FINITE PRODUCTS OF INTERPOLATING BLASCHKE PRODUCTS

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1. *Introduction*

Let H^∞ be the algebra of bounded analytic functions on the unit disc D of the complex plane. A function in H^∞ is called *inner* if it has radial limits of modulus one, almost everywhere on the unit circle. Given a sequence $\{z_n\}$ of points in D satisfying the Blaschke condition $\sum_n (1-|z_n|) < +\infty$ and a real number γ, the Blaschke product

$$B(z) = e^{i\gamma} z^m \prod_{z_n \neq 0} \frac{-\bar{z}_n}{|z_n|} \frac{z-z_n}{1-\bar{z}_n z} \quad \text{for } z \in D$$

is an inner function. Given a positive measure σ on the unit circle, singular to Lebesgue measure, the singular function

$$S(\sigma)(z) = \exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\sigma(\theta)\right) \quad \text{for } z \in D$$

is also inner. It is well known that any inner function can be factored into a Blaschke product and a singular function.

Let I be an inner function and $\alpha \in D$. It is clear that

$$\tau_\alpha(I)(z) = \frac{I(z)-\alpha}{1-\bar{\alpha} I(z)} \quad \text{for } z \in D$$

is also inner. Actually, Frostman proved that for all $\alpha \in D$, except possibly for a set of logarithmic capacity zero, the function $\tau_\alpha(I)$ is a Blaschke product.

See [**4**, Chapter II] for the proofs of these results.

A Blaschke product B is called *indestructible* if $\tau_\alpha(B)$ is a Blaschke product for all $\alpha \in D$, that is, if there is no exceptional set in Frostman's Theorem. As far as I know, the problem of characterizing the indestructible Blaschke products in terms of the distribution of their zeros remains open. In this paper we solve a conformal invariant version of that problem.

A positive measure μ on D is a Carleson measure if there is a constant $C = C(\mu)$ such that $\mu(Q) \leqslant C\, l(Q)$, for every sector

$$Q = \{z \in D\colon 1-|z| \leqslant h, |\operatorname{Arg} z - \theta| \leqslant h\}, \tag{1.1}$$

where $l(Q) = h$.

A sequence $\{z_n\}$ of points in D is called an *interpolating sequence* if, for every bounded sequence $\{w_n\}$ of complex numbers, there exists $f \in H^\infty$ such that $f(z_n) = w_n$

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for $n = 1, 2, \dots$. Carleson proved that $\{z_n\}$ is an interpolating sequence if and only if $\inf_{n \neq m} |(z_n - z_m)/(1 - \bar{z}_m z_n)| > 0$ and $\mu = \sum_n (1 - |z_n|^2) \delta_n$ is a Carleson measure, where δ_n is the Dirac measure at z_n. (See [4, Chapter VII].)

An interpolating Blaschke product is a Blaschke product whose zero set is an interpolating sequence. It is known that a Blaschke product with zeros $\{z_n\}$ is a finite product of interpolating Blaschke products if and only if the measure $\mu = \sum (1 - |z_n|^2) \delta_n$ is a Carleson measure [6]. This last condition is the conformal invariant version of the Blaschke condition; therefore the finite products of interpolating Blaschke products can be thought of, in terms of their zeros, as conformal invariant Blaschke products.

Our main result is a characterization of the Blaschke products B which are such that $\tau_\alpha(B)$ is a finite product of interpolating Blaschke products for all $\alpha \in D$. Given a sector Q and $N > 0$, we denote by NQ the dilatation of Q with factor N, that is, the sector defined by the right-hand side of (1.1) with h replaced by Nh.

THEOREM. *Let B be a finite product of interpolating Blaschke products. Let $\{z_n\}$ be the sequence of zeros of B, δ_n the Dirac measure at z_n and $\mu = \sum (1 - |z_n|^2) \delta_n$.*
The following are equivalent.

(i) *For all $\alpha \in D$, the function $\tau_\alpha(B)$ is a finite product of interpolating Blaschke products.*

(ii) *For every $M > 0$, there exist positive numbers $\delta = \delta(M), \varepsilon = \varepsilon(M)$ such that if Q is a sector satisfying $l(Q) < \delta$ and $\mu(Q) > Ml(Q)$, then there exists another sector Q' with $\varepsilon Q \subset Q' \subset \varepsilon^{-1} Q$ such that*

$$\left| \frac{\mu(Q)}{l(Q)} - \frac{\mu(Q')}{l(Q')} \right| \geqslant \varepsilon.$$

Condition (ii) is, in some sense, opposite to Bishop's condition characterizing the Blaschke products in the little Bloch space B_0 (see [2]). Since Blaschke products in B_0 are very far away from being interpolating, this should be not surprising. Actually, in the proof of (ii) $\Rightarrow$ (i), we use some of Bishop's ideas.

We prove the Theorem in the next section. In Section 3, given a number m satisfying $0 < m < 1$, we construct an interpolating Blaschke product $B = B(m)$ such that $\tau_\alpha(B)$ is not a finite product of interpolating Blaschke products, for all $\alpha \in D$ with $|\alpha| \geqslant m$. So there is no analogue of Frostman's Theorem for the class of finite products of interpolating Blaschke products. We use this result in order to answer in the negative a question in [10] about the Nevanlinna–Pick interpolation problem. The last section contains some remarks.

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2. *Proof of the Theorem*

Given a sector $Q = \{z \in D: 1 - |z| \leqslant h, |\text{Arg}(z) - \theta| \leqslant h\}$, define $z_Q = (1-h) e^{i\theta}$. For $z \in D$ and $0 < \delta < 1$, let

$$D_H(z, \delta) = \left\{ w \in D: \rho(z, w) = \left| \frac{z - w}{1 - \bar{w} z} \right| < \delta \right\}$$

be the pseudohyperbolic disc of centre z and radius δ. The following result follows easily from [5, Lemmas 1 and 3].

