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Inner functions in weak Besov spaces ☆

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Abstract

It is shown that inner functions in weak Besov spaces are precisely the exponential Blaschke products.

Keywords: Blaschke product; Hardy space; Inner function; Weak Lebesgue space

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of holomorphic functions in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. Recall that $I \in \mathcal{H}(\mathbb{D})$ is called inner provided that I is bounded in $\mathbb{D}$, and it satisfies $|I(e^{i\theta})| = 1$ for almost every (a.e.) $e^{i\theta} \in \partial\mathbb{D}$, where $I(e^{i\theta}) = \lim_{r\to 1^-} I(re^{i\theta})$. Every inner function I can be represented as a product $I = \xi BS$, where $|\xi| = 1$, B is a Blaschke product, and S is a singular inner function; see [19, p. 75]. It is a classical problem to classify those inner functions whose derivative belongs to a pre-given function space.

There is an extensive literature on inner functions whose derivative belongs to certain Hardy or Bergman spaces. In the seventies Ahern, Clark and other authors proved many seminal results [1,3,4,7,10,33], which have been extended and complemented later in various directions, see for instance [2,5,6,14–17,22,21,24,25,27,32,31,34,35,37]. The following discussion focuses on

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a couple of those results. The first one is [25, Theorem 1.1], see also [11, Theorem 3.1], stating that the only inner functions in any Besov space

$$\mathcal{B}^p = \left\{ f \in \mathcal{H}(\mathbb{D}) \colon \|f\|_{B^p} = \left(\int_{\mathbb{D}} \left| f'(z) \right|^p d\mu_p(z) \right)^{1/p} < \infty \right\}, \quad 1 < p < \infty,$$

are the finite Blaschke products. Here $d\mu_p(z) = (1-|z|)^{p-2}\, dA(z)$, and $dA(z)$ is the element of Lebesgue area measure in $\mathbb{D}$.

A Blaschke product B is said to be exponential if there exists a non-negative integer $M = M(B)$ such that each annulus

$$\mathcal{A}_j = \left\{ z \in \mathbb{D} \colon 2^{-j} < 1 - |z| \leqslant 2^{-j+1} \right\}, \quad j \in \mathbb{N}, \tag{1.1}$$

contains at most M zeros of B. If B is an exponential Blaschke product, then the zeros of B can be divided into a finite number of exponential sequences, that is, sequences $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$ with

$$\sup_{j\in\mathbb{N}} \frac{1-|z_{k+1}|}{1-|z_k|} < 1.$$

Conversely, any Blaschke product with zeros that can be divided into a finite number of exponential sequences, is exponential. In conclusion, a Blaschke product is exponential if and only if its zeros satisfy the weak Newman condition, see [29, p. 160]. Exponential Blaschke products can be described in terms of the growth of the integral means of their fractional derivatives [5, Theorem 3.1], see also [24,37]. Moreover, by [3, Theorem 3] and [8, Theorem 1] exponential Blaschke products are the only inner functions whose derivative belongs to the weak Hardy space H^1_w, which consists of those functions $f \in \mathcal{H}(\mathbb{D})$ for which there exists a constant $C = C(f)$ with $0 < C < \infty$ such that

$$\left| \left\{ e^{i\theta} \in \partial\mathbb{D} \colon \left| f\left(re^{i\theta}\right) \right| > \lambda \right\} \right| \leqslant C/\lambda, \quad 0 < \lambda < \infty,\ 0 \leqslant r < 1.$$

Here $|E|$ denotes the Euclidean length of the one-dimensional set E. The present paper extends [8, Theorem 1] by showing that the inner functions in weak Besov spaces are precisely the exponential Blaschke products.

1.1. Notation

We briefly recall the familiar concepts of L^p spaces of arbitrary measure spaces, and then proceed to consider certain specific spaces of analytic functions in $\mathbb{D}$. Let X be a measure space, and let μ be a positive measure on X. For $0 < p < \infty$, $L^p(X,\mu)$ denotes the space of all complex-valued μ-measurable functions on X whose modulus to the pth power is integrable. The shorter notation $L^p(X)$ is reserved for the space $L^p(X,\mu)$ provided that μ is the Lebesgue measure restricted to X. Correspondingly, $L^p_w(X,\mu)$ for $0 < p < \infty$ is the weak $L^p(X,\mu)$ space, which contains those complex-valued μ-measurable functions f for which the quasi-norm

$$\|f\|_{L^p_w(X,\mu)} = \sup_{0<\lambda<\infty} \lambda \left[\mu\left(\left\{ x \in X \colon \left| f(x) \right| > \lambda \right\} \right) \right]^{1/p}$$

is finite. Again, we write $L^p_w(X)$ instead of $L^p_w(X,\mu)$ in the case that μ is the Lebesgue measure restricted to X. For $0<p<\infty$, we have $L^p(X,\mu) \subsetneq L^p_w(X,\mu)$. If $0<r<p<\infty$, then the Kolmogorov condition [18, p. 485, Lemma 2.8] states that $f \in L^p_w(X,\mu)$ if and only if there exists a constant $C = C(p,r,f)$ with $0<C<\infty$ such that

$$\left(\int_E |f|^r \, d\mu\right)^{\frac{1}{r}} \leqslant C\mu(E)^{\frac{1}{r}-\frac{1}{p}}$$

for all μ-measurable subsets E of X. For more information on L^p spaces of arbitrary measure spaces, we refer to [18,23].

Instead of working with arbitrary measure spaces, we concentrate on holomorphic functions in $\mathbb{D}$. As usual, the Hardy space H^p for $0<p<\infty$ is defined as

$$H^p = \left\{ f \in \mathcal{H}(\mathbb{D}) \colon \|f\|_{H^p} = \sup_{0\leqslant r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p \, d\theta\right)^{1/p} < \infty \right\}.$$

The non-tangential maximal function of f is given by

$$M_{\triangleleft} f\left(e^{i\theta}\right) = \sup_{z\in\Gamma_\alpha(e^{i\theta})} \left|f(z)\right|, \quad e^{i\theta} \in \partial\mathbb{D},$$

for fixed $1<\alpha<\infty$, where the non-tangential region (Stolz angle)

$$\Gamma_\alpha\left(e^{i\theta}\right) = \left\{z \in \mathbb{D} \colon \left|z - e^{i\theta}\right| \leqslant \alpha\left(1-|z|\right)\right\}, \quad e^{i\theta} \in \partial\mathbb{D}, \tag{1.2}$$

has a vertex at $e^{i\theta} \in \partial\mathbb{D}$ with aperture equal to $2\arctan\sqrt{\alpha^2-1}$. A fundamental result by Hardy and Littlewood (for $1<p<\infty$) and by Burkholder, Gundy and Silverstein (for $0<p\leqslant 1$) states that Hardy spaces can be characterized by means of the non-tangential maximal function. That is, if $f \in \mathcal{H}(\mathbb{D})$ and $0<p<\infty$, then $f \in H^p$ if and only if $M_{\triangleleft} f \in L^p(\partial\mathbb{D})$. See [19, p. 57, Theorem 3.1].

