \mathcal{D}, \, J \cap E \neq \varnothing} |J| < \infty.$$

We want to find a Blaschke product B such that SB satisfies the WEP. We will describe now the first family of zeros of B. By (7), one can pick a sequence of integers k_j increasing to infinity such that

$$\sum_{j=1}^{\infty} k_j \sum_{J \in \mathcal{F} \cap \mathcal{D}_j} |J| < \infty.$$

Next, for each $J \in \mathcal{F} \cap \mathcal{D}_j$, we choose a set $\Lambda(J) = \{z_n(J) : 1 \leq n \leq k_j\}$ uniformly distributed in $T(J)$. Uniform distribution means that

$$\min_{z, w \in \Lambda(J), \, z \neq w} |z - w| \asymp \max_{z \in T(J)} \text{dist}\,(z, \Lambda(J)) \asymp \min_{z \in \Lambda(J)} \text{dist}\,(z, \partial T(J)).$$

Let B_1 be the Blaschke product with zeros $Z(B_1) = \cup_{J \in \mathcal{F}} \Lambda(J)$. Since k_j tends to infinity, for any $0 < \delta < 1$ there exists $r = r(\delta) < 1$ such that

$$(8) \qquad \{z \in \mathbb{D} : \rho(z, Z(B_1)) > \delta\} \subset \{z \in \mathbb{D} : |z| < r\} \bigcup \cup_{J \in \mathcal{G}} Q(J).$$

Next we will construct certain additional zeros of B which are contained in $\cup_{J \in \mathcal{G}} Q(J)$. Set $f = SB_1$. Since $\inf_{J \in \mathcal{G}} \rho(z(J), Z(B_1)) > 0$, we have $\log |f(z(J))|^{-1} \lesssim (1 - |z(J)|)^{-1}$, $J \in \mathcal{G}$. Now, applying Lemma 4 (b), we obtain

$$\sum_{J \in \mathcal{G}} |J| \log \log |f(z(J))|^{-1} < \infty.$$

Next, for every $J \in \mathcal{G}$ and every $z \in Q(J)$ with $1 - |z| < |J|/8$, we have

$$(9) \qquad \log |f(z)|^{-1} \lesssim \frac{1 - |z|}{1 - |z(J)|} \log |f(z(J))|^{-1}.$$

To prove (9) consider the measure $d\sigma = d\mu + \sum_{z \in Z(B_1)} (1 - |z|^2) \delta_z$, where $d\mu$ is the positive singular measure associated to S. Fix for a

moment $J \in \mathcal{G}$. There exists a constant $C > 0$ such that if $z \in Q(J)$ and $0 < 1 - |z| < |J|/8$, then $\rho(z, Z(B_1)) \geq C$. Then, by (6), there exists a constant $C_1 > 0$ such that

$$\log |f(z)|^{-1} \leq C_1 \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \overline{w}z|^2} d\sigma(w).$$

Applying Lemma 4 (c) and using that the support of μ does not intersect $2J$, for every w in the support of σ and every $z \in Q(J)$ we obtain that $|1 - \overline{w}z| \geq C_2 |1 - \overline{w}z(J)|$, where $C_2 > 0$ is a constant. Hence,

$$\log |f(z)|^{-1} < \frac{C_1(1 - |z|^2)}{C_2^2(1 - |z(J)|^2)} \int_{\mathbb{D}} \frac{1 - |z(J)|^2}{|1 - \overline{w}z(J)|^2} d\sigma(w).$$

By (5), this integral is bounded by a fixed multiple of $\log |f(z(J))|^{-1}$, and estimate (9) follows.

In particular, for any constant $K > 0$ there exists $K_1 = K_1(K) > 0$ such that for every $J \in \mathcal{G}$ we have

$$(10) \qquad |f(z)| > K_1 \quad \text{if} \quad z \in Q(J),\ 1 - |z| < \frac{K|J|}{\log |f(z(J))|^{-1}}.$$

Let $M(J) = \log_2 |J|^{-1}$. Pick a sequence of positive integers t_k increasing to infinity and such that $t_{k+1} \leq t_k + 1$ and

$$\sum_{J \in \mathcal{G}} |J| \sum_{k=1}^{M(J)} t_k < \infty.$$

For $1 \leq k \leq M(J)$ consider the strips

$$\Omega_k = \Omega_k(J) = \{z \in Q(J) : 2^{k-1}|J|^2 < 1 - |z| \leq 2^k|J|^2\}$$

and choose sets $\Lambda_k(J) = \{z_j(\Omega_k)\}_{1 \leq j \leq s_k}$ of $s_k = t_k 2^{M(J)-k}$ points uniformly distributed in Ω_k. Let $\Lambda(J) = \cup_{1 \leq k \leq M(J)} \Lambda_k(J)$. Then

$$\sum_{z \in \Lambda(J)} (1 - |z|) = \sum_{k=1}^{M(J)} \sum_{j=1}^{s_k} (1 - |z_j(\Omega_k)|) \leq |J| \sum_{k=1}^{M(J)} t_k.$$

Thus, the set $\cup_{J \in \mathcal{G}} \Lambda(J)$ satisfies the Blaschke condition. Let B_2 be the Blaschke product with these zeros.