LEMMA 1 [5]. *Let B be a Blaschke product with zeros $\{z_n\}$. Then B is a finite product of interpolating Blaschke products if and only if there exists a number m satisfying $0 < m < 1$ and a subsequence $\{c_n\}$ of $\{z_n\}$ such that the discs $D_H(c_n, m)$ are pairwise disjoint and*

$$\inf\{|B(z)|: z \notin \bigcup_n D_H(c_n, m)\} > 0.$$

Now let us go into the proof of the Theorem.

(i) ⇒ (ii) Let B be a finite product of interpolating Blaschke products and assume that (ii) fails. Then there exist $M > 0$, ε_j tending to zero, and sectors Q_j such that

$$\mu(Q_j) > Ml(Q_j), \quad l(Q_j) \underset{j\to\infty}{\longrightarrow} 0,$$

$$\sup\left\{\left|\frac{\mu(Q_j)}{l(Q_j)} - \frac{\mu(Q_j')}{l(Q_j')}\right| : \varepsilon_j^2 Q_j \subset Q_j' \subset \varepsilon_j^{-2} Q_j\right\} \underset{j\to\infty}{\longrightarrow} 0. \tag{2.1}$$

Consider $\alpha_j = z_{Q_j}$. Using the inequality $\log x^{-1} \geqslant 2^{-1}(1-x^2)$ for $0 < x < 1$, and the identity

$$1 - \left|\frac{z-w}{1-\bar{w}z}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\bar{w}|^2} \quad \text{for } z, w \in D,$$

one can get

$$\log|B(\alpha_j)|^{-1} \geqslant \frac{1}{2} \sum_{z_n \in Q_j} \frac{(1-|z_n|^2)(1-|\alpha_j|^2)}{|1-\bar{z}_n \alpha_j|^2}$$

$$\geqslant \frac{1}{8(1-|\alpha_j|)} \sum_{z_n \in Q_j} (1-|z_n|^2) = \frac{1}{8}\frac{\mu(Q_j)}{l(Q_j)} > \tfrac{1}{8}M.$$

Thus,

$$|B(\alpha_j)| \leqslant \exp(-\tfrac{1}{8}M) < 1.$$

We claim that it is sufficient to show that for each m with $0 < m < 1$, one has

$$\sup\{(1-|z|)|B'(z)| : z \in D_H(\alpha_j, m)\} \underset{j\to\infty}{\longrightarrow} 0. \tag{2.2}$$

Assume that (2.2) holds. Taking a subsequence if necessary, one can assume that

$$\lim_{j\to\infty} B(\alpha_j) = a \in D.$$

Now using (2.2), for each $0 < r < 1$ one has

$$\sup\left\{\left|\frac{B(z)-a}{1-\bar{a}B(z)}\right| : z \in D_H(\alpha_j, r)\right\} \underset{j\to\infty}{\longrightarrow} 0.$$

Applying Lemma 1, one gets that $\tau_a(B)$ is not a finite product of interpolating Blaschke products and this finishes the proof of (i) ⇒ (ii). Thus, it suffices to show that (2.2) holds.

We shall omit the index j, writing $Q = Q_j$, $\alpha = \alpha_j$, $N = [\varepsilon_j^{-1/2}]$ and $l(Q) \to 0$, $N \to \infty$ when $j \to \infty$. Consider the collection $\{Q^{(k)}: k = 1, \ldots, N^2\}$ of sectors with pairwise disjoint interiors lying inside NQ, with $l(Q^{(k)}) = N^{-1} l(Q)$ and $R = \bigcup_k Q^{(k)}$. If $l(Q)$ is sufficiently small, one has

$$\mu(NQ \backslash R) = 0. \tag{2.3}$$

Otherwise, there would exist $z_n \in NQ \backslash R$, and taking

$$T = \{z \in D\colon 1-|z| \leqslant 1-|z_n|, |\operatorname{Arg} z - \operatorname{Arg} z_n| < 1-|z_n|\}$$

and T_1, T_2 the disjoint sectors inside T with $l(T_i) = 2^{-1} l(T)$ for $i = 1, 2$, it would follow that

$$\frac{\mu(T)}{l(T)} - \frac{\mu(T_1)}{l(T_1)} \geqslant 1 + \frac{\mu(T_1)+\mu(T_2)}{l(T)} - \frac{\mu(T_1)}{l(T_1)} = 1 + \frac{1}{2}\left[\frac{\mu(T_2)}{l(T_2)} - \frac{\mu(T_1)}{l(T_1)}\right]$$

and this would contradict (2.1). So (2.3) holds.