For $0<p<\infty$, the weak Hardy space H^p_w is defined as

$$H^p_w = \left\{ f \in \mathcal{H}(\mathbb{D}) \colon \|f\|_{H^p_w} = \sup_{0\leqslant r<1} \|f_r\|_{L^p_w(\partial\mathbb{D})} < \infty \right\}, \quad f_r(z) = f(rz).$$

The weak Hardy spaces can be also characterized in terms of the non-tangential maximal function; if $f \in \mathcal{H}(\mathbb{D})$ and $0<p<\infty$, then $f \in H^p_w$ if and only if $M_{\triangleleft} f \in L^p_w(\partial\mathbb{D})$. We refer to [9, p. 36] for further discussion.

1.2. Weak Bergman spaces

Define

$$\mathcal{L}^p_w = \mathcal{H}(\mathbb{D}) \cap L^p_w(\mathbb{D},\mu_p), \quad 1<p<\infty.$$

The definition above does not extend naturally to the case $p = 1$, since $\mathcal{L}_w^1$ does not contain non-zero constant functions. This is a consequence of the fact that $\mu_1(\mathbb{D}) = \infty$. We write $f \in \widetilde{\mathcal{L}}_w^1$ provided that $f \in \mathcal{H}(\mathbb{D})$ and $(1-|z|)^{-1}|f(z)| \in L_w^1(\mathbb{D}, A)$.

The nesting property of the spaces $\mathcal{L}_w^p$, which is given by Proposition 1(ii), relies on a certain point-wise growth estimate. If $0 \leqslant p < \infty$, then we define the growth spaces A^{-p}, and A_0^{-p}, to consist of functions $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\|f\|_{A^{-p}} = \sup_{z\in\mathbb{D}} (1-|z|)^p |f(z)| < \infty, \quad \text{and} \quad \lim_{|z|\to 1^-} (1-|z|)^p |f(z)| = 0,$$

respectively; see [26].

Proposition 1. *We have*

(i) $H^1 \subset \widetilde{\mathcal{L}}_w^1 \subset H_w^1 \subset \mathcal{L}_w^p \subset A^{-1}$ *for all* $1 < p < \infty$;
(ii) $\mathcal{L}_w^p \subset \mathcal{L}_w^q$ *for all* $1 < p < q < \infty$.

It is worth mentioning that $\mathcal{L}_w^p$ is strictly larger than H_w^1 for any $1 < p < \infty$. For instance, since $L^p(\mathbb{D}, \mu_p)$ contains the Bloch space, $\mathcal{L}_w^p$ contains functions in $\mathcal{H}(\mathbb{D})$ having finite radial limit at no point of the unit circle for any $1 < p < \infty$. In regard to Proposition 1(i), we point out that $H^1 \subset A_0^{-1}$ [12, Theorem 5.9].

1.3. Derivatives of inner functions

The weak Besov spaces $\mathcal{B}_w^p$ are defined as

$$\mathcal{B}_w^p = \{ f \in \mathcal{H}(\mathbb{D}) \colon\ f' \in L_w^p(\mathbb{D}, \mu_p) \}, \quad 1 < p < \infty.$$

Our main result characterizes inner functions in weak Besov spaces. This should be compared with the result mentioned in the Introduction, according to which the only inner functions in the corresponding Besov spaces are the finite Blaschke products.

Theorem 2. *Let B be an inner function, and let $1 < p < \infty$. Then, $B \in \mathcal{B}_w^p$ if and only if B is an exponential Blaschke product.*

In view of [8, Theorem 1] and Proposition 1(i), our contribution in Theorem 2 is to prove that, if B is inner and $B' \in \mathcal{L}_w^p$ for some $1 < p < \infty$, then B is an exponential Blaschke product. The following result characterizes inner functions whose derivative belongs to $\widetilde{\mathcal{L}}_w^1$, which is a space intermediate to H^1 and H_w^1 by Proposition 1(i).

Theorem 3. *Let B be an inner function. Then, $B' \in \widetilde{\mathcal{L}}_w^1$ if and only if B is a finite Blaschke product.*

The subsequent sections are devoted to proving these results.

2. Proof of Proposition 1

2.1. Proof of Proposition 1(i)

We proceed to prove each inclusion separately.

Proof of $H^1 \subset \widetilde{\mathcal{L}}^1_w$. Let $f \in H^1$. This implies that $M_\triangleleft f \in L^1(\partial\mathbb{D})$, and hence by Chebyshev's inequality

$$\lambda\left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > \lambda\right\}\right| \leqslant \|M_\triangleleft f\|_{L^1(\partial\mathbb{D})} < \infty, \quad 0 < \lambda < \infty. \tag{2.1}$$

We proceed to show that $f \in \widetilde{\mathcal{L}}^1_w$. Suppose that $2 < \lambda < \infty$ is given. Then,

$$\begin{aligned}
&\mathrm{Area}\left\{z \in \mathbb{D}\colon\ \left|f(z)\right| > \lambda\left(1-|z|\right)\right\} \\
&\quad = \int_0^{1-\lambda^{-1}} \left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ \left|f\left(re^{i\theta}\right)\right| > \lambda(1-r)\right\}\right| r\,dr \qquad (2.2)\\
&\qquad + \int_{1-\lambda^{-1}}^{1} \left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ \left|f\left(re^{i\theta}\right)\right| > \lambda(1-r)\right\}\right| r\,dr. \qquad (2.3)
\end{aligned}$$

Let I_1 and I_2 be the integrals in (2.2) and (2.3), respectively. To compute I_1, let $K = K(\lambda)$ be the largest natural number such that $K \leqslant \log_2 \lambda$. This implies that $1 - 2^K\lambda^{-1} < 1/2$, and hence by (2.1) we deduce

$$\begin{aligned}
I_1 &\leqslant \int_0^{1/2} \left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > \lambda/2\right\}\right| dr \\
&\quad + \sum_{k=1}^{K} \int_{1-2^k\lambda^{-1}}^{1-2^{k-1}\lambda^{-1}} \left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > 2^{k-1}\right\}\right| dr \\
&\leqslant \frac{1}{\lambda}\|M_\triangleleft f\|_{L^1(\partial\mathbb{D})} + \frac{1}{\lambda}\sum_{k=1}^{K} 2^{k-1}\left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > 2^{k-1}\right\}\right|. \qquad (2.4)
\end{aligned}$$