We will now show that SB_1B_2 satisfies the WEP. Fix $\eta > 0$ and take $z \in \mathbb{D}$ such that $\rho(z, Z(B_1B_2)) > \eta$. Applying (8) we obtain that either $|z| \leq r(\eta) < 1$ or $z \in Q(J)$ for some $J \in \mathcal{G}$. In the first case $|S(z)B_1(z)B_2(z)| > \varepsilon(\eta) > 0$. Assume that $z \in Q(J)$ for some fixed $J \in \mathcal{G}$. Since $\lim_{k \to \infty} t_k = \infty$ and $\rho(z, Z(B_2)) > \eta$, there

exists $C_1 = C_1(\eta) > 0$ such that $1 - |z| < C_1|J|^2$. Furthermore, $z \in \cup_{0 \leq k \leq k(\eta)} \Omega_k(J)$, where

$$\Omega_0(J) = \{z \in Q(J) : 1 - |z| \leq |J|^2\}.$$

Since $\log |f(z(J))|^{-1} < C/|J|$, applying (10) we obtain that $|f(z)| \geq C_2$ for some $C_2 = C_2(\eta) > 0$. Let J_- and J_+ be two arcs of the family $\mathcal{G}$ contiguous to J. Factor $B_2 = B_3 B_4$, where B_3 is the Blaschke product with zeros $\Lambda(J_-) \cup \Lambda(J) \cup \Lambda(J_+)$. By Lemma 4 (c), dist $(Z(B_4), Q(J)) \gtrsim |J|$. Hence there exists a constant $C > 0$ such that $|1 - \overline{w}z| > C|J|$ for any zero w of B_4. Therefore, by (6),

$$\log |B_4(z)|^{-1} \leq C_3 \sum_{w \in Z(B_4)} \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w}z|^2} \leq C_4 \sum_{w \in Z(B_4)} (1 - |w|^2).$$

On the other hand, if we reorder the zeros of B_3 as $Z(B_3) = (z_s)_s$ and use that $\rho(z, Z(B_2)) > \eta$, estimate (6) gives that there exists a constant $C_5 = C_5(\eta) > 0$ such that

$$\log |B_3(z)|^{-1}$$
$$\leq C_5 \frac{\sum_{z_s \in Q(z)} (1 - |z_s|)}{1 - |z|} + C_5 \sum_{j=1}^{\infty} \frac{1}{2^{2j}(1 - |z|)} \sum_{z_s \in 2^j Q(z)} (1 - |z_s|),$$

where $Q(z) = Q(\tilde{J})$ is the Carleson box defined by the arc $\tilde{J}$ satisfying $z = z(\tilde{J})$, $2^j Q(z) = Q(2^j \tilde{J})$.

Next, again by Lemma 4 (c), $\log_2 |J|^{-1} \asymp \log_2 |J_\pm|^{-1}$. Hence, the densities of zeros $z_s \in Q(z)$ are bounded by $t_{k(\eta)}$,

$$\frac{1}{1 - |z|} \sum_{z_s \in Q(z)} (1 - |z_s|) \leq \sum_{1 \leq k \leq k(\eta)} t_k \leq C_6(\eta).$$

Similarly,

$$\frac{1}{1 - |z|} \sum_{z_s \in 2^j Q(z)} (1 - |z_s|) \leq 2^j \sum_{1 \leq k \leq k(\eta)+j} t_k$$
$$\leq 2^j (C_6(\eta) + j t_{k(\eta)} + j(j+1)/2),$$

and

$$\sum_{j=1}^{\infty} \frac{1}{2^{2j}(1 - |z|)} \sum_{z_s \in 2^j Q(z)} (1 - |z_s|) \leq C_7(\eta).$$