Applying (2.1), one gets

$$\sup\left\{\left|\frac{\mu(Q^{(i)})}{l(Q^{(i)})} - \frac{\mu(Q^{(k)})}{l(Q^{(k)})}\right| : i, k = 1, \ldots, N^2\right\} \longrightarrow 0 \quad \text{as } l(Q) \longrightarrow 0. \tag{2.4}$$

Fix m such that $0 < m < 1$. One can check that $\rho(D_H(\alpha, m), D \backslash NQ) \to 1$ as $l(Q) \to 0$, $N \to \infty$. Also

$$\sup\left\{\frac{l(Q^{(k)})}{1-|z|} \colon z \in D_H(\alpha, m)\right\} \leqslant \frac{N^{-1} l(Q)}{2^{-1}(1-m)\, l(Q)} = 2N^{-1}(1-m)^{-1}. \tag{2.5}$$

Then (2.3) shows that

$$\rho(D_H(\alpha, m), \{z_n\}) \longrightarrow 1 \tag{2.6}$$

as $l(Q) \to 0, N \to \infty$. Applying Lemma 1, there exists a constant $C > 0$ such that

$$\inf\{|B(z)| \colon z \in D_H(\alpha, m)\} \geqslant C > 0. \tag{2.7}$$

Now, let us prove (2.2). Fixing $z \in D_H(\alpha, m)$, one has

$$(1-|z|)|B'(z)| \leqslant (1-|z|)\left|\frac{B'(z)}{B(z)}\right| = \left|\sum_n \frac{(1-|z|)(1-|z_n|^2)}{(z-z_n)(1-\bar{z}_n z)}\right| \leqslant A + B,$$

where

$$A = \left|\sum_{z_n \in NQ} \frac{(1-|z|)(1-|z_n|^2)}{(z-z_n)(1-\bar{z}_n z)}\right|, \quad B = \left|\sum_{z_n \in D\backslash NQ} \frac{(1-|z|)(1-|z_n|^2)}{(z-z_n)(1-\bar{z}_n z)}\right|.$$

Consider $\|\mu\|_c = \sup\{\mu(Q)\, l(Q)^{-1} \colon Q$ is a sector of the form (1.1)$\}$. Applying (2.6) and the fact that μ is a Carleson measure, one gets

$$B \leqslant 2 \sum_{z_n \notin NQ} \frac{(1-|z|)(1-|z_n|^2)}{|1-\bar{z}_n z|^2} \leqslant 2 \sum_{k=\log_2(N)}^{\infty} (1-|z|) \sum_{z_n \in 2^{k+1}Q \backslash 2^k Q} \frac{1-|z_n|^2}{|1-\bar{z}_n z|^2}$$

$$\leqslant 2 \sum_{k=\log_2(N)}^{\infty} \frac{\mu(2^{k+1}Q)}{2^{2k}(1-|z|)} \leqslant \frac{4\|\mu\|_c\, l(Q)}{N(1-|z|)} \leqslant \frac{8\|\mu\|_c}{N(1-m)} \longrightarrow 0 \tag{2.8}$$

as $l(Q) \to 0, N \to \infty$, because $1-|z| \geqslant 2^{-1}(1-m)\, l(Q)$.

On the other hand, (2.3) gives

$$A = \left|\sum_{z_n \in \bigcup Q^{(k)}} \frac{(1-|z|)(1-|z_n|^2)}{(z-z_n)(1-\bar{z}_n z)}\right|.$$

Take $\zeta_k = z_{Q^{(k)}}$. Given $z_n \in Q^{(k)}$, a computation and (2.5) show that

$$\left|\frac{1}{(z-z_n)(1-\bar{z}_n z)} - \frac{1}{(z-\zeta_k)(1-\bar{\zeta}_k z)}\right| \leqslant \frac{4N^{-1}l(Q)}{|1-\bar{z}_n z|^2(1-|z|)} \leqslant \frac{8N^{-1}}{|1-\bar{z}_n z|^2(1-m)}.$$

Then, applying (2.7),

$$\begin{aligned}
&\left|\sum_k \sum_{z_n \in Q^{(k)}} \left(\frac{1}{(z-z_n)(1-\bar{z}_n z)} - \frac{1}{(z-\zeta_k)(1-\bar{\zeta}_k z)}\right)(1-|z|)(1-|z_n|^2)\right| \\
&\quad \leqslant 8N^{-1}(1-m)^{-1} \sum_k \sum_{z_n \in Q^{(k)}} \frac{(1-|z|)(1-|z_n|^2)}{|1-\bar{z}_n z|^2} \\
&\quad \leqslant 16N^{-1}(1-m)^{-1} \log|B(z)|^{-1} \\
&\quad \leqslant 16N^{-1}(1-m)^{-1} \log(C^{-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
A &\leqslant 16N^{-1}(1-m)^{-1}\log(C^{-1}) + \left|\sum_k \sum_{z_n \in Q^{(k)}} \frac{(1-|z|)(1-|z_n|^2)}{(z-\zeta_k)(1-\bar{\zeta}_k z)}\right| \\
&= 16N^{-1}(1-m)^{-1}\log(C^{-1}) + \left|\sum_k \frac{(1-|z|)\,\mu(Q^{(k)})}{(z-\zeta_k)(1-\bar{\zeta}_k z)}\right|.
\end{aligned} \tag{2.9}$$

Applying (2.1) one gets

$$\sup\left\{\frac{|\mu(Q^{(k)})-\mu(Q^{(1)})|}{l(Q^{(1)})} : k = 1, \ldots, N^2\right\} \longrightarrow 0 \quad \text{as } l(Q) \longrightarrow 0, N \longrightarrow \infty.$$

