Bull. London Math. Soc. 38 (2006) 250–260
doi:10.1112/S0024609306018339

HARMONIC AND SUPERHARMONIC MAJORANTS ON THE DISK

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Abstract

A proof is given to show that a positive function on the unit disk admits a harmonic majorant if and only if it has a certain explicit upper envelope that admits a superharmonic majorant. The (logarithmic) Lipschitz regularity of this superharmonic majorant is discussed.

1. *Definitions and statements*

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and $H^+(\mathbb{D})$ the cone of positive harmonic functions on $\mathbb{D}$. We would like to describe the nonnegative functions $\varphi : \mathbb{D} \longmapsto \mathbb{R}_+$ which admit a harmonic majorant: that is, $h \in H^+(\mathbb{D})$ such that $h \geqslant \varphi$. This question arises in problems about the decrease of bounded holomorphic functions in the unit disk, as well as in the description of free interpolating sequences for the Nevanlinna class. See [**4**], where an answer is given in terms of duality with the measures that act on positive harmonic functions. The aim of this paper is to reduce this problem first to the finiteness of a certain best Lipschitz majorant function, and then to the existence of a merely *super*harmonic (nontrivial) majorant.

Let the hyperbolic (or Poincaré) distance ρ on the disk be defined by $d\rho(z) := (1-|z|^2)^{-1}|dz|$. This is invariant under biholomorphic maps from the disk to itself. Explicitly, if we first define the pseudohyperbolic (or Gleason) distance by

$$d(z,w) := \left|\frac{z-w}{1-z\overline{w}}\right|,$$

then

$$\rho(z,w) = \frac{1}{2}\log\frac{1+d(z,w)}{1-d(z,w)}. \tag{1.1}$$

For $h \in H^+(\mathbb{D})$, the classical Harnack inequality reads, for $0<r<1$, $\theta \in \mathbb{R}$:

$$\frac{1-r}{1+r}h(0) \leqslant h(re^{i\theta}) \leqslant \frac{1+r}{1-r}h(0).$$

This implies that the function $\log h$ is Lipschitz with constant 2 with respect to the hyperbolic distance. We will say that a positive valued function F is *Log-Lipschitz* (with constant C) if and only if $|\log F(z) - \log F(w)| \leqslant C\rho(z,w)$ for all $z, w \in \mathbb{D}$.

Received 2 October 2003; revised 22 March 2005.

2000 *Mathematics Subject Classification* 31A05.

All the authors were supported by the PICS program no 1019 of Generalitat de Catalunya and CNRS. The first and second authors were also supported by the European Commission Research Training Network HPRN-CT-2000-00116. The second author was supported by DGICYT grant BFM2002-00571 and the CIRIT grant 2001-SGR00431.

THEOREM 1. *A nonnegative function φ admits a superharmonic and Log-Lipschitz majorant with constant $C \geqslant 2$ if and only if φ admits a harmonic majorant.*

Since the infimum of two harmonic functions is not in general harmonic, there is no smallest harmonic majorant for a given function. On the other hand, the cone $\mathrm{Sp}(\mathbb{D})$ of superharmonic functions is stable under finite infima. Denote by $R(\varphi)$ the reduced function of φ; that is,

$$R(\varphi)(z) := \inf\{u(z) : u \in \mathrm{Sp}(\mathbb{D}), u \geqslant \varphi \text{ on } \mathbb{D}\}$$

(see, for example, [**2**]). We use the convention that $R(\varphi) \equiv \infty$ if there is no (non identically infinite) superharmonic majorant. The reduced function could also be called the 'superharmonic envelope', as in [**8**]. The reduced function is not in general superharmonic, because it can fail to be lower semicontinuous. J. W. Green [**3**, Theorem 2] has proved that when φ is continuous, then $R(\varphi)$, if finite, is also continuous, and therefore superharmonic, so the infimum in its definition is really a minimum; furthermore, $R(\varphi)$ is harmonic in the open set $\{z : R(\varphi)(z) > \varphi(z)\}$.