Let S be the sum in (2.4). By the summation by parts, and (2.1), we obtain

$$\begin{aligned}
S &= \sum_{k=0}^{K-1}\left(2^{k+1}-2^k\right)\left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > 2^k\right\}\right| \\
&= \left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > 2^K\right\}\right|2^K - \left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > 1\right\}\right| \\
&\quad - \sum_{k=0}^{K-1} 2^{k+1}\left(\left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > 2^{k+1}\right\}\right| - \left|\left\{e^{i\theta} \in \partial\mathbb{D}\colon\ M_\triangleleft f\left(e^{i\theta}\right) > 2^k\right\}\right|\right)
\end{aligned}$$

$$\leqslant \|M_{\sphericalangle} f\|_{L^1(\partial\mathbb{D})} + \sum_{k=0}^{K-1} 2^{k+1} \bigl|\bigl\{e^{i\theta} \in \partial\mathbb{D}\colon 2^k < M_{\sphericalangle} f\bigl(e^{i\theta}\bigr) \leqslant 2^{k+1}\bigr\}\bigr|.$$

It follows that

$$S \leqslant \|M_{\sphericalangle} f\|_{L^1(\partial\mathbb{D})} + 2\sum_{k=0}^{K-1} \int\limits_{\{e^{i\theta}\in\partial\mathbb{D}\colon 2^k < M_{\sphericalangle} f(e^{i\theta}) \leqslant 2^{k+1}\}} M_{\sphericalangle} f\bigl(e^{i\theta}\bigr)\, d\theta \leqslant 3\|M_{\sphericalangle} f\|_{L^1(\partial\mathbb{D})}.$$

Since $I_2 \leqslant 2\pi/\lambda$, we conclude that $f \in \widetilde{\mathcal{L}}^1_w$. $\square$

Proof of $\widetilde{\mathcal{L}}^1_w \subset H^1_w$. If $f \in \widetilde{\mathcal{L}}^1_w$, then there exists a constant $C = C(f)$ with $0 < C < \infty$ such that

$$\operatorname{Area}\bigl\{z \in \mathbb{D}\colon \bigl|f(z)\bigr| > \lambda\bigl(1-|z|\bigr)\bigr\} \leqslant C/\lambda, \quad 0 < \lambda < \infty.$$

Write $r_k = 1 - 2^{-k}$ for $k \in \mathbb{N}$, for short. Now

$$\begin{aligned}\frac{C}{\lambda 2^k} &\geqslant \operatorname{Area}\bigl\{z \in \mathbb{D}\colon \bigl|f(z)\bigr| > \lambda 2^k\bigl(1-|z|\bigr)\bigr\}\\ &= \int_0^1 \bigl|\bigl\{e^{i\theta} \in \partial\mathbb{D}\colon \bigl|f\bigl(re^{i\theta}\bigr)\bigr| > \lambda 2^k(1-r)\bigr\}\bigr| r\, dr\\ &\geqslant \int_{r_k}^{r_{k+1}} \bigl|\bigl\{e^{i\theta} \in \partial\mathbb{D}\colon \bigl|f\bigl(re^{i\theta}\bigr)\bigr| > \lambda\bigr\}\bigr| r\, dr, \quad 0 < \lambda < \infty,\ k \in \mathbb{N}.\end{aligned}$$

Since $r_{k+1} - r_k = 2^{-k-1}$ for all $k \in \mathbb{N}$, we conclude that

$$\frac{1}{r_{k+1}-r_k}\int_{r_k}^{r_{k+1}} \bigl|\bigl\{e^{i\theta} \in \partial\mathbb{D}\colon \bigl|f\bigl(re^{i\theta}\bigr)\bigr| > \lambda\bigr\}\bigr| r\, dr \leqslant \frac{2C}{\lambda}, \quad 0 < \lambda < \infty,\ k \in \mathbb{N}. \tag{2.5}$$

Fix $0 < p < 1$. By using the distribution function, Fubini's theorem implies that

$$\begin{aligned}&\frac{1}{r_{k+1}-r_k}\int_{r_k}^{r_{k+1}} \left(\int_0^{2\pi} \bigl|f\bigl(re^{i\theta}\bigr)\bigr|^p\, d\theta\right) r\, dr\\ &\leqslant \frac{1}{r_{k+1}-r_k}\int_{r_k}^{r_{k+1}} \left(2\pi + \int_1^{\infty} \bigl|\bigl\{e^{i\theta} \in \partial\mathbb{D}\colon \bigl|f\bigl(re^{i\theta}\bigr)\bigr| > \lambda\bigr\}\bigr| p\lambda^{p-1}\, d\lambda\right) r\, dr\\ &= 2\pi + \int_1^{\infty} \left(\frac{1}{r_{k+1}-r_k}\int_{r_k}^{r_{k+1}} \bigl|\bigl\{e^{i\theta} \in \partial\mathbb{D}\colon \bigl|f\bigl(re^{i\theta}\bigr)\bigr| > \lambda\bigr\}\bigr| r\, dr\right) p\lambda^{p-1}\, d\lambda, \quad k \in \mathbb{N}.\end{aligned}$$

By (2.5) there exists a constant $K = K(p, f)$ with $0 < K < \infty$ such that

$$\frac{1}{r_{k+1} - r_k} \int_{r_k}^{r_{k+1}} \left(\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta \right) r\,dr \leqslant K < \infty, \quad k \in \mathbb{N}. \tag{2.6}$$

Since the mapping $r \mapsto \|f_r\|^p_{L^p(\partial\mathbb{D})}$ is non-decreasing [12, p. 9], (2.6) yields

$$\int_0^{2\pi} \left| f\left(r_k e^{i\theta}\right) \right|^p d\theta \leqslant \frac{2K}{r_{k+1} + r_k}, \quad k \in \mathbb{N},$$

from which we deduce that $f \in H^p$.