Then, since $l(Q^{(1)}) \leqslant 2^{-1}|\zeta_k - \zeta_{k-1}|$ for $k = 1, \ldots, N^2$, one has

$$\begin{aligned}
&\left|\sum_k \frac{(1-|z|)(\mu(Q^{(k)})-\mu(Q^{(1)}))}{(z-\zeta_k)(1-\bar{\zeta}_k z)}\right| \\
&\quad \leqslant 2^{-1}\sup\left\{\frac{|\mu(Q^{(k)})-\mu(Q^{(1)})|}{l(Q^{(1)})} : k = 1, \ldots, N^2\right\} \sum_k \frac{(1-|z|^2)|\zeta_k - \zeta_{k-1}|}{|1-\bar{\zeta}_k z|^2} \\
&\quad \leqslant \pi \sup\left\{\left|\frac{\mu(Q^{(k)})-\mu(Q^{(1)})}{l(Q^{(1)})}\right| : k = 1, \ldots, N^2\right\} \longrightarrow 0 \quad \text{as } l(Q) \longrightarrow 0, N \longrightarrow \infty,
\end{aligned} \tag{2.10}$$

because the last sum is a Riemann sum of the Poisson kernel. Also

$$\left|\sum_k \frac{(1-|z|)\,\mu(Q^{(1)})}{(z-\zeta_k)(1-\bar{\zeta}_k z)}\right| = \frac{\mu(Q^{(1)})}{l(Q^{(1)})}\left|\sum_k \frac{l(Q^{(1)})(1-|z|)}{(z-\zeta_k)(1-\bar{\zeta}_k z)}\right|$$

is a Riemann sum of the integral

$$\frac{\mu(Q^{(1)})}{l(Q^{(1)})}\int_\Gamma \frac{(1-|z|)\,d\zeta}{(z-\zeta)(1-\bar{\zeta}z)\zeta},$$

where $\Gamma = \{\zeta \in D : |\zeta| = 1 - N^{-1}l(Q), \zeta \in NQ\}$. Since

$$\left|\int_{\{|\zeta|=1-N^{-1}l(Q)\}\setminus\Gamma} \frac{(1-|z|)}{(z-\zeta)(1-\bar{\zeta}z)}\frac{d\zeta}{\zeta}\right| \leqslant 2\int_{\{|\zeta|=1-N^{-1}l(Q)\}\setminus\Gamma} \frac{(1-|z|)}{|z-\zeta|^2}\,d|\zeta| \longrightarrow 0$$

as $l(Q) \to 0, N \to \infty$, and

$$\int_{\{|\zeta|=1-N^{-1}l(Q)\}} \frac{(1-|z|)\,d\zeta}{(z-\zeta)(1-\bar{\zeta}z)\zeta} = 0,$$

one gets

$$\left|\sum_k \frac{(1-|z|)\,\mu(Q^{(1)})}{(z-\zeta_k)(1-\bar{\zeta}_k z)}\right| \longrightarrow 0 \quad \text{as } l(Q) \longrightarrow 0, N \longrightarrow \infty. \tag{2.11}$$

Now, (2.9), (2.10) and (2.11) give that $A \to 0$ as $l(Q) \to 0, N \to \infty$. This shows that (2.2) holds and finishes the proof that (i) $\Rightarrow$ (ii). (Recently, in a private communication, K. Oyma showed me a different proof of this implication, where he studies $|B(z)|$ using harmonic measure techniques.)

(ii) $\Rightarrow$ (i) If (i) fails, there exists $\alpha \in D$ with $\alpha \neq 0$, such that $\tau_\alpha(B)$ is not a finite product of interpolating Blaschke products. By Lemma 1, there exist $\alpha_j \in D$ with $|\alpha_j| \to 1$, and m_j satisfying $0 < m_j \to 1$, such that

$$\sup\{|\log|B(\zeta)|^{-1} - \log|\alpha|^{-1}| : \zeta \in D_H(\alpha_j, m_j)\} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{2.12}$$

We shall show that for each $0 < t < 1$ one has

$$\sup\left\{\left|\pi\frac{\mu(Q_\zeta)}{l(Q_\zeta)} - \log|\alpha|^{-1}\right| : \zeta \in D_H(\alpha_j, t)\right\} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \tag{2.13}$$

where $Q_\zeta = \{z \in D : 1-|z| \leqslant 1-|\zeta|, |\operatorname{Arg} z - \operatorname{Arg}\zeta| \leqslant 1-|\zeta|\}$. Since (2.13) contradicts (ii), this will finish the proof of the theorem.

Fix t with $0 < t < 1$, $\zeta \in D_H(\alpha_j, t)$ and $Q = Q_\zeta$. Take s_j with $0 \leqslant s_j \to 1$ such that $(1-s_j)(1-m_j)^{-1} \to \infty$, and ε_j with $0 < \varepsilon_j \to 0$ such that $\varepsilon_j(1-s_j)^{-1} \to \infty$. Consider

$$R = R_j(\zeta) = \{z \in Q : 1-|z| \leqslant (1-s_j)(1-|\zeta|)\},$$
$$L = L_j(\zeta) = (1-\varepsilon_j)Q \cap \{z : 1-|z| = (1-s_j)(1-|\zeta|)\}.$$

From (2.12) it follows that B has no zeros in $D_H(\alpha_j, m_j)$. The choice of the constants and a computation with the pseudohyperbolic distance, gives that for j sufficiently large, $Q \backslash R \subset D_H(\alpha_j, m_j)$. Thus

$$\mu(Q\backslash R) = 0, \tag{2.14}$$

$$\inf\{\rho(z, \{z_n\}) : z \in L = L_j(\zeta)\} \longrightarrow 1 \quad \text{as } j \longrightarrow \infty. \tag{2.15}$$