If, instead of taking the infimum of all superharmonic functions above φ, we restrict ourselves to those that are Log-Lipschitz with a given constant C, then the corresponding infimum $R_C(\varphi)$ (if finite) will again be Log-Lipschitz with constant C, and hence it will be the smallest Log-Lipschitz superharmonic majorant of φ with constant C. Theorem 1 means that φ admits a harmonic majorant if and only if $R_C(\varphi)$ is finite for any given $C \geqslant 2$.

In order to study the finiteness of $R_C(\varphi)$, it would be nice to be able to proceed in two steps, dealing first with the Lipschitz property, and then with superharmonicity. The smallest Log-Lipschitz majorant with constant C of a given nonnegative function φ is (see [**7**], or [**5**] for a more recent survey)

$$L_C(\varphi)(z) = \exp\left\{\sup_{w\in\mathbb{D}}(\varphi(w) - C\rho(w, z))\right\}.$$

Clearly, $L_C(\varphi) \leqslant R_C(\varphi)$, so $R(L_C(\varphi)) \leqslant R_C(\varphi)$. That inequality can be strict when $C < 2$.

PROPOSITION 2. *For every $0 < \gamma \leqslant \beta < 2$ there exists a bounded positive function H on the disc such that*

$$R(L_\gamma(H)) \neq R_\beta(H).$$

However, we do not know whether $R(L_C(\varphi)) = R_C(\varphi)$ holds for $C \geqslant 2$. Nevertheless, if a Log-Lipschitz function admits a superharmonic majorant s, then the invariant averages of s will provide us with a superharmonic majorant with a weak Log-Lipschitz property, and so we obtain the following result.

THEOREM 3. *If there exists $C > 0$ such that $L_C(\varphi)$ admits a superharmonic majorant, then φ admits a harmonic majorant.*

Note that it would not help to perform our two steps in the reverse order. Typical data for many problems of harmonic majorants are functions φ which vanish everywhere except on a discrete subset [**4**]. For such φ, we have $R(\varphi) = \varphi$.

Of course, Theorem 3 raises the question of criteria to ensure the finiteness of the reduced function of Log-Lipschitz data. In this general direction, one should note results of Koosis [**6**, p. 77] and Cole and Ransford [**1**, Theorem 1.3], which imply that when φ is merely continuous, then

$$R(\varphi)(x) = \sup\left\{ \int \varphi d\mu : \mu \in I_x \right\} = \sup\left\{ \int \varphi d\mu : \mu \in H_x \right\},$$

where I_x denotes the set of Jensen measures for x, and H_x denotes the set of harmonic measures for x with respect to a domain $\omega \subset\subset \mathbb{D}$. A perhaps more computable characterization of $R(\varphi)$ is given by [**6**, Théorème, p. 80]: let $D_H(z,r)$ stand for the disc of center z and radius r with respect to the hyperbolic distance ρ, and let $d\beta(z) := (1-|z|^2)^{-2} dm(z)$ be the invariant measure on the disk, where m denotes two-dimensional Lebesgue measure. Given a real-valued continuous function F on $\mathbb{D}$, let

$$MF(z) := \sup_{r>0} \frac{1}{\beta(D_H(z,r))} \int_{D_H(z,r)} F(w)\, d\beta(w),$$

and define $F^{(0)} := F$ and $F^{(k+1)} := MF^{(k)}$. Then, arguing as in [**6**], one can check that

$$R\varphi(z) = \lim_{n\to\infty} \varphi^{(n)}(z).$$

So Theorem 3 says that φ admits a harmonic majorant if and only if the sequence $L_C(\varphi)^{(n)}(0)$ remains bounded.

The paper is organized as follows. The next section is devoted to studying the dyadic analogue of the problem of harmonic majorants. Theorems 1 and 3 are proved in Sections 3 and 5 respectively. Proposition 2 is proved in Section 4.