Suppose that $0 < \lambda < \infty$ is given. Since f belongs to H^p for $0 < p < 1$, the radial limit $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ exists for a.e. $e^{i\theta} \in \partial\mathbb{D}$. By applying Egorov's theorem [36, p. 73], we conclude that this convergence is uniform outside a set of arbitrarily small Lebesgue measure. In particular, there exist a constant $r^\star = r^\star(\lambda, f)$ with $0 < r^\star < 1$, and a set $E = E(\lambda, f) \subset \partial\mathbb{D}$ of length $|E| \leqslant 1/\lambda$, such that

$$\left| f\left(re^{i\theta}\right) - f\left(e^{i\theta}\right) \right| \leqslant \lambda/2, \quad r^\star < r < 1,\ e^{i\theta} \in \partial\mathbb{D} \setminus E.$$

Observe that $\{e^{i\theta} \in \partial\mathbb{D}\colon |f(e^{i\theta})| > \lambda\} \subset A \cup B$, where $A = A(\lambda, f)$ and $B = B(\lambda, f)$ are subsets of $\partial\mathbb{D}$ given by

$$A = \left\{ e^{i\theta} \in \partial\mathbb{D}\colon \left| f\left(re^{i\theta}\right) - f\left(e^{i\theta}\right) \right| > \lambda/2 \text{ for some } r \text{ with } r^\star < r < 1 \right\},$$
$$B = \left\{ e^{i\theta} \in \partial\mathbb{D}\colon \left| f\left(e^{i\theta}\right) \right| > \lambda,\ \left| f\left(re^{i\theta}\right) - f\left(e^{i\theta}\right) \right| \leqslant \lambda/2 \text{ for all } r \text{ with } r^\star < r < 1 \right\}.$$

Since $A \subset E$, we conclude that $|A| \leqslant 1/\lambda$. We proceed to consider the size of B. Note that $B \subset \{e^{i\theta} \in \partial\mathbb{D}\colon |f(re^{i\theta})| > \lambda/2\}$ for any r with $r^\star < r < 1$. Consequently, if $k \in \mathbb{N}$ is sufficiently large such that $r^\star \leqslant r_k < 1$, then (2.5) implies that

$$|B| \leqslant \frac{2}{r_{k+1}^2 - r_k^2} \int_{r_k}^{r_{k+1}} \left| \left\{ e^{i\theta} \in \partial\mathbb{D}\colon \left| f\left(re^{i\theta}\right) \right| > \lambda/2 \right\} \right| r\,dr \leqslant \frac{8C}{(r_{k+1} + r_k)\lambda}.$$

In conclusion, $|\{e^{i\theta} \in \partial\mathbb{D}\colon |f(e^{i\theta})| > \lambda\}| \leqslant |A| + |B| \leqslant (1 + 32C/5)/\lambda$, where C is independent of λ. This proves that $f(e^{i\theta}) \in L^1_w(\partial\mathbb{D})$. Since $f \in H^p$ for $0 < p < 1$, we have $f \in H^1_w$ by [9, Theorem 1.10.4]. $\square$

Proof of $H^1_w \subset \mathcal{L}^p_w$ for $1 < p < \infty$. Let $f \in H^1_w$, and hence $M_\triangleleft f \in L^1_w(\partial\mathbb{D})$, and let $1 < p < \infty$ be fixed. Suppose that $\lambda_0 < \lambda < \infty$ is given, where

$$\lambda_0 = \lambda_0(\alpha, f) = \max\left\{ \sup\left\{ \left| f(z) \right|\colon |z| \leqslant \frac{\alpha - 1}{\alpha + 1} \right\}, \frac{\|M_\triangleleft f\|_{L^1_w(\partial\mathbb{D})}}{2\sqrt{\alpha^2 - 1}} \right\}.$$

Here $\alpha > 1$ is the (fixed) constant in (1.2), which determines the aperture of the non-tangential region. Since $\{e^{i\theta} \in \partial\mathbb{D}\colon M_\lhd f(e^{i\theta}) > \lambda\}$ is an open subset of $\partial\mathbb{D}$, there exists a family of pairwise disjoint open arcs $I_k = I_k(\lambda, f) \subset \partial\mathbb{D}$ for $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} I_k = \{e^{i\theta} \in \partial\mathbb{D}\colon M_\lhd f(e^{i\theta}) > \lambda\}$, and

$$\sum_{k\in\mathbb{N}} |I_k| \leqslant \frac{\|M_\lhd f\|_{L^1_w(\partial\mathbb{D})}}{\lambda}. \tag{2.7}$$

For each arc $I \subset \partial\mathbb{D}$ we define a corresponding (sectorial) domain $T(I)$ by

$$T(I) = \left\{z \in \mathbb{D}\colon z/|z| \in I \text{ and } 1 - |z| \leqslant \frac{|I|}{2\sqrt{\alpha^2 - 1}}\right\}.$$

If $z \in \mathbb{D}$ is a point such that $|f(z)| > \lambda$, where

$$\lambda > \lambda_0 \geqslant \sup\bigl\{\bigl|f(z)\bigr|\colon |z| \leqslant (\alpha - 1)/(\alpha + 1)\bigr\},$$

then $\{e^{i\theta} \in \partial\mathbb{D}\colon M_\lhd f(e^{i\theta}) > \lambda\}$ contains an arc I_z of length $|I_z| \geqslant 2\sqrt{\alpha^2 - 1}(1 - |z|)$, centered at $e^{i \arg z}$. Evidently, such point z belongs to $T(I_z)$, which is in turn contained in $T(I_k)$ for some $k \in \mathbb{N}$. We conclude that

$$\bigl\{z \in \mathbb{D}\colon \bigl|f(z)\bigr| > \lambda\bigr\} \subset \bigcup_{k\in\mathbb{N}} T(I_k),$$

which yields $\mu_p(\{z \in \mathbb{D}\colon |f(z)| > \lambda\}) \leqslant \sum_{k\in\mathbb{N}} \mu_p(T(I_k))$. Since there exists a constant $C = C(\alpha, p)$ with $0 < C < \infty$, such that

$$\mu_p\bigl(T(I_k)\bigr) = \int_{T(I_k)} \bigl(1 - |z|\bigr)^{p-2}\, dA(z) \leqslant C|I_k|^p, \quad k \in \mathbb{N},$$

we deduce that

$$\mu_p\bigl(\bigl\{z \in \mathbb{D}\colon \bigl|f(z)\bigr| > \lambda\bigr\}\bigr) \leqslant C\sum_{k\in\mathbb{N}} |I_k|^p \leqslant C\Bigl(\sum_{k\in\mathbb{N}} |I_k|\Bigr)^p \leqslant C\frac{\|M_\lhd f\|^p_{L^1_w(\partial\mathbb{D})}}{\lambda^p}$$

by (2.7). Hence $f \in \mathcal{L}^p_w$. □

Proof of $\mathcal{L}^p_w \subset A^{-1}$ for $1 < p < \infty$. Suppose that $f \in \mathcal{L}^p_w$ for some $1 < p < \infty$. Let $a \in \mathbb{D}$, and suppose that $E_a = D(a, (1 - |a|)/2)$ is a Euclidean disc of radius $(1 - |a|)/2$. The Kolmogorov condition mentioned in Section 1.1 (with $r = 1$) shows that there exists a constant $C_1 = C_1(p, f)$ with $0 < C_1 < \infty$ such that

$$\int_{E_a} \bigl|f(z)\bigr|\bigl(1 - |z|\bigr)^{p-2}\, dA(z) \leqslant C_1\Bigl(\int_{E_a} \bigl(1 - |z|\bigr)^{p-2}\, dA(z)\Bigr)^{1-1/p}, \quad a \in \mathbb{D}.$$