$\square$

Proof of Corollary 1. (a) Fix $0 < \varepsilon < 1$ and consider $A(\varepsilon) = \{z \in \mathbb{D} : |S_\mu(z)| < \varepsilon\}$. Let $\mathcal{L}_n$ be the collection of dyadic arcs $J \in \mathcal{D}_n$ such

that $\mu(J) > 10|J|\log\varepsilon^{-1}$ and let $\mathcal{L} = \cup_{n\geq 0}\mathcal{L}_n$. Since $\log|S_\mu(z)|^{-1} > \mu(J)/(10|J|)$ for $z \in T(J)$, we deduce that

$$\int_{A(\varepsilon)} \frac{dA(z)}{1-|z|^2} \geq C \sum_{J\in\mathcal{L}} |J| = C \sum_n 2^{-n}a_n,$$

where a_n is the number of dyadic arcs in the collection $\mathcal{L}_n$. Since μ is singular, there exists $n_0 = n_0(\varepsilon) > 0$ such that for $n \geq n_0$, one has

$$\sum_{J\in\mathcal{L}_n} \mu(J) \geq \frac{1}{2}\mu(\mathbb{T}).$$

We deduce that $a_n w(2^{-n}) \geq \mu(\mathbb{T})/2$. Thus,

$$\int_{A(\varepsilon)} \frac{dA(z)}{1-|z|^2} \geq C\mu(\mathbb{T}) \sum_{n\geq n_0} \frac{2^{-n}}{w(2^{-n})},$$

which finishes the proof of (a).

(b) We may assume that $w(1) = 1$. Set $n_1 = 0$ and for $k = 2, 3, \ldots$ let n_k be the smallest positive integer such that $w(2^{-n_k}) < 2^{-k+1}$. Since $2^{-k+1} \leq w(2^{-n_k+1}) < 2w(2^{-n_k})$, we have $2^{-k} < w(2^{-n_k}) < 2^{-k+1}$. Hence for any $k = 1, 2 \ldots$ and any integer n with $n_k \leq n < n_{k+1}$ we have $2^{-k} \leq w(2^{-n}) < 2^{-k+1}$. Let $\mathcal{L}_k = \mathcal{D}_{n_k}$, $k \geq 1$. The measure μ will be defined by prescribing inductively its mass $\mu(J)$ over any dyadic arc $J \in \cup_k\mathcal{L}_k$. Define $\mu(\mathbb{T}) = 1$. Assume that $\mu(J)$ has been defined for any arc $J \in \mathcal{L}_k$ and we will define the mass of μ over dyadic arcs of $\mathcal{L}_{k+1}$. Fix $J \in \mathcal{L}_k$ and let $\mathcal{G}(J)$ be the family of arcs in $\mathcal{L}_{k+1}$ contained in J. If $\mu(J) = 0$, define $\mu(L) = 0$ for any dyadic arc $L \in \mathcal{G}(J)$. If $\mu(J) > 0$, pick two (arbitrary) arcs $J_i \in \mathcal{G}(J)$, $i = 1, 2$, define $\mu(J_i) = \mu(J)/2$ for $i = 1, 2$ and $\mu(L) = 0$ for any other $L \in \mathcal{G}(J)$ with $L \neq J_i$, $i = 1, 2$. In other words, in each dyadic arc of generation n_k of positive measure μ, this measure distributes its mass among two (arbitrary) arcs of generation n_{k+1} and gives no mass to the others. Let J be a dyadic arc of the unit circle with $2^{-n_{k+1}} < |J| \leq 2^{-n_k}$ for a certain integer $k = k(J)$. By construction, $\mu(J) \leq 2^{-k} \leq w(|J|)$. Since any arc is contained in the union of two dyadic arcs of comparable length, there exists a constant $C > 0$ such that $\mu(J) < Cw(|J|)$ for any arc $J \subset \mathbb{T}$. Next we will show that the support of μ has finite entropy. For any integer $k \geq 1$ there are 2^{k-1} arcs J of $\mathcal{L}_k$ with $\mu(J) > 0$. Similarly if $n_k < n \leq n_{k+1}$, then there are at most 2^k dyadic arcs J of normalised

length 2^{-n} with $\mu(J) > 0$. Then

$$\sum_{J \in \mathcal{D}, \, \mu(J) > 0} |J| \leq 1 + \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} 2^{-n} 2^k$$

$$\leq 1 + \sum_{k=1}^{\infty} 2^k 2^{-n_k} \leq 1 + 2 \sum_{k=1}^{\infty} \frac{2^{-n_k}}{w(2^{-n_k})} \leq 1 + 2 \sum_{n \geq 0} \frac{2^{-n}}{w(2^{-n})}.$$

Applying Lemma 1 we deduce that the support of μ has finite entropy and zero Lebesgue measure. By Theorem 1, S_μ is wepable.   □

Remark 1. *Let μ be a positive singular measure in the Zygmund class, that is, there exists a constant $C > 0$ such that for any pair of contiguous arcs $J, J' \subset \mathbb{T}$ of the same length one has*

$$|\mu(J) - \mu(J')| \leq C|J|.$$

As in Corollary 1, consider $w(t) = \sup \mu(J)$, where the supremum is taken over all arcs $J \subset \mathbb{T}$ with $|J| = t$. Then there exists a constant $C > 0$ such that $w(2^{-n}) \leq Cn2^{-n}$ for any positive integer n and Corollary 1 gives that S_μ is non wepable.