2. *A discrete model*

Recall that any positive harmonic function h is the Poisson integral of a finite positive measure μ on the boundary of the disk:

$$h(z) = \int_0^{2\pi} P_z(e^{i\theta})\, d\mu(\theta), \qquad \text{where } P_z(e^{i\theta}) := \frac{1}{2\pi} \frac{(1-|z|^2)}{|z-e^{i\theta}|^2}.$$

The following considerations concern the simpler case of functions that are generated by the 'square' kernel

$$K_z(e^{i\theta}) := \frac{1}{|I_z|} \chi_{I_z}(e^{i\theta}), \qquad \text{where } I_z := \{e^{i\theta} : z \in \Gamma_\alpha(e^{i\theta})\}.$$

Here, χ_E stands for the characteristic function of the set E, and $\Gamma_\alpha(e^{i\theta})$ for the Stolz angle of aperture α and vertex $e^{i\theta}$. The 'square' integral of a finite measure μ is defined by

$$\int_0^{2\pi} K_z(e^{i\theta})\, d\mu(\theta) = \frac{\mu(I_z)}{|I_z|}.$$

Consider the usual partition of $\partial\mathbb{D}$ in dyadic arcs, for any n in $\mathbb{Z}_+$:

$$I_{n,k} := \{e^{i\theta} : \theta \in [2\pi k 2^{-n}, 2\pi(k+1)2^{-n})\}, \qquad 0 \leqslant k < 2^n. \tag{2.1}$$

Note that $|I_{n,k}| = 2\pi 2^{-n}$. To this subdivision we associate the Whitney partition in 'dyadic squares' of the unit disk :

$$Q_{n,k} := \{re^{i\theta} : e^{i\theta} \in I_{n,k}, 1 - 2^{-n} \leqslant r < 1 - 2^{-n-1}\}.$$

It is well known and easy to see that there exists a constant c_α such that for any $z \in Q_{n,k}$, we have $P_z \geqslant c_\alpha K_{I_{n,k}}$. This implies that sufficient conditions for majorization by 'K-harmonic functions' yield sufficient conditions for majorization by (true) harmonic functions.

THEOREM 4. *Given a collection of nonnegative data $\{p_{n,k}\} \subset \mathbb{R}_+$, there exists a finite positive measure μ on $\partial\mathbb{D}$ such that*

$$\frac{\mu(I_{n,k})}{|I_{n,k}|} \geqslant p_{n,k} \tag{2.2}$$

if and only if there exists a constant S such that

$$\sum_{n,k} p_{n,k}|I_{n,k}| \leqslant S, \tag{2.3}$$

where the sum is taken over any disjoint subfamily of the whole family of dyadic arcs $\{I_{n,k}\}$ defined in (2.1).

Proof. Condition (2.3) is clearly necessary with $S = \mu(\partial\mathbb{D})$. To prove the converse direction, let us consider the following modified data:

$$\tilde{p}_{n,k} := \frac{1}{|I_{n,k}|} \sup \left\{ \sum p_{q,j}|I_{q,j}| \right\},$$

where the supremum is taken over any disjoint subfamily $\{I_{q,j}\}$ of the family of all dyadic subarcs of the given $I_{n,k}$. Observe that $\tilde{p}_{n,k} \geqslant p_{n,k}$, and that $\tilde{p}_{n,k}$ satisfies the following discrete superharmonicity property:

$$\tilde{p}_{n,k} \geqslant \tfrac{1}{2}(\tilde{p}_{n+1,2k} + \tilde{p}_{n+1,2k+1}).$$

Assuming that (2.3) holds, we will construct a sequence of positive measures μ_n of bounded total mass, uniformly distributed on each arc $I_{n,j}$, such that for all $m \leqslant n$,

$$\mu_n(I_{m,k}) \geqslant |I_{m,k}|\tilde{p}_{m,k}. \tag{2.4}$$

Let μ_0 be the uniform measure of total mass S on the arc $I_{0,0} = \partial\mathbb{D}$. The hypothesis (2.3) coincides with (2.4) in this case. Assuming that μ_m, $m \leqslant n$, have already been constructed satisfying (2.4), we will choose μ_{n+1}. Fix j, $0 \leqslant j < 2^n$. Then $I_{n,j} = I_{n+1,2j} \cup I_{n+1,2j+1}$, so (2.4) implies in particular that

$$\mu_n(I_{n,j}) \geqslant |I_{n+1,2j}|\tilde{p}_{n+1,2j} + |I_{n+1,2j+1}|\tilde{p}_{n+1,2j+1}.$$