Now using (2.15) and the facts that $\varepsilon_j(1-s_j)^{-1} \to \infty$ and μ is a Carleson measure, one can see, as in (2.8), that

$$\inf\{|B_{D\backslash R}(z)| : z \in L\} \longrightarrow 1 \quad \text{as } j \longrightarrow \infty,$$

where $B_{D\backslash R}$ is the Blaschke product with zeros $\{z_n : z_n \in D\backslash R\}$. Then, using (2.15) and the fact that $(1-x^2)^{-1}\log x^{-2} \to 1$ as $x \to 1$, one gets

$$\sup\left\{\left|2\log|B(z)|^{-1} - \int_R P_z(w)\,d\mu(w)\right| : z \in L\right\} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty,$$

where $P_z(w) = (1-|z|^2)|1-\bar{w}z|^{-2}$. Since $L \subset D_H(\alpha_j, m_j)$ for j sufficiently large, (2.12) shows that

$$\sup\left\{\left|2\log|\alpha|^{-1} - \int_R P_z(w)\,d\mu(w)\right| : z \in L\right\} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.$$

Now, parametrizing L by $z = re^{i\theta}$, where $1-r = (1-s_j)(1-|\zeta|)$, and integrating, one gets

$$\left|2|L|\log|\alpha|^{-1} - \int_R \int_L P_{re^{i\theta}}(w)\, d\theta\, d\mu(w)\right| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \tag{2.16}$$

where $|L|$ is the Euclidian length of L. If w is a zero of B satisfying $w \in (1-\varepsilon_j^{1/2})\,Q$, one can check that

$$\int_L P_{re^{i\theta}}(w)\, d\theta = \int_L \frac{1-r^2}{|re^{i\theta} - w|^2}\, d\theta \longrightarrow 2\pi \quad \text{as } r \longrightarrow 1, \tag{2.17}$$

because (2.14) and (2.15) show that the zeros of B in Q are much closer to the circle than the points of L are. Also, since μ is a Carleson measure,

$$\left|\int_{R\setminus(1-\varepsilon_j^{1/2})\,Q} \int_L P_{re^{i\theta}}(w)\, d\theta\, d\mu(w)\right| \leqslant 2\pi\mu(R\setminus(1-\varepsilon_j^{1/2})\,Q) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{2.18}$$

Now, introducing (2.17) and (2.18) in (2.16), one gets

$$|\,2|L|\log|\alpha|^{-1} - 2\pi\mu(R)\,| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.$$

By (2.14), $\mu(R) = \mu(Q)$ and $|L| = (1-\varepsilon_j)\, l(Q)$. Therefore

$$\left|\log|\alpha|^{-1} - \pi\frac{\mu(Q)}{l(Q)}\right| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.$$

This proves (2.13) and finishes the proof of the theorem.

3. *An example*

Let B be a Blaschke product. It follows from Lemma 1, that the set

$$\{\alpha \in D \colon \tau_\alpha(B) \text{ is not a finite product of interpolating Blaschke products}\}$$

is closed. Let us remark that the exceptional set appearing in Frostman's Theorem is not, in general, closed. In fact, it can even be dense on the unit disc (see [**8**, p. 714]).

In this section we shall show that there is no analogue of Frostman's Theorem for the finite products of interpolating Blaschke products.

PROPOSITION. *For each m with $0 < m < 1$, there exists an interpolating Blaschke product $B = B_m$, such that $\tau_\alpha(B)$ is not a finite product of interpolating Blaschke products, for all $\alpha \in D$ and $|\alpha| \geqslant m$.*

For the proof of the Proposition, we need the following results. Let $f \in H^\infty(D)$ and $e^{i\theta} \in \partial D$, then the radial cluster set of f at $e^{i\theta}$ is the set of complex numbers w such that there exists $r_k e^{i\theta}$ with $1 > r_k \to 1$, such that $f(r_k e^{i\theta}) \to w$.

LEMMA 2. *There exists an interpolating Blaschke product whose radial cluster set at the point* 1 *contains the unit circle.*

LEMMA 3. *Let $f \in H^\infty(D)$, $z_k \in D$ and t_k satisfying $0 < t_k < 1$, $t_k \to 1$ be such that*

$$\sup\{|\,|f(z)| - \alpha\,| \colon z \in D_H(z_k, t_k)\} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Then there exist l_k with $0 < l_k < 1, l_k \longrightarrow 1$, such that

$$\sup\{|f(z) - f(z_k)| \colon z \in D_H(z_k, l_k)\} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Proof of Lemma 2. Let $\{e^{i\theta_k}\}$ be a dense sequence in the unit circle. We shall choose the zeros $\{z_n\}$ of the Blaschke product by induction and we shall denote by B_N the Blaschke product with zeros $z_1, \ldots, z_N$.

Choose r_1 and z_1 with $0 < r_1 < 1$ and $\rho(z_1, r_1) = 2^{-1}$ such that

$$|B_1(r_1) - e^{i\theta_1}| = 2^{-1}.$$

Assume that we have defined $r_1, \ldots, r_k$ with $1 - r_{j+1}^2 \leqslant 2^{-3j}(1 - r_j^2)$ and $z_1, \ldots, z_k$ with $\rho(z_j, r_j) = 1 - 2^{-j}$ for $j = 1, \ldots, k$, such that

$$|B_k(r_k) - e^{i\theta_k}| \leqslant 2^{-k+1}.$$

Then, choose $r_{k+1} < 1$ with $1 - r_{k+1}^2 < 2^{-3k}(1 - r_k^2)$ such that

$$1 - |B_k(r_{k+1})| \leqslant 2^{-k-1}$$

and z_{k+1} with $\rho(z_{k+1}, r_{k+1}) = 1 - 2^{-k-1}$ such that

$$\left| \frac{-\bar{z}_{k+1}}{|z_{k+1}|} \frac{r_{k+1} - z_{k+1}}{1 - \bar{z}_{k+1} r_{k+1}} - e^{i\theta_{k+1}} \frac{\overline{B_k(r_{k+1})}}{|B_k(r_{k+1})|} \right| = 2^{-k-1}.$$