4. Atomic Measures and Easily Wepable Singular Functions

We start with a modification of a construction from the proof of Proposition 6 in [2].

Lemma 5. *Given a large A, there exists $m \gg 1$ satisfying the following property: if $J \in \mathcal{D}_k$, J is the union of 2^m arcs $J_s \in \mathcal{D}_{k+m}$, $x_s \in J_s$, $0 \leq s < 2^m$, then there exists a set $Y \subset (x_s)_{0 \leq s < 2^m}$, Card $Y \gtrsim 2^m/A^2$, such that if*

$$\mu = \sum_{y \in Y} c_y \delta_y$$

with $10 \cdot 2^{-k-m} A \leq c_y \leq 20 \cdot 2^{-k-m} A$ (and we have $\|\mu\| \lesssim |J|/A$), if Λ is a subset of $\mathbb{D}$ such that

$$P[\mu](z(I)) \geq A \implies \Lambda \cap T(I) \neq \varnothing, \qquad I \in \mathcal{D},$$

and if B_Λ is the corresponding Blaschke product, then

$$\log |B_\Lambda(z(J))| \lesssim -A^2.$$

Proof. Let $J = [0, 2\pi \cdot 2^{-k})$. Fix $q \in \mathbb{N}$, $q \geq A^2$, $n = q^2$, and $m \geq 1$ such that $2^{m-1} \leq q2^{2n} < 2^m$ and define

$$Y = \{x_s : s = jq2^n + \ell, \, 0 \leq j < 2^n, \, 0 \leq \ell < 2^n\}.$$

Then $\|\mu\| \asymp 2^{-k}A/q \lesssim |J|/A$, and

$$2^{-k-m} \le 1 - r \le 2^{n-k-m} \ \& \ 0 \le \theta - jq2^{n-k-m} \le 2^{n-k-m}$$
$$\implies P[\mu](re^{2\pi i\theta}) \ge A.$$

Hence, for every $I \in \cup_{k+m-n \le p \le k+m}\mathcal{D}_p$ such that

$$I \subset \cup_{0 \le j < 2^n}[2\pi \cdot jq2^{n-k-m}, 2\pi \cdot (jq+1)2^{n-k-m}]$$

we have

$$\Lambda \cap T(I) \ne \varnothing.$$

Thus, applying (5),

$$\log|B_\Lambda(z(J))| \lesssim -q.$$

$\square$

Proof of Theorem 3. $(a) \Rightarrow (b)$ If (4) does not hold, then we can choose a sequence of groups $(b_{s_n}, \ldots, b_{s_n+n})_{n\ge 1}$ such that $b_{s_n} \le 2b_{s_n+n}$. By Lemma 5, passing to a subsequence of (s_n) denoted also by (s_n) we construct a sequence of dyadic arcs J_n and measures

$$\mu_n = \sum_{s_n \le y < s_n+2^{mn}} b_y \delta_{x_y}$$

such that

$$\text{supp } \mu_n \subset J_n, \quad \|\mu_n\| = o(|J_n|), \quad \text{dist}\,(J_n, J_{n'}) \gtrsim \max(|J_n|, |J_{n'}|).$$

Furthermore, if the sets Λ_n satisfy the property

(11) $\qquad P[\mu_n](z(I)) \ge n \implies \Lambda_n \cap T(I) \ne \varnothing, \qquad I \in \mathcal{D},$

then the corresponding Blaschke products B_{Λ_n} satisfy the estimate

$$|B_{\Lambda_n}(z(J_n))| \le \exp(-n^2).$$

Finally, we take $x \in \mathbb{T} \setminus \cup_n J_n$ and set

$$\mu = \sum_{n\ge 1}\mu_n + \Big(\sum_{s\ge 1}b_s - \sum_{n\ge 1}\|\mu_n\|\Big)\delta_x.$$

Suppose that B_Λ is a Blaschke product with zero set Λ such that $S_\mu B_\Lambda$ has the WEP. Then for every $n \ge n_0$, the set $\Lambda \cap Q(J_n)$ satisfies the property (11), and hence,

$$|B_\Lambda(z(J_n))| \le \exp(-n^2).$$

Therefore, B_Λ should have a zero in $T(J_n)$, $n \ge n_1$. However,

$$P[\mu](z(J_n)) \to 0, \qquad n \to \infty.$$

Thus, S_μ is not easily wepable.