Now choose $\alpha, \beta \geqslant 0$ such that

$$\mu_n(I_{n,j}) = \alpha + \beta, \qquad \alpha \geqslant |I_{n+1,2j}|\tilde{p}_{n+1,2j}, \qquad \beta \geqslant |I_{n+1,2j+1}|\tilde{p}_{n+1,2j+1},$$

and set $\mu_{n+1}(I_{n+1,2j}) = \alpha$ and $\mu_{n+1}(I_{n+1,2j+1}) = \beta$. This defines a measure μ_{n+1} which satisfies (2.4) at rank $n+1$ for $m = n+1$. It also satisfies

$$\mu_{n+1}(I_{m,j}) = \mu_n(I_{m,j}), \qquad \forall m \leqslant n,$$

so (2.4) is satisfied by μ_{n+1} for all $m \leqslant n+1$. This bounded sequence of measures contains a weakly convergent subsequence, whose limit μ will satisfy (2.2). □

3. *Proof of Theorem* 1

We will prove the following slightly more general fact, which will be useful in Section 5.

PROPOSITION 5. *Let u be a positive superharmonic function on $\mathbb{D}$ such that, for any $z, w \in \mathbb{D}$,*

$$|\log u(z) - \log u(w)| \leqslant C_1(1 + \rho(z,w)). \tag{3.1}$$

Then there exists $h \in H^+(\mathbb{D})$ such that $h \geqslant u$.

Proof. The Riesz representation theorem tells us that there exists a positive measure $\nu = -\Delta u$ in the sense of distributions, and a positive harmonic function h_0 such that

$$u(z) = h_0(z) + \int_{\mathbb{D}} \log \frac{1}{d(z,w)}\, d\nu(w) =: h_0(z) + u_0(z).$$

Let $\delta \in (0,1)$, to be chosen later. Then define

$$\begin{aligned} u_0(z) &= \int_{\{w: d(z,w) \leqslant \delta\}} \log \frac{1}{d(z,w)}\, d\nu(w) + \int_{\{w: d(z,w) > \delta\}} \log \frac{1}{d(z,w)}\, d\nu(w) \\ &=: u_1(z) + u_2(z). \end{aligned}$$

For $d(z,w) > \delta$, we have

$$\log \frac{1}{d(z,w)} \leqslant C_\delta \frac{1 - |zw|^2}{|1 - z\overline{w}|^2}(1 - |w|^2),$$

which is harmonic in z. Note that

$$\infty > u(0) \geqslant \int_{\mathbb{D}} \log \frac{1}{|w|}\, d\nu(w) \geqslant \int_{\mathbb{D}} (1 - |w|)\, d\nu(w), \tag{3.2}$$

so that the harmonic function

$$h_2(z) := C_\delta \int_{\mathbb{D}} \frac{1 - |zw|^2}{|1 - z\overline{w}|^2}(1 - |w|^2) d\nu(w)$$

is bigger than u_2. Now we need only to to find a harmonic majorant for the remaining term, u_1.

A sequence $\{z_k\} \subset \mathbb{D}$ is called *uniformly dense* if there exists $0 < r_1 < r_2$ satisfying

(i) $D_H(z_j, r_1) \cap D_H(z_k, r_1) = \emptyset$ for any $j \neq k$,
(ii) $\mathbb{D} \subset \cup_k D_H(z_k, r_2)$.

Here, $D_H(z,r)$ denotes the hyperbolic disk with center z and radius r.

LEMMA 6. *For $\delta \in (0,1)$, a properly chosen absolute constant, there exist a uniformly dense sequence $\{z_k\}$ and a positive harmonic function h_1 such that for any k, we have $u_1(z_k) \leqslant h_1(z_k)$.*

Conclusion of the proof of Proposition 5. Accepting this lemma, if we write $h_3 := h_0 + h_1 + h_2$, we see that we have $u(z_k) \leqslant h_3(z_k)$, for any k. Now by Harnack's inequality, for any $z \in D_H(z_k, r_2)$, we have $h_3(z) \geqslant e^{-2r_2} h_3(z_k)$, while by (3.1) — note that this is the only step in this argument where this hypothesis is used — we have

$$u(z) \leqslant \exp[C_1(1+r_2)]u(z_k) \leqslant \exp[C_1(1+r_2)]h_3(z_k) \leqslant \exp[C_1(1+r_2)]e^{2r_2}h_3(z).$$