Now,

$$|B_{k+1}(r_{k+1}) - e^{i\theta_{k+1}}| \leqslant 1 - |B_k(r_{k+1})| + \left| \frac{-\bar{z}_{k+1}}{z_{k+1}} \frac{r_{k+1} - z_{k+1}}{1 - \bar{z}_{k+1} r_{k+1}} - e^{i\theta_{k+1}} \frac{\overline{B_k(r_{k+1})}}{|B_k(r_{k+1})|} \right| \leqslant 2^{-k}.$$

Since $\rho(z_i, r_i) = 1 - 2^{-i}$ and $1 - r_{k+1}^2 \leqslant 2^{-3k}(1 - r_k^2)$, using the inequality

$$\frac{|z| - |w|}{1 - |w|\,|z|} \leqslant \rho(z, w) \leqslant \frac{|z| + |w|}{1 + |w|\,|z|} \quad \text{for } z, w \in D$$

(see [4, p. 4]), one gets

$$\begin{aligned} 1 - |z_{k+1}|^2 &\leqslant 1 - \left(\frac{r_{k+1} - (1 - 2^{-k-1})}{1 - (1 - 2^{-k-1}) r_{k+1}} \right)^2 = \frac{(1 - r_{k+1}^2)\, 2^{2(-k-1)}}{(1 - (1 - 2^{-k-1}) r_{k+1})^2} \\ &\leqslant 2^{-3k} \frac{(1 - r_k^2)\, 2^{-2k}}{2^{-2k}} = 2^{-k}(1 - r_k^2)\, 2^{-2k} \leqslant 2^{-k+2} \left(1 - \left(\frac{r_k + (1 - 2^{-k})}{1 + (1 - 2^{-k}) r_k} \right)^2 \right) \\ &\leqslant 2^{-k+2}(1 - |z_k|^2). \end{aligned}$$

This shows that $\{z_n\}$ is an interpolating sequence. Let B be the Blaschke product with zeros $\{z_n\}$. One has

$$\begin{aligned} |B(r_k) - e^{i\theta_k}| &\leqslant |B_k(r_k) - e^{i\theta_k}| + \left| \frac{B(r_k)}{B_k(r_k)} - 1 \right| \leqslant 2^{-k+1} + \sum_{j=k+1}^{\infty} \left| \frac{-\bar{z}_j}{|z_j|} \frac{r_k - z_j}{1 - \bar{z}_j r_k} - 1 \right| \\ &\leqslant 2^{-k+1} + 2 \sum_{j=k+1}^{\infty} \frac{1 - |z_j|}{1 - r_k} \leqslant 2^{-k+1} + 4 \frac{1 - |z_{k+1}|}{1 - r_k} \leqslant 2^{-k+1} + 8 \cdot 2^{-3k}. \end{aligned}$$

Since $\{e^{i\theta_k}\}$ is dense in the unit circle, the radial cluster set of B at the point 1 contains the unit circle.

Proof of Lemma 3. Assume that the conclusion fails. Then taking a subsequence if necessary, there exist l with $0<l<1$ and points $c_k \in D_H(z_k, l)$ such that $|f(c_k)-f(z_k)| \geqslant \delta > 0$. Since

$$\delta \leqslant |f(c_k)-f(z_k)| \leqslant \int_{z_k}^{c_k} |f'(w)|\, d|w|,$$

there exist $\zeta_k \in D_H(z_k, l)$ and a constant $c = c(l) > 0$, such that

$$(1-|\zeta_k|)|f'(\zeta_k)| \geqslant c\delta.$$

Applying Bloch's Theorem [3, p. 295] to the function

$$\frac{f(\tau_{\zeta_k}(z)) - f(\zeta_k)}{(1-|\zeta_k|^2) f'(\zeta_k)} \quad \text{for } z \in D,$$

one gets

$$f(D_H(\zeta_k, 2^{-1})) \supset D(f(\zeta_k), (72)^{-1} 2^{-1}(1-|\zeta_k|^2)|f'(\zeta_k)|) \supset D(f(\zeta_k), 144^{-1} c\delta)$$

and this contradicts the hypothesis of Lemma 3.

Proof of the Proposition. Let B_1 be an interpolating Blaschke product satisfying the conditions of Lemma 2. Choose r_k with $0 < r_k < 1, r_k \to 1$, such that

$$\{B_1(r_k) : k = 1, 2, \ldots\}$$

is dense on the unit circle and

$$\frac{1-r_{k+1}}{1-r_k} \leqslant 2^{-2k} \quad \text{for } k = 1, 2, \ldots . \tag{3.1}$$

Take α_k with $m < \alpha_k < 1 - k^{-1}$, such that $\{\alpha_k B_1(r_k) : k = 1, 2, \ldots\}$ is dense in

$$\{z : m < |z| < 1\}.$$

We shall construct an interpolating Blaschke product B_2 such that $B_1 B_2$ is interpolating and for all $k = 1, 2, \ldots$, the function

$$\frac{B_1 B_2(z) - \alpha_k B_1(r_k)}{1 - \alpha_k \overline{B_1(r_k)}\, B_1 B_2(z)} \quad \text{for } z \in D$$

is not a finite product of interpolating Blaschke products. Then, the observation of the beginning of this section will give the proof of the Proposition.