$(b) \Rightarrow (a)$ Given $z, w \in \mathbb{D}$, set

$$\beta(z, w) = \log_2 \frac{1}{1 - \rho(z, w)}.$$

Let $J_1, J_2 \in \mathcal{D}$, $2^{-n} = |J_1| \leq |J_2| = 2^{-m}$, $J \in \mathcal{D}$, $J_1 \subset J$, $|J| = |J_2|$, $J = [k2^{-m}, (k+1)2^{-m}]$, $J_2 = [k'2^{-m}, (k'+1)2^{-m}]$. Then

$$\beta_{J_1, J_2} \stackrel{\text{def}}{=} \beta\big(z(J_1), z(J_2)\big) = n - m + 2\log_2(|k - k'| + 1) + O(1).$$

Furthermore, if $T(J_1) \cap T(J_2) = \varnothing$, then

$$\min_{z_1 \in T(J_1), \, z_2 \in T(J_2)} \beta(z, w) = \beta_{J_1, J_2} + O(1).$$

Let ϕ be an increasing subadditive function on $(0, +\infty)$ such that $\phi(x) = x$, $0 < x \leq 1$, $\phi(x) \asymp \log x$, $x \to \infty$.

Let $\mu = \sum_{s \geq 1} b_s \delta_{x_s}$. For every $J \in \mathcal{D}$ we set $\lambda(J) = P[\mu](z(J))$. Harnack's principle gives us a Lipschitz type estimate

(12) $$\qquad |\phi(\lambda(J_1)) - \phi(\lambda(J_2))| \lesssim \beta_{J_1, J_2}.$$

Now, for every $J \in \mathcal{D}$ we denote by k_J the integer part of $\phi(\lambda(J))$, and choose k_J points $z_{J,1}, \ldots, z_{J,k_J}$ uniformly distributed in $T(J)$. Let B be the Blaschke product with zeros in the points $z_{J,1}, \ldots, z_{J,k_J}$, $J \in \mathcal{D}$. To check that BS_μ has the WEP (and incidentally that B exists) we need only to verify that for every $J \in \mathcal{D}$ we have

(13) $$\sum_{I \in \mathcal{D} \setminus \{J\}} 2^{-\beta_{J,I}} \phi(\lambda(I)) \lesssim \max(\lambda(J), 1).$$

Fix $J \in \mathcal{D}$. Let $|J| = 2^{-n}$. By (12), we have

$$\sum_{I \in \mathcal{D}, \, |I| > |J|} 2^{-\beta_{J,I}} \phi(\lambda(I)) \lesssim \sum_{I \in \mathcal{D}, \, |I| > |J|} 2^{-\beta_{J,I}} \big(\phi(\lambda(J)) + \beta_{J,I}\big)$$

$$\lesssim \sum_{0 \leq k < n} \sum_{s=1}^{2^k} 2^{k-n} s^{-2} \big(\phi(\lambda(J)) + n - k + \log s + O(1)\big) \lesssim \max(\lambda(J), 1).$$

Next, we set $\mu' = \chi_{10J} \mu$, $\mu'' = \mu - \mu'$, and define $\lambda'(I) = P[\mu'](z(I))$, $\lambda''(I) = P[\mu''](z(I))$, $I \in \mathcal{D}$. To prove (13), we need only to check that

$$\sum_{I \in \mathcal{D} \setminus \{J\}, \, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(\lambda'(I)) \lesssim \max(\lambda'(J), 1),$$

(14) $$\sum_{I \in \mathcal{D} \setminus \{J\}, \, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(\lambda''(I)) \lesssim \max(\lambda''(J), 1).$$

We have $\mu' = \sum_{s \in N'} b_s \delta_{x_s}$, and we set $a = \max\{b_s : s \in N'\}$. (If $\mu' = 0$, we just pass to μ''.) By (4), $P[\mu'](z(J)) \asymp 2^n a$, and hence,

$$\sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(\lambda'(I)) \lesssim \sum_{s \in N'} \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I))).$$

Fix for a moment $s \in N'$. Without loss of generality, $x_s = 0$, and for $m \geq n$, $I = [2\pi \cdot t 2^{-m}, 2\pi \cdot (t+1) 2^{-m})$, $I \neq J$, we have

$$\beta_{J,I} \geq n - m + O(1), \qquad P[\delta_{x_s}](z(I)) \lesssim 2^m t^{-2}.$$