By the subharmonicity of $|f|$, and the Kolmogorov condition above, there exists a constant $C_2 = C_2(p, f)$ with $0 < C_2 < \infty$, such that

$$\begin{aligned}\left|f(a)\right| &\leqslant \frac{4}{\pi(1-|a|)^2}\int_{E_a}\left|f(z)\right| dA(z)\\ &\leqslant \frac{2^p}{\pi(1-|a|)^p}\int_{E_a}\left|f(z)\right|\bigl(1-|z|\bigr)^{p-2} dA(z) \leqslant \frac{C_2}{(1-|a|)}, \quad a\in\mathbb{D},\end{aligned}$$

and hence $f \in A^{-1}$. $\square$

2.2. Proof of Proposition 1(ii)

Suppose that $f \in \mathcal{L}_w^p$ for some $1 < p < \infty$, and let $p < q < \infty$. We proceed to prove that $f \in \mathcal{L}_w^q$. Let $2 < \lambda < \infty$ be given, and take $K = K(\lambda) \in \mathbb{N}$ such that $2^K < \lambda \leqslant 2^{K+1}$. Now

$$\mu_q\bigl(\bigl\{z\in\mathbb{D}\colon \left|f(z)\right| > \lambda\bigr\}\bigr) \leqslant \sum_{k=K}^{\infty} \int_{\{z\in\mathbb{D}\colon 2^{k+1}\geqslant|f(z)|>2^k\}} \bigl(1-|z|\bigr)^{q-2} dA(z).$$

According to Proposition 1(i) we may suppose that $f \in A^{-1}$. Consequently, if $k \in \mathbb{N}$ and $2^k < |f(z)|$, then $1 - |z| < 2^{-k}\|f\|_{A^{-1}}$. We conclude that

$$\begin{aligned}\mu_q\bigl(\bigl\{z\in\mathbb{D}\colon \left|f(z)\right| > \lambda\bigr\}\bigr) &\leqslant \|f\|_{A^{-1}}^{q-p}\sum_{k=K}^{\infty} 2^{kp-kq} \int_{\{z\in\mathbb{D}\colon |f(z)|>2^k\}} \bigl(1-|z|\bigr)^{p-2} dA(z)\\ &\leqslant \frac{\|f\|_{A^{-1}}^{q-p}\|f\|_{L_w^p(\mathbb{D},\mu_p)}^p}{1-2^{-q}}\frac{2^q}{\lambda^q}.\end{aligned}$$

This means that $f \in \mathcal{L}_w^q$, and we are done.

3. Proof of Theorem 2

The point of departure is a discussion of certain auxiliary results, which show that inner functions whose derivative belongs to $\mathcal{L}_w^p$ for some $1 < p < \infty$ reduce to finite products of interpolating Blaschke products.

Proposition 4. *If B is inner, and if $B' \in \mathcal{L}_w^p$ for some $1 < p < \infty$, then B is a Blaschke product.*

Proof. By the assumption $B' \in L_w^p(\mathbb{D}, \mu_p)$, and consequently $B' \in L^q(\mathbb{D}, \mu_p)$ for any $0 < q < p$. That is,

$$\int_{\mathbb{D}} \left|B'(z)\right|^q \bigl(1-|z|\bigr)^{p-2} dA(z) < \infty, \quad 0 < q < p.$$

If we choose q to satisfy $p - 1/2 \leqslant q < p$, then we conclude that B is a Blaschke product by [2, p. 736]. $\square$

A Blaschke product B is called interpolating provided that its zeros $\{z_k\}_{k\in\mathbb{N}}$ form a uniformly separated sequence in $\mathbb{D}$. That is to say that there exists a constant $\delta = \delta(B)$ with $0 < \delta < 1$ such that

$$\inf_{k\in\mathbb{N}} \prod_{n\in\mathbb{N}\setminus\{k\}} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| = \inf_{k\in\mathbb{N}} \left(1 - |z_k|^2\right) \left|B'(z_k)\right| \geqslant \delta. \tag{3.1}$$

Proposition 5. *If B is a Blaschke product, and if $B' \in \mathcal{L}^p_w$ for some $1 < p < \infty$, then B is a product of finitely many interpolating Blaschke products.*

Before the proof of Proposition 5 we consider a standard lemma, whose proof bears similarity to that of [20, Lemma 1]. The proof of Lemma 6 is given for the convenience of the reader. For $\zeta \in \mathbb{D}$, let

$$\Delta(\zeta, m) = \left\{z \in \mathbb{D}\colon\ |z - \zeta| < m|1 - \bar{z}\zeta|\right\}$$

be an open pseudo-hyperbolic disc of the pseudo-hyperbolic radius $0 < m < 1$.

Lemma 6. *Suppose that B is a Blaschke product, which is not a finite product of interpolating Blaschke products. Then there exist sequences $\{\zeta_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$, and $\{m_k\}_{k\in\mathbb{N}} \subset (0, 1)$, such that*

$$\lim_{k\to\infty} m_k = 1, \quad \textit{and} \quad \lim_{k\to\infty} \Big(\sup_{z\in\Delta(\zeta_k, m_k)} \left|B(z)\right| \Big) = 0. \tag{3.2}$$

Proof. Since B is not a finite product of interpolating Blaschke products, we know that $\mu = \sum_{k\in\mathbb{N}} (1 - |z_k|)\delta_{z_k}$ is not a Carleson measure [28, Lemma 21]. Here δ_{z_k} is the Dirac measure with unit point mass at $z_k \in \mathbb{D}$, and $\{z_k\}_{k\in\mathbb{N}}$ is the zero-sequence of B. Consequently, there exists a sequence $\{Q_j\}_{j\in\mathbb{N}}$ of Carleson boxes $Q_j = Q_j(B)$ of the form

$$Q_j = \left\{z \in \mathbb{D}\colon\ z/|z| \in I_{Q_j} \text{ and } 1 - |z| \leqslant |I_{Q_j}|\right\}, \quad j \in \mathbb{N},$$

where I_{Q_j} is an arc on $\partial\mathbb{D}$ of length $|I_{Q_j}| = l(Q_j)$, such that the corresponding sequence $\{S_j\}_{j\in\mathbb{N}}$, where $S_j = S_j(B) \in (0, \infty)$ is defined by

$$S_j = \frac{1}{l(Q_j)} \sum_{z_k \in Q_j} \left(1 - |z_k|\right) = \frac{\mu(Q_j)}{l(Q_j)},$$

satisfies $S_j \to \infty$, as $j \to \infty$. Without loss of generality, we may suppose that $Q_j \cap \{z_k\}_{k\in\mathbb{N}} \neq \emptyset$, and $l(Q_j) \leqslant 1/2$, for every $j \in \mathbb{N}$.