Consider $Q_k = \{z \in D : 1 - |z| \leqslant 1 - r_k, |\operatorname{Arg} z| \leqslant 1 - r_k\}$, then

$$k\bar{Q}_k \cap \{|z| = 1\} = \{e^{i\theta} : -a_k \leqslant \theta \leqslant a_k\},$$

where $a_k = k(1-r_k)$. Define t_k by $1 - t_k = k^{-1}(1-r_k)$ and s_k with $0 < s_k < \pi$ by $|\exp(is_k) - 1| = 2\pi(1-t_k)(\log|\alpha_k|^{-1})^{-1}$, and put

$$z_n^{(k)} = t_k \exp(is_k n) \quad \text{for } n = -[a_k s_k^{-1}], \ldots, 0, \ldots, [a_k s_k^{-1}].$$

Thus,

$$|z_n^{(k)} - z_{n+1}^{(k)}| = t_k 2\pi(1-t_k)(\log|\alpha_k|^{-1})^{-1} = t_k(1-|z_n^{(k)}|)\, 2\pi(\log|\alpha_k|^{-1})^{-1}. \tag{3.2}$$

If Q is a sector in the unit disc, one has

$$\sum_{n : z_n^{(k)} \in Q} (1 - |z_n^{(k)}|) \leqslant 2(2\pi)^{-1} \log|\alpha_k|^{-1} l(Q) \leqslant 2(2\pi)^{-1} \log m^{-1} l(Q). \tag{3.3}$$

Let I_k be the Blaschke product with zeros $\{z_n^{(k)}: n = -[a_k s_k^{-1}], \ldots, [a_k s_k^{-1}]\}$. Let $Z(B)$ denote the zero set of the Blaschke product B. One can choose r_k in such a way that $\rho(Z(I_k), Z(B_1)) \geqslant 2^{-1}$.

Let $a_k \sim b_k$ mean that $|a_k - b_k| \to 0$, as $k \to \infty$. Using (3.2), for each $M < 1$, one has

$$\sup\{|\log|I_k(z)|^{-1} - \log|\alpha_k|^{-1}|: z \in D_H(r_k, M)\}$$

$$\sim \sup\left\{\left|\frac{1}{2}\sum_n \frac{(1-|z_n^{(k)}|^2)(1-|z|^2)}{|1-\bar{z}_n^{(k)} z|^2} - \log|\alpha_k|^{-1}\right|: z \in D_H(r_k, M)\right\}$$

$$\sim \sup\left\{\left|(2\pi)^{-1}(\log|\alpha_k|^{-1})\sum_n \frac{|z_{n+1}^{(k)} - z_n^{(k)}|(1-|z|^2)}{|1-\bar{z}_n^{(k)} z|^2} - \log|\alpha_k|^{-1}\right|: z \in D_H(r_k, M)\right\} \longrightarrow 0 \tag{3.4}$$

as $k \to \infty$, because the last sum is a Riemann sum of the integral of the Poisson kernel at the point z along the arc $\{t_k e^{i\theta}: -a_k \leqslant \theta \leqslant a_k\}$. Since the points $\{z_n^{(k)}\}$ are symmetric with respect to the real axis, one has $I_k(r_k) > 0$. So (3.4) and Lemma 3 give that

$$\sup\{|I_k(z) - \alpha_k|: z \in D_H(r_k, M)\} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \tag{3.5}$$

for each M with $0 < M < 1$. Consider $B_2 = \prod_k I_k$. Using (3.1) and (3.3) one can easily show that B_2 is an interpolating Blaschke product. Since $\rho(Z(B_2), Z(B_1)) \geqslant 2^{-1}$, it follows that $B_1 B_2$ is an interpolating Blaschke product. Also, using (3.1) and the symmetry of $\{z_n^{(k)}\}$, one can check that

$$\prod_{j \neq k} I_j(r_k) \longrightarrow 1 \quad \text{as } k \longrightarrow \infty.$$

So, from (3.5) and Schwarz's Lemma, it follows that

$$\sup\{|B_2(z) - \alpha_k|: z \in D_H(r_k, M)\} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \tag{3.6}$$

for each M with $0 < M < 1$. Since $|B_1(r_k)| \to 1$ as $k \to \infty$, another application of Schwarz's Lemma gives that

$$\sup\{|B_1(z) - B_1(r_k)|: z \in D_H(r_k, M)\} \longrightarrow 0 \quad \text{as } K \longrightarrow \infty \tag{3.7}$$

for each $0 < M < 1$. Now for fixed $a = \alpha_k B_1(r_k)$, (3.6) and (3.7) imply that there exists a subsequence $\{p_k\}$ of $\{r_k\}$ such that

$$\sup\left\{\left|\frac{B_1 B_2(z) - a}{1 - \bar{a} B_1 B_2(z)}\right|: z \in D_H(p_k, M)\right\} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty$$

for each M with $0 < M < 1$, and Lemma 1 shows that the function

$$\frac{B_1 B_2(z) - a}{1 - \bar{a} B_1 B_2(z)}$$

is not a finite product of interpolating Blaschke products. This finishes the proof of the Proposition.

Now, we use the Proposition in order to answer in the negative a question in [**10**, p. 515]. First, we recall some results.

Give two sequences of points $\{z_n\}$, $\{w_n\}$ in D, the Nevanlinna–Pick interpolation problem consists in finding analytic functions $f \in H^\infty$ satisfying

$$\|f\|_\infty = \sup\{|f(z)|: z \in D\} \leqslant 1 \quad \text{and} \quad f(z_n) = w_n \text{ for } n = 1, 2, \ldots.$$

We shall denote it by

(*) Find $f \in H^\infty$, $\|f\|_\infty \leqslant 1, f(z_n) = w_n, n = 1, 2, \dots$.