So we have found a harmonic majorant of u. $\square$

Proof of Lemma 6. First, let $T_{n,j} := \{z \in Q_{n,j} : \rho(z, \partial Q_{n,j}) > \delta\}$. For δ small enough, we have

$$m(Q_{n,j}) \geqslant m(T_{n,j}) \geqslant C(\delta) m(Q_{n,j}),$$

where m is two-dimensional Lebesgue measure, and $0 < C(\delta) < 1$.

Choose in each $T_{n,j}$ a point $z_{n,j}$ such that

$$u_1(z_{n,j}) \leqslant \frac{1}{m(T_{n,j})} \int_{T_{n,j}} u_1(\zeta)\, dm(\zeta).$$

It is enough to estimate this last average. We apply Fubini's theorem:

$$\begin{aligned}
\frac{1}{m(T_{n,j})} \int_{T_{n,j}} u_1(z)\, dm(z) &= \frac{1}{m(T_{n,j})} \int_{T_{n,j}} \int_{\{w : d(z,w) \leqslant \delta\}} \log \frac{1}{d(z,w)}\, d\nu(w)\, dm(\zeta) \\
&\leqslant \frac{1}{m(T_{n,j})} \int_{Q_{n,j}} \left(\int_{T_{n,j}} \log \frac{1}{d(z,w)} dm(z) \right) d\nu(w) \\
&\leqslant C \int_{Q_{n,j}} d\nu(w),
\end{aligned}$$

where the last inequality is due to the following explicit estimate: for $w \in Q_{n,j}$,

$$\begin{aligned}
\int_{T_{n,j}} \log \frac{1}{d(z,w)}\, dm(z) &\leqslant \sum_{k=0}^{\infty} \log \frac{1}{2^{-k-1}}\, m\left(\{z : 2^{-k-1} < d(w,z) \leqslant 2^{-k}\}\right) \\
&\leqslant C \sum_{k=0}^{\infty} k 2^{-2n-2k} \\
&\leqslant C 2^{-2n} \\
&\leqslant C m(T_{n,j}).
\end{aligned}$$

Now we set

$$p_{n,j} := \frac{1}{|I_{n,j}|} \int_{Q_{n,j}} (1 - |w|)\, d\nu(w) \approx \int_{Q_{n,j}} d\nu(w).$$

Since the 'squares' $Q_{n,j}$ are disjoint, the condition (3.2) implies that

$$\sum_{n,j} |I_{n,j}| p_{n,j} < \infty. \tag{3.3}$$

Hence $\{p_{n,j}\}$ satisfies (2.3), and Theorem 4 provides a positive measure μ on the unit circle such that for any dyadic arc $I_{n,j}$,

$$\mu(I_{n,j}) \geqslant |I_{n,j}| p_{n,j}. \tag{3.4}$$

However, whenever condition (3.3) is satisfied, a more direct construction can be applied. Namely, one may take $d\mu = f(e^{i\theta})\, d\theta$ where

$$f(e^{i\theta}) := \sum_{I_{n,k} \ni e^{i\theta}} p_{n,k}.$$

Observe that $\int_0^{2\pi} f(e^{i\theta})d\theta = \sum |I_{n,k}|p_{n,k}$, and since $f(e^{i\theta}) \geqslant p_{n,k}$ whenever $e^{i\theta} \in I_{n,k}$, the measure μ satisfies (3.4).

Then, by the remarks before Theorem 4, there exists a harmonic function h_1 such that

$$h_1(z_{n,j}) \geqslant Cp_{n,j} \geqslant C\int_{Q_{n,j}} d\nu(w) \geqslant u_1(z_{n,j}). \qquad \square$$

4. *Proof of Proposition* 2

The gist of this proof is that the Poisson kernel itself cannot be log-Lipschitz with a constant better than 2.