Hence,

$$\sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(\lambda'(I)) \lesssim \sum_{s \in N'} \sum_{m \geq n} \sum_{t \geq 1} 2^{n-m} \phi(b_s 2^m t^{-2})$$

$$= \sum_{s \in N'} \sum_{m \geq n} \sum_{t^2 \leq 2^m b_s} 2^{n-m} \phi(2^m b_s t^{-2}) + \sum_{s \in N'} \sum_{m \geq n} \sum_{t^2 > 2^m b_s} 2^{n-m} \phi(2^m b_s t^{-2})$$

$$\lesssim \sum_{s \in N'} \sum_{m \geq n} 2^{m/2} \sqrt{b_s} 2^{n-m} (\phi(2^m b_s) + 1)$$

$$\lesssim \sum_{s \in N'} \sum_{m \geq n} 2^{n-m/2} \sqrt{b_s} \max(m + \log b_s, 1)$$

$$\lesssim \sum_{s \in N'} 2^{n/2} \sqrt{b_s} \max(n + \log b_s, 1)$$

$$\lesssim 2^{n/2} \sqrt{a} \max(n + \log a, 1) \lesssim \max(2^n a, 1) \lesssim \max(\lambda'(J), 1).$$

We have $\mu'' = \sum_{s \in N''} b_s \delta_{x_s}$. To prove (14), we need to verify that

$$(15) \quad \sum_{s \in N''} \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I)))$$

$$\lesssim 1 + \sum_{s \in N''} P[b_s \delta_{x_s}](z(J)).$$

Fix $s \in N''$ and choose $r = r(s) \geq 2$ such that

$$2r|J| = \text{dist}(x_s, J).$$

Then

$$\sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I)))$$

$$= \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|, \text{dist}(I,J) \leq r|J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I)))$$

$$+ \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|, \text{dist}(I,J) > r|J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I))) = A_1 + A_2.$$

Next, we can assume that $J = [0, 2^{-n})$ and for $I = [2\pi \cdot t 2^{-m}, 2\pi \cdot (t+1)2^{-m})$, $m \geq n$, we have $\beta_{J,I} = m - n + 2\log_2(1 + t 2^{n-m}) + O(1)$. Hence,

$$A_1 \lesssim \sum_{m \geq n} \sum_{t \geq 1} 2^{n-m}(1 + t 2^{n-m})^{-2} \phi(b_s 2^{-m} r^{-2} 2^{2n})$$

$$\lesssim \sum_{m \geq n} \phi(2^{2n-m} b_s r^{-2}) \lesssim 2^n b_s r^{-2} \asymp P[b_s \delta_{x_s}](z(J)).$$

Furthermore,

$$A_2 \lesssim \sum_{m \geq n} \sum_{t \geq 1} 2^{n-m} r^{-2} \phi(b_s 2^m t^{-2})$$

$$= \sum_{m \geq n} \sum_{t^2 \leq 2^m b_s} 2^{n-m} r^{-2} \phi(2^m b_s t^{-2}) + \sum_{m \geq n} \sum_{t^2 > 2^m b_s} 2^{n-m} r^{-2} \phi(2^m b_s t^{-2})$$

$$\lesssim \sum_{m \geq n} 2^{m/2} \sqrt{b_s} 2^{n-m} r^{-2} \phi(2^m b_s) + \sum_{m \geq n} 2^{n-m} r^{-2} 2^m b_s 2^{-m/2} b_s^{-1/2}$$

$$\lesssim 2^{n/2} \sqrt{b_s} r^{-2} \max(n + \log b_s, 1) + 2^{n/2} \sqrt{b_s} r^{-2}.$$

If $b_s \geq 2^{-n}$, then

$$A_2 \lesssim 2^n b_s r^{-2} \asymp P[b_s \delta_{x_s}](z(J)).$$

Thus, to complete the proof, we need only to estimate

$$H = \sum_{s \in N'', \, b_s < 2^{-n}} 2^{n/2} \sqrt{b_s} r(s)^{-2}.$$

By (4),

$$\sum_{b_s < 2^{-n}} \sqrt{b_s} \lesssim 2^{-n/2}.$$

Hence,

$$H \lesssim 1,$$

and (15) follows. $\qquad\square$

5. Porous Sets and Easily Wepable Singular Functions

Proof of Theorem 2. $(b) \Rightarrow (a)$. Suppose that E is porous and set $u = P[\mu]$. Set

$$\mathcal{G}_k := \{I \in \mathcal{D} : k^2 < u(z(I)) \leq (k+1)^2\}, \qquad k \geq 1.$$

We claim that there exists a constant $C > 0$ such that

(16) $$\sum_{k \geq 1} k \sum_{I \in \mathcal{G}_k} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2} \leq C u(z(J)), \qquad J \in \mathcal{D}.$$

Indeed, by Lemma 2 (c),

$$\sum_{k \geq 1} k \sum_{I \in \mathcal{G}_k} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2}$$

$$\leq \sum_{k \geq 1} \sum_{I \in \mathcal{D},\, k^2 < u(z(I))} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2} \leq C \sum_{k \geq 1} \frac{u(z(J))}{k^2}.$$

Now for each integer $k \geq 1$ and each $I \in \mathcal{G}_k$ we consider the set $\Lambda(I)$ consisting of k points, uniformly distributed in $T(I)$. Let $\Lambda := \cup_{k \geq 1} \cup_{I \in \mathcal{G}_k} \Lambda(I)$. Taking $J = \partial \mathbb{D}$ in (16), we see that Λ is a Blaschke sequence. Let B be the Blaschke product with the zero set Λ.