Let $\zeta_j \in \mathbb{D}$ be the point such that $|\zeta_j| = 1 - l(Q_j)$ and $\zeta_j/|\zeta_j|$ is the center of I_{Q_j}, for all $j \in \mathbb{N}$. Let $\{m_j\}_{j\in\mathbb{N}}$ be a sequence of numbers $0 < m_j < 1$ satisfying the asymptotic properties $m_j \to 1^-$ and $(1-m_j)^3 S_j \to \infty$, as $j \to \infty$. If $w_j \in \Delta(\zeta_j, m_j)$, then $1-|w_j| > (1-m_j)l(Q_j)/8$ by [13, p. 42], and moreover,

$$|\zeta_j - w_j| < \frac{2m_j}{1-m_j^2}\left(1-|\zeta_j|^2\right).$$

Consequently, for all $z_k \in Q_j$ we have

$$|1-\bar{z}_k w_j|^2 = |z_k|^2 |1/\bar{z}_k - w_j|^2 < \frac{8^2}{(1-m_j^2)^2} l(Q_j)^2,$$

which implies that for every $w_j \in \Delta(\zeta_j, m_j)$

$$\begin{aligned}\log\frac{1}{|B(w_j)|} &\geqslant \frac{1}{2}\sum_{z_k \in Q_j} \frac{(1-|w_j|^2)(1-|z_k|^2)}{|1-\bar{z}_k w_j|^2}\\ &\geqslant \frac{(1-m_j)^3}{2\cdot 8^3 \cdot l(Q_j)} \sum_{z_k\in Q_j}\left(1-|z_k|^2\right) \geqslant \frac{(1-m_j)^3 S_j}{2\cdot 8^3} \to \infty, \quad j\to\infty.\end{aligned}$$

Here we have applied the inequality $\log 1/x \geqslant (1-x^2)/2$ for $0<x<1$ to each factor of $|B(w_j)|$. $\square$

Proof of Proposition 5. Suppose on the contrary to the claim that B is a Blaschke product, which is not a finite product of interpolation Blaschke products. By Lemma 6 there exist $\{\zeta_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$, and $\{m_k\}_{k\in\mathbb{N}} \subset (0,1)$, such that (3.2) holds.

Let $1 < M < \infty$ be a large fixed number, whose exact value is to be determined later. Let $I_k \subset \partial\mathbb{D}$ for $k \in \mathbb{N}$ be the arc, which is centered at $\zeta_k/|\zeta_k|$, and whose length is $|I_k| = 1-|\zeta_k|$. After removing finitely many points from $\{\zeta_k\}_{k\in\mathbb{N}}$ we may assume that $|I_k| \leqslant 1/2$ for all $k\in\mathbb{N}$. Correspondingly, MI_k denotes the concentric arc of length $M|I_k| = M(1-|\zeta_k|)$. Define

$$E_k = \left\{re^{i\theta} \in \mathbb{D}\colon\ e^{i\theta} \in MI_k,\ \text{and } 1-r \leqslant |I_k|\right\}, \quad k \in \mathbb{N}.$$

On one hand, (3.2) shows that $|B(z)| \leqslant 1/2$ holds for all $z \in \mathbb{D}$ such that $|z| = 1-|I_k|$, and $z/|z| \in MI_k$, provided that k is sufficiently large. This follows from the fact that the pseudo-hyperbolic distance between ζ_k and $\zeta_k \exp(iM(1-|\zeta_k|)/2)$ is uniformly bounded by $M/\sqrt{M^2+16} < 1$ for all $k \in \mathbb{N}$. Thus

$$\frac{1}{2} \leqslant \left|B\left(e^{i\theta}\right) - B\left(\left(1-|I_k|\right)e^{i\theta}\right)\right| \leqslant \int_{1-|I_k|}^{1} \left|B'\left(re^{i\theta}\right)\right| dr, \quad \text{a.e. } e^{i\theta} \in MI_k,$$

and consequently $\int_{E_k} |B'(z)|\, dA(z) \geqslant M|I_k|/4$ for all $k\in\mathbb{N}$ large enough.

On the other hand, if $\max\{1, p-1\} < r < p$, then the Kolmogorov condition stated in Section 1.1 shows that there exists a constant $C_1 = C_1(p, r, B)$ with $0 < C_1 < \infty$ such that

$$\left(\int_{E_k} \left|B'(z)\right|^r (1-|z|)^{p-2}\, dA(z)\right)^{\frac{1}{r}} \leqslant C_1\left(\int_{E_k} (1-|z|)^{p-2}\, dA(z)\right)^{\frac{1}{r}-\frac{1}{p}}, \quad k \in \mathbb{N}.$$

Now Hölder's inequality with indices r and $r/(r-1)$ implies that there exists a constant $C_2 = C_2(p, r)$ with $0 < C_2 < \infty$ such that

$$\begin{aligned}
\int_{E_k} \left|B'(z)\right| dA(z) &\leqslant \left(\int_{E_k} \left|B'(z)\right|^r (1-|z|)^{p-2}\, dA(z)\right)^{\frac{1}{r}} \left(\int_{E_k} (1-|z|)^{\frac{2-p}{r-1}}\, dA(z)\right)^{\frac{r-1}{r}} \\
&\leqslant C_1\left(\int_{E_k} (1-|z|)^{p-2}\, dA(z)\right)^{\frac{1}{r}-\frac{1}{p}} \left(\int_{E_k} (1-|z|)^{\frac{2-p}{r-1}}\, dA(z)\right)^{\frac{r-1}{r}} \\
&\leqslant C_1 C_2 M^{1-\frac{1}{p}} |I_k|, \quad k \in \mathbb{N}.
\end{aligned}$$

We conclude that, if $1 < M < \infty$ is sufficiently large, then the obtained upper and lower bounds for $\int_{E_k} |B'(z)|\, dA(z)$ lead to a contradiction. The assertion follows. □

Before the proof of Theorem 2, we consider a standard lemma. Similar results can be found in the literature, see for instance [30, Lemma 5].