Denote

$$f(z) = \mathrm{Re}\frac{1+z}{1-z} = P_z(1).$$

Fix $\gamma > 0$. For small $\delta > 0$, consider the function

$$g(z) = f((1-\delta)z),$$

positive and harmonic in $(1/(1-\delta))\mathbb{D}$. For $0 < \varepsilon < \delta/2$, consider

$$g_\varepsilon(z) = g\Big(\frac{z}{1-\varepsilon}\Big), \qquad z \in \mathbb{D}.$$

Denote

$$M_\delta = \sup_{0<\varepsilon<\delta/2,\, z\in\mathbb{D}} \|\nabla(\log g_\varepsilon)(z)\|.$$

Put

$$h(z) = g_\varepsilon(z), \qquad z \in (1-\varepsilon)\partial\mathbb{D}.$$

LEMMA 7. *For sufficiently small $\varepsilon < \delta/2$, and for any $w \in (1-\varepsilon)\partial\mathbb{D}$, $z \in \mathbb{D}$, we have*

$$|\log g_\varepsilon(w) - \log g_\varepsilon(z)| \leqslant M_\delta|z-w| \leqslant \gamma\rho(z,w),$$

where ρ is defined as in (1.1).

Applying this to the special case where $z \in (1-\varepsilon)\partial\mathbb{D}$, we see that for sufficiently small ε, $\log h$ satisfies the Lipschitz condition with respect to ρ with constant γ.

Consider the γ-log-Lipschitz (with respect to ρ) extension of h to $\mathbb{D}$:

$$H(z) = \sup_{w\in(1-\varepsilon)\partial\mathbb{D}} h(w)e^{-\gamma\rho(w,z)}.$$

It also follows from the lemma that for sufficiently small $\varepsilon < \delta/2$, we have

$$H \leqslant g_\varepsilon \qquad \text{on } \mathbb{D}.$$

Proof of Lemma 7. To prove the second inequality, note that if

$$\left|\frac{z-w}{1-z\bar{w}}\right| \leqslant \frac{1}{2},$$

then $|1-z\bar{w}| \leqslant u(\varepsilon) \to 0$ as $\varepsilon \to 0$, and then (since $\log((1+s)/(1-s)) \geqslant 2s$, $0 \leqslant s < 1$) we have

$$2\rho(w,z) \geqslant 2\left|\frac{z-w}{1-z\bar{w}}\right| \geqslant \frac{2}{u(\varepsilon)}|z-w|.$$

If
$$\left|\frac{z-w}{1-z\bar{w}}\right| > \frac{1}{2}, \qquad \text{and} \qquad |z-w| \leqslant \frac{\gamma}{2M_\delta},$$
then
$$2\rho(w,z) \geqslant \log 3 > 1 \geqslant \frac{2M_\delta}{\gamma}|z-w|.$$
Finally, if $|z-w| > \gamma/2M_\delta$, then
$$\left|\frac{z-w}{1-z\bar{w}}\right| \geqslant 1 - u_1\left(\varepsilon, \frac{\gamma}{2M_\delta}\right),$$
where $u_1(\varepsilon, \gamma/2M_\delta) \to 0$ as $\varepsilon \to 0$, and
$$2\rho(w,z) \geqslant \log \frac{1}{u_1(\varepsilon,\gamma/2M_\delta)} \geqslant \frac{2M_\delta}{\gamma},$$
as $\varepsilon \to 0$. This completes the proof of Lemma 7. □

By definition of H, we have $L_\gamma(H) = H$.

Since $R(H)$ is superharmonic, and H coincides with g_ε on $(1-\varepsilon)\partial\mathbb{D}$, with g_ε harmonic in $\mathbb{D}$, we obtain
$$R(H) \geqslant g_\varepsilon \qquad \text{on } (1-\varepsilon)\mathbb{D}.$$
Furthermore, since $H \leqslant g_\varepsilon$ on $\mathbb{D}$,
$$R(H) \leqslant g_\varepsilon \qquad \text{on } \mathbb{D}.$$
As a result,
$$R(H) = g_\varepsilon \qquad \text{on } (1-\varepsilon)\mathbb{D},$$
and, writing $z = x + iy$, we see that
$$\begin{aligned}\frac{\partial}{\partial x}(\log R(H))(0) &= \frac{\partial}{\partial x}(\log g_\varepsilon)(0)\\ &= \frac{1-\delta}{1-\varepsilon}\frac{\partial}{\partial x}(\log f)(0)\\ &= 2\frac{1-\delta}{1-\varepsilon}.\end{aligned}$$
Therefore, the function $R(H)$ is no better than $2((1-\delta)/(1-\varepsilon)$-log-Lipschitz, and
$$R(L_\gamma(H)) = R(H) \neq R_c(H), \qquad c < 2\frac{1-\delta}{1-\varepsilon}.$$
This completes the proof of Proposition 2. □