Notice that the zeros of B are restricted to the sets $T(I)$ where the modulus of S_μ is small, so if we prove that BS_μ has the WEP, we will have shown that S_μ is easily wepable.

Furthermore, the zeros of B are more and more densely packed as $k \to \infty$, i.e. as the modulus of S_μ gets smaller; thus for any $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that $|S_\mu(z)| > \eta$ whenever $\rho(z, \Lambda) > \varepsilon$. Thus, to prove that BS_μ has the WEP, we only need to show that $\inf\{|B(z)| : \rho(z, \Lambda) > \varepsilon\} > 0$. Fix z such that $\rho(z, \Lambda) > \varepsilon$ and let $J \in \mathcal{D}$ be such that $z \in T(J)$. Then by Harnack's inequality,

$$(17) \qquad\qquad u(z(J)) \leq C \log \eta^{-1}.$$

By (6),

$$\log |B(z)|^{-1} \leq C(\varepsilon) \sum_{\lambda \in \Lambda} \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \bar{\lambda}z|^2}$$

$$= C(\varepsilon) \sum_{k=1}^{\infty} \sum_{I \in \mathcal{G}_k} \sum_{\lambda \in \Lambda \cap T(I)} \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \bar{\lambda}z|^2}$$

$$\leq C(\varepsilon) \sum_{k=1}^{\infty} k \sum_{I \in \mathcal{G}_k} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2}.$$

Applying (16) and (17), we conclude that $\log |B(z)|^{-1} \leq C(\varepsilon) \log \eta^{-1}$.
□

Proof of Theorem 2. $(a) \Rightarrow (b)$. Assume now that E is not porous. We can find a sequence of arcs $J_n \in \mathcal{D}_{k_n}$, $k_n \to \infty$, and a sequence of numbers $M_n \to \infty$, $n \to \infty$, such that every $J \in \mathcal{D}_{k_n + M_n}$, $J \subset J_n$, meets E. Passing to a subsequence and using Lemma 5 we obtain a sequence of arcs $J_n \in \mathcal{D}_{k_n}$, $k_n \to \infty$, and a sequence of measures μ_n such that $\text{supp}\, \mu_n \subset J_n \cap E$, $\|\mu_n\| = o(|J_n|)$, and $\text{dist}\,(J_n, J_{n'}) \gtrsim$

$\max(|J_n|, |J_{n'}|)$. Furthermore, if sets $\Lambda_n \subset \mathbb{D}$ satisfy the property

$$P[\mu_n](z(I)) \geq n \implies \Lambda_n \cap T(I) \neq \varnothing, \qquad I \in \mathcal{D},$$

then the corresponding Blaschke products B_{Λ_n} satisfy the estimate

$$|B_{\Lambda_n}(z(J_n))| \leq \exp(-n^2).$$

Let

$$\mu = \sum_{n \geq 1} \mu_n.$$

To conclude that S_μ is not easily wepable we use the same argument as in the part $(a) \Rightarrow (b)$ of the proof of Theorem 3. $\qquad \square$

6. CORONA TYPE CONSTANTS

First, we make an easy remark: $c_n(\delta, I) \geq c_{n-1}(\delta, I)$, thus $\delta_n(I) \geq \delta_{n-1}(I)$.

Indeed, suppose that $\gamma < c_{n-1}(\delta, I)$, then there are $(f_1, \dots, f_{n-1}) =: f$ such that $\delta^2 \leq \sum_{j=1}^{n-1} |f_j(\lambda)|^2 \leq \|f\|_{\infty,n}^2 \leq 1$, and that

$$\gamma < \inf\{\|g\|_{\infty,n-1} : \exists h \in H^\infty : \sum_{j=1}^{n-1} g_j f_j + hI \equiv 1\}.$$

Given $\tilde{f} := (f_1, \dots, f_{n-1}, 0)$, for every $g \in (H^\infty)^n$ we obtain that $\sum_{j=1}^{n} g_j f_j = \sum_{j=1}^{n-1} g_j f_j$, so that $\chi_I(\tilde{f}) \geq \gamma$. Since $\tilde{f}$ fulfils the condition to be a candidate in the supremum, we obtain that $c_n(\delta, I) \geq \gamma$, q.e.d.