Lemma 7. *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic. If there exist a constant $0 < \delta \leqslant 1$, and a sequence $\{\zeta_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$, such that $\inf_{k\in\mathbb{N}}(1-|\zeta_k|^2)|f'(\zeta_k)| \geqslant \delta$, then there are constants $\delta^\star = \delta^\star(\delta)$ and $\eta = \eta(\delta)$ satisfying $0 < \delta^\star \leqslant 1$ and $0 < \eta < 1$ such that*

$$\left(1-|z|^2\right)\left|f'(z)\right| \geqslant \delta^\star, \quad z \in \Delta(\zeta_k, \eta),\ k \in \mathbb{N}.$$

Proof. If $h : \mathbb{D} \to \mathbb{D}$ is analytic and $|h'(0)| \geqslant \delta$ for some $0 < \delta \leqslant 1$, then, by applying Cauchy's integral formula to $h'(z) - h'(0)$, there exists a constant $\eta = \eta(\delta)$ with $0 < \eta < 1$ such that $|h'(z)| \geqslant \delta/2$ for all $|z| < \eta$. The assertion of Lemma 7 follows by applying this observation to the functions $g_k = f \circ \varphi_{\zeta_k}$ for $k \in \mathbb{N}$, where $\varphi_{\zeta_k}(z) = (\zeta_k - z)/(1-\overline{\zeta}_k z)$. □

Finally we are in a position to prove our main result.

Proof of Theorem 2. As we remarked directly after the statement of Theorem 2, it suffices to show that, if B is inner and $B' \in \mathcal{L}^p_w$ for some $1 < p < \infty$, then B is an exponential Blaschke product.

By Propositions 4 and 5, we may assume that B is a finite product of interpolating Blaschke products. Observe that, if $\{z_k\}_{k\in\mathbb{N}}$ is the zero-sequence of B, then it is possible that (3.1) fails to be true for any $0 < \delta < 1$ due to zeros with multiplicities. However, according to [20, Lemma 1] there exists a sequence $\{\zeta_k\}_{k\in\mathbb{N}}$, where $\zeta_k = \zeta_k(B) \in \mathbb{D}$ for $k \in \mathbb{N}$, as well as constants $\rho = \rho(B)$ and $\delta = \delta(B)$ satisfying $0 < \rho < 1$ and $0 < \delta \leqslant 1$ such that

$$\inf_{k\in\mathbb{N}} \bigl(1 - |\zeta_k|^2\bigr)\bigl|B'(\zeta_k)\bigr| \geqslant \delta, \tag{3.3}$$

while $\zeta_k \in \Delta(z_k, \rho)$ for all $k \in \mathbb{N}$. By Lemma 7 inequality (3.3) can be extended to sufficiently small pseudo-hyperbolic neighborhoods of the points ζ_k for $k \in \mathbb{N}$. In particular, there exist constants $\eta = \eta(B)$ and $\delta^\star = \delta^\star(B)$ satisfying $0 < \eta < 1$ and $0 < \delta^\star \leqslant 1$, such that

$$\bigl(1 - |z|^2\bigr)\bigl|B'(z)\bigr| \geqslant \delta^\star, \quad z \in \Delta(\zeta_k, \eta),\ k \in \mathbb{N}. \tag{3.4}$$

Since B is a finite product of interpolating Blaschke products, we can write $B = B_1 B_2 \cdots B_N$, where the zero-sequence $\{z_{n,k}\}_{k\in\mathbb{N}}$ of each sub-product B_n for $n = 1, \dots, N$ satisfies (3.1) for some strictly positive constant $\delta = \delta_n$. By taking smaller η if necessary, we may assume that η in (3.4) is sufficiently small to satisfy $\eta < \min\{\delta_1, \delta_2, \dots, \delta_N\}/2$ and $\eta < 1/3$. Consequently, the pseudo-hyperbolic discs $\{\Delta(z_{n,k}, \eta)\}_{k\in\mathbb{N}}$ are pairwise disjoint for each $n = 1, \dots, N$. Although the discs $\{\Delta(z_k, \eta)\}_{k\in\mathbb{N}}$ are not necessarily pairwise disjoint, they are quasi-disjoint in the sense that the characteristic functions $\chi_{\Delta(z_k,\eta)}$ of the discs $\Delta(z_k, \eta)$ for $k \in \mathbb{N}$ satisfy

$$\sum_{k\in\mathbb{N}} \chi_{\Delta(z_k,\eta)}(z) \leqslant N, \quad z \in \mathbb{D}.$$

This implies that the sequence $\{\zeta_k\}_{k\in\mathbb{N}}$ in (3.3) satisfies

$$\sum_{k\in\mathbb{N}} \chi_{\Delta(\zeta_k,\eta)}(z) \leqslant M, \quad z \in \mathbb{D}, \tag{3.5}$$

for some constant $M = M(B)$ with $0 < M < \infty$, and hence the discs $\{\Delta(\zeta_k, \eta)\}_{k\in\mathbb{N}}$ are quasi-disjoint.

Let $\mathcal{A}_j$ for $j \in \mathbb{N}$ be the annuli in (1.1), and observe that their pseudo-hyperbolic widths are always greater than $1/3$. Take λ such that $\delta^\star \leqslant \lambda < \infty$, and denote $J(\lambda) = \log_2 \lambda/\delta^\star + 3$. If $k \in \mathbb{N}$ such that $\zeta_k \in \mathcal{A}_j$ for some $j > J(\lambda)$, then $\Delta(\zeta_k, \eta) \subset (\mathcal{A}_{j-1} \cup \mathcal{A}_j \cup \mathcal{A}_{j+1})$, and by (3.4)

$$\bigl|B'(z)\bigr| > \frac{\delta^\star/2}{1 - |z|} \geqslant \delta^\star 2^{j-3} > \lambda, \quad z \in \Delta(\zeta_k, \eta).$$