5. *Comparison of the upper envelopes*

The aim of this section is to prove Theorem 3. Proposition 5 shows that it will follow from the following result.

LEMMA 8. *Given $C_0 > 0$, there is a $C_1 \geqslant C_0$ such that if φ is Log-Lipschitz with constant C_0 and admits a non-trivial superharmonic majorant, then there exists a superharmonic function v such that $v \geqslant \varphi$ and v satisfies* (3.1) *with constant C_1.*

Proof. Let $u \in \mathrm{Sp}(\mathbb{D})$ such that $u \geqslant \varphi$. We note (see Lemma 9) that an averaged version of u always satisfies (3.1), and that (up to a multiplicative constant) it provides the regular superharmonic majorant that we are seeking.

For any $\delta \in (0, \frac{1}{2}]$, let $D_H(z,\delta)$ stand, as above, for a hyperbolic disk of radius δ centered at z. Let $d\beta(z) := (1-|z|^2)^{-2} dm(z)$ be the invariant measure on the disk. For any Möbius automorphism ϕ of the disk, any measurable function f and any measurable set E, we have (see, for example, [**9**, (2.19), p. 19])

$$\int_E f \circ \phi \, d\beta = \int_{\phi(E)} f \, d\beta.$$

For any measurable function g on the unit disk, let

$$g_\delta(z) := \frac{1}{\beta(D_H(z,\delta))} \int_{D_H(z,\delta)} g \, d\beta.$$

Since φ is Log-Lipschitz with constant C_0, we have $\varphi(w) \geqslant e^{-C_0\delta}\varphi(z)$ for any $w \in D_H(z,\delta)$, so that, being an average of such values, $\varphi_\delta(z) \geqslant e^{-C_0\delta}\varphi(z)$. Now $\varphi \leqslant u$ implies that

$$\varphi(z) \leqslant e^{C_0\delta}\varphi_\delta(z) \leqslant e^{C_0\delta} u_\delta(z).$$

The proof of Lemma 8 will conclude with the next two lemmas. □

LEMMA 9. *There exists an absolute constant κ such that for any positive-valued superharmonic function u, and for any z, w in $\mathbb{D}$ such that $d(z,w) \leqslant \delta/4$, one has $u_\delta(w) \leqslant \kappa u_\delta(z)$, and therefore u_δ satisfies* (3.1).

Proof. Recall that since u is superharmonic, for any $z \in \mathbb{D}$ and $r_1 < r_2$,

$$\frac{1}{\beta(D_H(z,r_1))} \int_{D_H(z,r_1)} u \, d\beta \geqslant \frac{1}{\beta(D_H(z,r_2))} \int_{D_H(z,r_2)} u \, d\beta. \tag{5.1}$$

This fact is clear when $z = 0$, because $d\beta$ has radial density, and we can reduce ourselves to this case by composing u with an appropriate Möbius automorphism of $\mathbb{D}$.

Pick a constant $K < 1$ such that $\rho(z,w) \leqslant \delta/4$ implies that $D_H(w, K\delta) \subset D_H(z,\delta)$. Then (5.1) implies that

$$u_\delta(w) \leqslant u_{K\delta}(w) \leqslant \frac{1}{\beta(D_H(w,K\delta))} \int_{D_H(z,\delta)} u \, d\beta,$$

by the inclusion of discs and the positivity of u. But

$$\frac{1}{\beta(D_H(w,K\delta))} \int_{D_H(z,\delta)} u \, d\beta \leqslant \kappa \frac{1}{\beta(D_H(z,\delta))} \int_{D_H(z,\delta)} u \, d\beta = \kappa u_\delta(z),$$

since $\beta(D_H(w,K\delta))$ and $\beta(D_H(z,\delta))$ are comparable. □

LEMMA 10. *Let u be a positive superharmonic function on $\mathbb{D}$. Then u_δ is also superharmonic.*

Proof. If we were dealing with superharmonic functions on the plane, we could simply use the invariance of harmonicity under translations to define suitable averaged functions. In $\mathbb{D}$, we need invariance under Möbius automorphisms. The appropriate machinery happens to have been developed in the more general case of the unit ball of $\mathbb{C}^n$.