Pick and Nevanlinna found necessary and sufficient conditions in order that the problem (*) has a solution. Let G be the set of all solutions of the problem (*). Nevanlinna showed that if G consists of more than one element, there is a parametrization of the form

$$G = \left\{ f \in H^\infty : f = \frac{p\phi + q}{r\phi + s}, \phi \in H^\infty, \|\phi\|_\infty \leqslant 1 \right\},$$

where p, q, r, s are certain analytic functions in D, depending on $\{z_n\}, \{w_n\}$ and satisfying $ps - qr = B$, the Blaschke product with zeros $\{z_n\}$.

Later, Nevanlinna showed that for each unimodular constant $e^{i\theta}$, the function

$$I_\theta = \frac{pe^{i\theta} + q}{re^{i\theta} + s}$$

is inner. Therefore, if the problem (*) has more than one solution, then there are inner functions solving it. See [**4**, pp. 6, 165] for the proofs of these results.

Recently, A. Stray [**9**] has proved that, in fact, for all unimodular constants $e^{i\theta}$ except possibly for a set of zero logarithmic capacity, the function I_θ is a Blaschke product. Also [**10**, Theorem 3], if $\{z_n\}$ is an interpolating sequence, then there exists a number $r > 0$ depending only on $\{z_n\}$, such that if

$$\inf\{\|f\|_\infty : f \in G\} \leqslant r,$$

then the function I_θ is a finite product of interpolating Blaschke products for all unimodular constants $e^{i\theta}$.

In [**10**, p. 515], the question is asked if the same result is valid with some numerical constant r independent of $\{z_n\}$. We now answer this question in the negative.

For each m with $0 < m < 1$, let $B = B_m$ be the interpolating Blaschke product given by the Proposition. Let $\{z_n\}$ be the sequence of zeros of B. Now, choose $\alpha = \alpha_m \in D$ with $|\alpha| = m$, and consider the following Nevanlinna–Pick problem.

$(*)_m$ Find $f \in H^\infty$, $\|f\|_\infty \leqslant 1, f(z_n) = -\alpha$, $n = 1, 2, \dots$.

Let G_m be the set of all solutions of $(*)_m$. It is clear that

$$G_m = \left\{ \frac{B\phi - \alpha}{1 - \bar{\alpha} B\phi} : \phi \in H^\infty, \|\phi\|_\infty \leqslant 1 \right\},$$

$$\inf\{\|f\|_\infty : f \in G_m\} = m.$$

Now the Proposition gives that the function

$$I_\theta = \frac{Be^{i\theta} - \alpha}{1 - \bar{\alpha} Be^{i\theta}}$$

is not a finite product of interpolating Blaschke products, for all $\theta \in [0, 2\pi]$. Since one can choose m with $0 < m < 1$ to be arbitrarily small, this shows that the constant r cannot be chosen independently of $\{z_n\}$.

4. *Remarks*

Let I be an inner function. A computation shows that I is a finite product of interpolating Blaschke products if and only if there exists r with $0 < r < 1$ such that

$$\sup_{|w|<1} \int_0^{2\pi} \log \left| I\left(\frac{re^{i\theta}+w}{1+\bar{w}re^{i\theta}}\right)\right|^{-1} d\theta < +\infty.$$

This could be understood as the conformal invariant version of Frostman's condition characterizing Blaschke products among inner functions (see [**4**, p. 56]).

Using the techniques of [**1**], one can show that conditions (i) or (ii) in the Theorem are also equivalent to any of the following.

(iii) For each m with $0 < m < 1$, there exists r with $0 < r < 1$ such that

$$\inf\left\{\int_{D_H(z,r)} |B'(w)|^2\, dm(w) : |B(z)| \leqslant m\right\} > 0.$$

(iv) For each m with $0 < m < 1$, there exists r with $0 < r < 1$ such that

$$\inf\{\text{diameter}\,(B(D_H(z,r))) : |B(z)| \leqslant m\} > 0.$$

(v) For each m with $0 < m < 1$, there exists r with $0 < r < 1$ such that

$$\inf\left\{\frac{1}{(1-|z|)^2}\int_{D_H(z,r)} |B(w)-B(z)|^2\, dm(w) : |B(z)| \leqslant m\right\} > 0.$$

H. Morse [**7**] constructed a destructible Blaschke product which becomes indestructible when a single point is deleted from its zero-set. So no asymptotic condition on the measure μ can characterize indestructible Blaschke products.

References

1. S. Axler, 'The Bergman space, the Bloch space and commutators of multiplication operators', *Duke Math. J.* 53 (1986) 315–332.
2. C. J. Bishop, 'Bounded functions in the little Bloch space', *Pacific J. Math.* 142 (1990) 209–225.
3. J. B. Conway, *Functions of one complex variable*, Graduate Texts in Mathematics 11 (2nd edition) (Springer, New York–Heidelberg, 1978).
4. J. B. Garnett, *Bounded analytic functions* (Academic Press, New York, 1981).
5. A. Kerr-Lawson, 'Some lemmas on interpolating Blaschke products and a correction', *Canad. J. Math.* 21 (1969) 531–534.
6. G. McDonald and C. Sundberg, 'Toeplitz operators on the disc', *Indiana Univ. Math. J.* 28 (1979) 595–611.
7. H. S. Morse, 'Destructible and indestructible Blaschke products', *Trans. Amer. Math. Soc.* 257 (1980) 547–253.
8. K. Stephenson, 'Construction of an inner function in the little Bloch space', *Trans. Amer. Math. Soc.* 308 (1988) 713–720.
9. A. Stray, 'Minimal interpolation by Blaschke products, II', *Bull. London Math. Soc.* 20 (1988) 329–333.
10. A. Stray, 'Interpolating sequences and the Nevanlinna–Pick problem', *Publ. Mat.* 35 (1991) 507–516.

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