Define $\mathcal{N}_j = \#(\mathcal{A}_j \cap \{\zeta_k\}_{k\in\mathbb{N}})$ for $j \in \mathbb{N}$. Since $B' \in \mathcal{L}^p_w$, and the discs $\{\Delta(\zeta_k, \eta)\}_{k\in\mathbb{N}}$ satisfy (3.5), we obtain

$$\begin{aligned}
\frac{\|B'\|^p_{L^p_w(\mathbb{D},\mu_p)}}{\lambda^p} &\geqslant \mu_p\bigl(\bigl\{z \in \mathbb{D}\colon \bigl|B'(z)\bigr| > \lambda\bigr\}\bigr) \\
&\geqslant \frac{1}{3} \sum_{j>J(\lambda)} \mu_p\bigl(\bigl\{z \in (\mathcal{A}_{j-1} \cup \mathcal{A}_j \cup \mathcal{A}_{j+1})\colon \bigl|B'(z)\bigr| > \lambda\bigr\}\bigr) \\
&\geqslant \frac{1}{3} \sum_{j>J(\lambda)} \mu_p\Biggl(\bigcup_{\zeta_k\in\mathcal{A}_j} \Delta(\zeta_k, \eta)\Biggr) \geqslant \frac{1}{3M} \sum_{j>J(\lambda)} \sum_{\zeta_k\in\mathcal{A}_j} \mu_p\bigl(\Delta(\zeta_k, \eta)\bigr)
\end{aligned}$$

$$= \frac{1}{3M} \sum_{j>J(\lambda)} \sum_{\zeta_k \in \mathcal{A}_j} \int_{\Delta(\zeta_k,\eta)} \left(1-|z|\right)^{p-2} dA(z).$$

We deduce

$$\frac{\|B'\|^p_{L^p_w(\mathbb{D},\mu_p)}}{\lambda^p} \geqslant \frac{C}{M} \sum_{j>J(\lambda)} \mathcal{N}_j 2^{-jp}$$

for all $\delta^\star \leqslant \lambda < \infty$, where $C = C(\eta, p)$ is a constant such that $0 < C < \infty$. Consequently, each annulus $\mathcal{A}_j$ for $j \in \mathbb{N}$ contains at most a fixed number of points from the sequence $\{\zeta_k\}_{k\in\mathbb{N}}$. This means that B is an exponential Blaschke product, since $\zeta_k \in \Delta(z_k, \rho)$ for all $k \in \mathbb{N}$. This concludes the proof of Theorem 2. $\square$

4. Proof of Theorem 3

It is sufficient to prove that, if B is inner and $B' \in \widetilde{\mathcal{L}}^1_w$, then B is a finite Blaschke product. To this end, suppose that B is an inner function such that $B' \in \widetilde{\mathcal{L}}^1_w$, and assume that B has infinitely many zeros. According to Proposition 1(i) we know that $B' \in H^1_w$, and hence B is an exponential Blaschke product [8, Theorem 1]. Consequently, B' has radial limits almost everywhere. If $\{z_k\}_{k\in\mathbb{N}}$ is the zero-sequence of B, then

$$\left|B'(\xi)\right| = \sum_{k\in\mathbb{N}} \frac{1-|z_k|^2}{|\xi - z_k|^2}, \quad \text{a.e. } \xi \in \partial\mathbb{D},$$

by [3, Corollary 3]. Let I_k for $k \in \mathbb{N}$ be the arc on $\partial\mathbb{D}$, which is centered at $z_k/|z_k|$, and whose length is $2\pi(1-|z_k|)$. If $z_k \in \mathbb{D}$ is a zero of B, then

$$\left|B'(\xi)\right| \geqslant \frac{1-|z_k|^2}{|\xi - z_k|^2} > \frac{1}{(1+\pi)^2} \frac{1}{1-|z_k|}, \quad \text{a.e. } \xi \in I_k.$$

For any fixed $k \in \mathbb{N}$ we may consider the family of functions $\{g_r\}$, where $0 \leqslant r < 1$, defined by $g_r(\xi) = B'(r\xi)$ for $\xi \in I_k^\star$. Here $I_k^\star$ consists of those values $\xi \in I_k$ for which $|B'(\xi)|$ exists. By Egorov's theorem [36, p. 73] we conclude that there is a subset $E_k \subset I_k^\star$ of length $|E_k| > |I_k|/2$ such that $g_r \to g$ uniformly in E_k, as $r \to 1^-$. Consequently, there exists a sequence $\{r_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that

$$\left|B'(r\xi)\right| > \frac{1}{(1+\pi)^2} \frac{1}{1-|z_k|}, \quad r_k < r < 1,\ \xi \in E_k.$$

By extracting a subsequence of $\{z_k\}_{k\in\mathbb{N}}$, and by choosing the sets E_k accordingly, we may assume that the sets $H_k = \{r\xi \in \mathbb{D}\colon r_k < r < 1,\ \xi \in E_k\}$ for $k \in \mathbb{N}$ are pairwise disjoint, while $|E_k| > |I_k|/4$. For instance, we may take a subsequence $\{z_{k_j}\}_{j\in\mathbb{N}}$ of $\{z_k\}_{k\in\mathbb{N}}$, which satisfies $1-|z_{k_{j+1}}| \leqslant 5^{-1}(1-|z_{k_j}|)$ for all $j \in \mathbb{N}$, and choose

$$E^\star_{k_j} = E_{k_j} \Big\backslash \left(\bigcup_{n=1}^{\infty} E_{k_{j+n}} \right), \quad j \in \mathbb{N}.$$

Then, the sets H_k are pairwise disjoint by construction, while

$$\left|E^{\star}_{k_j}\right| > \frac{|I_{k_j}|}{2} - \sum_{n=1}^{\infty} |I_{k_{j+n}}| > \frac{|I_{k_j}|}{4}, \qquad j \in \mathbb{N}.$$

Fix $0 < \lambda < \infty$, and define

$$K(\lambda) = \left\{k \in \mathbb{N}:\ \lambda > \frac{1}{(1+\pi)^2(1-r_k)(1-|z_k|)}\right\}.$$

Observe that $K(\lambda)$ is a bounded subset of $\mathbb{N}$, while $\#K(\lambda) \to \infty$, as $\lambda \to \infty$. Now

$$\left\{z \in \mathbb{D}:\ \frac{|B'(z)|}{1-|z|} > \lambda\right\} \supset \bigcup_{k \in K(\lambda)} \left\{z \in H_k:\ \frac{1}{(1+\pi)^2(1-|z_k|)(1-|z|)} > \lambda\right\},$$

which implies that

$$\begin{aligned}&\operatorname{Area}\left\{z \in \mathbb{D}:\ \frac{|B'(z)|}{1-|z|} > \lambda\right\}\\ &\quad\geqslant \sum_{k \in K(\lambda)} \operatorname{Area}\left\{z \in H_k:\ \frac{1}{(1+\pi)^2(1-|z_k|)(1-|z|)} > \lambda\right\} \geqslant \frac{\#K(\lambda)\cdot\pi}{2(1+\pi)^2\lambda}.\end{aligned}$$

Consequently $B' \notin \widetilde{\mathcal{L}}^1_w$, which concludes the proof.

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