Lemma 6. *For any n, $\delta_n(I) \leq \tilde{\delta}(I)$.*

Proof. Pick any number $\varepsilon_0 > \tilde{\delta}(I)$, then choose ε_1 such that $\varepsilon_0 > \varepsilon_1 > \tilde{\delta}(I)$. Suppose that $f := (f_1, \dots, f_n) \in (H^\infty)^n$ satisfies the estimates $\varepsilon_0^2 \leq \inf_k \sum_{\lambda \in Z(I)}^{n} |f_j(\lambda)|^2$, $\|f\|_{\infty,n} \leq 1$. Take $z \in \mathbb{D}$ such that for some $\lambda \in Z(I)$ we have $\rho(z, \lambda) < \varepsilon_1$. Then, applying the Schwarz-Pick Lemma to the function $\varphi := f \cdot \bar{v}$, where v is a unit vector in $\mathbb{C}^n$ parallel to $f(\lambda)$, we see that

$$\left(\sum_{j=1}^{n} |f_j(z)|^2\right)^{1/2} \geq |\varphi(z)| \geq \frac{\varepsilon_0 - \varepsilon_1}{1 - \varepsilon_0 \varepsilon_1} =: \varepsilon_2.$$

On the other hand, suppose that $\rho(z, Z(I)) \geq \varepsilon_1$, then $|I(z)| \geq \eta_I(\varepsilon_1) > 0$. Finally,

$$\inf_{z \in \mathbb{D}} \left(\sum_{j=1}^{n} |f_j(z)|^2 + |I(z)|^2\right) \geq \min(\varepsilon_2^2, \eta_I(\varepsilon_1)^2).$$

By Carleson's Corona Theorem, we can find $g \in (H^\infty)^n$, $h \in H^\infty$ with $\|g\|_{\infty,n} \leq C(\varepsilon_2, \eta_I(\varepsilon_1))$ such that $\sum_{j=1}^n g_j f_j + hI \equiv 1$, therefore $c_n(\varepsilon_0, I) < \infty$. Since this holds for any $\varepsilon_0 > \tilde{\delta}(I)$, we are done.   □

The following will end the proof of Proposition 1.

Lemma 7. $\delta_1(I) \geq \tilde{\delta}(I)$.

Proof. Let $\varepsilon_0 < \tilde{\delta}(I)$. We want to prove that $c_1(\varepsilon_0, I) = \infty$. Pick ε_1 such that $\varepsilon_0 < \varepsilon_1 < \tilde{\delta}(I)$. Then there exists an infinite sequence $(\zeta_n)_n \subset \mathbb{D}$ such that $\rho(\zeta_n, Z(I)) \geq \varepsilon_1$ and $|I(\zeta_n)| \to 0$.

Choose a subsequence (ξ_n) of this sequence, with

$$1 - \inf_k \prod_{j:j\neq k} \rho(\xi_j, \xi_k)$$

so small that the Blaschke product B with zeros (ξ_n) satisfies the property $|B(z)| > \varepsilon_0$ if $\rho(z, Z(B)) > \varepsilon_1$ (see, for instance [6, p. 395]). Then for any $\lambda \in Z(I)$ we have $|B(\lambda)| \geq \varepsilon_0$. On the other hand, for any $g, h \in H^\infty$,

$$g(\xi_n)B(\xi_n) + h(\xi_n)I(\xi_n) = h(\xi_n)I(\xi_n) \to 0, \qquad n \to \infty.$$

This proves that $gf + hI \not\equiv 1$.   □

Proof of Proposition 2. The argument is anologous to that in the proof of Lemma 7. Take a strictly increasing function $\psi : (0,1) \to (0,1)$ such that $\psi \cdot (\phi+1) \leq 1$. Using the above mentioned result from [2, p. 1199] we find a Blaschke product B satisfying the WEP and such that for every $\delta \in (0,1)$ there exists $z_\delta \in \mathbb{D}$ satisfying

$$\rho(z_\delta, Z(B)) = \delta, \qquad |B(z_\delta)| \leq \psi(\delta).$$

Denote $b_\delta(z) = (z - z_\delta)/(1 - \bar{z}_\delta z)$. We have $\min_{Z(B)} |b_\delta| = \delta$. If $g, h \in H^\infty$, $gb_\delta + hB \equiv 1$, then

$$\|h\|_\infty \geq 1 + \phi(\delta),$$

and hence,

$$c_1(\delta, B) \geq \inf_{gb_\delta+hB\equiv 1} \|g\|_\infty \geq \phi(\delta), \qquad \delta \in (0,1).$$

□

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