We will follow the notation and use the results of [**9**, Chapter 4, pp. 34–39], which was itself inspired by [**10**]. Let ϕ_z denote the unique involutive Möbius automorphism of the disk that exchanges z and 0. Define the invariant convolution of two measurable functions by

$$(f * g)(z) := \int_{\mathbb{D}} f(w)\,(g \circ \phi_z)(w)\, d\beta(w),$$

whenever the integral makes sense. This operation is commutative (see [**9**, bottom of page 34]). If we set

$$\Omega_\delta(z) := \frac{1}{\beta(D_H(0,\delta))} \chi_{D_H(0,\delta)}(z),$$

then $f_\delta = f * \Omega_\delta$. Note that $D_H(0,\delta) = D(0, \tanh\delta)$, so that Ω_δ is a radial function.

The $\mathcal{M}$-subharmonic functions defined in [**9**, Chapter 4, (4.1)] reduce for $n = 1$ to ordinary subharmonic functions. The invariant Laplacian (the Laplace–Beltrami operator for the Bergman metric of the ball) reduces in the case $n = 1$ to

$$\tilde{\Delta} = 2(1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}},$$

so that $\mathcal{C}^2$ superharmonic functions g can be characterized as those such that $\tilde{\Delta} g \leqslant 0$. Since our function Ω_δ is not smooth, we need to perform an approximation argument. It will be enough to show that $u * \Omega_\delta$ can be approximated from below by an increasing sequence of $\mathcal{C}^2$ superharmonic functions. Pick an increasing sequence of smooth, nonnegative, *radial* functions $\Omega_{\delta,n}$ so that $\lim_{n\to\infty} \Omega_{\delta,n} = \Omega_\delta$ almost everywhere. Then the monotone convergence theorem tells us that $u * \Omega_{\delta,n}$ converges to $u * \Omega_\delta$, and the sequence is clearly increasing. For $f \in \mathcal{C}^2(\mathbb{D})$, by [**9**, (4.11), p. 36] we have

$$(\tilde{\Delta} f)(a) = \lim_{r\to 0} \frac{4}{r^2} [(f * \Omega_r)(a) - f(a)].$$

Now, by applying, twice, Ulrich's lemma about the associativity of the invariant convolution when the middle element is radial [**9**, Lemma 4.5, p. 36], we have

$$\begin{aligned}
\tilde{\Delta}(u * \Omega_{\delta,n}) &= \lim_{r\to 0} \frac{4}{r^2} \left[(u * \Omega_{\delta,n}) * \Omega_r - u * \Omega_{\delta,n} \right] \\
&= \lim_{r\to 0} \frac{4}{r^2} \left[u * (\Omega_{\delta,n} * \Omega_r) - u * \Omega_{\delta,n} \right] \\
&= \lim_{r\to 0} \frac{4}{r^2} \left[u * (\Omega_r * \Omega_{\delta,n}) - u * \Omega_{\delta,n} \right] \\
&= \lim_{r\to 0} \frac{4}{r^2} \left[(u * \Omega_r) * \Omega_{\delta,n} - u * \Omega_{\delta,n} \right] \\
&= \lim_{r\to 0} \frac{4}{r^2} \left[(u * \Omega_r - u) * \Omega_{\delta,n} \right].
\end{aligned}$$

Since u is superharmonic, $u * \Omega_r - u \leqslant 0$, and since $\Omega_{\delta,n} \geqslant 0$, we finally have $\tilde{\Delta}(u * \Omega_{\delta,n}) \leqslant 0$. □

Acknowledgements. We would like to thank the referee for the very careful reading of our paper.

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