INNER FUNCTIONS, MÖBIUS DISTORTION AND ANGULAR DERIVATIVES

KONSTANTINOS BAMPOURAS AND ARTUR NICOLAU

Abstract. We prove that an inner function has finite $\mathcal{L}(p)$-entropy if and only if its accumulated Möbius distortion is in L^p, $0 < p < \infty$. We also study the support of the positive singular measures such that their corresponding singular inner functions have finite $\mathcal{L}(p)$-entropy.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane and let $d_h(z, w)$ denote the hyperbolic distance between the points $z, w \in \mathbb{D}$ given by

$$d_h(z, w) = \inf_{\gamma} \int_{\gamma} \frac{2|d\zeta|}{1 - |\zeta|^2} = \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

where the infimum is taken over all curves γ contained in $\mathbb{D}$ joining z and w and $\rho(z, w) = |z - w|/|1 - \overline{w}z|$, $z, w \in \mathbb{D}$. The Schwarz lemma says that any analytic self-mapping f of the unit disc is a contraction in the hyperbolic metric, that is, $d_h(f(z), f(w)) \leq d_h(z, w)$ for any $z, w \in \mathbb{D}$, or equivalently, the hyperbolic derivative of f, denoted by $D_h f$, satisfies

$$D_h f(z) = \frac{(1 - |z|^2)|f'(z)|}{1 - |f(z)|^2} \leq 1, \quad z \in \mathbb{D}. \tag{1}$$

Moreover, equality at a single point implies equality at every point in the unit disc and that f is an automorphism of $\mathbb{D}$.

An analytic self-mapping f of the unit disc is said to have a *finite angular derivative* (in the sense of Carathéodory) at a point $\xi \in \mathbb{T}$ if $\lim_{r \to 1} f(r\xi)$ exists and belongs to the unit circle and $\lim_{r \to 1} f'(r\xi)$ exists. In this case, we write $|f'(\xi)| = \lim_{r \to 1} |f'(r\xi)|$. It is well known that f has a finite angular derivative at a point $\xi \in \mathbb{T}$ if and only if

$$\liminf_{z \to \xi} \frac{1 - |f(z)|}{1 - |z|} < \infty. \tag{2}$$

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If f does not have a finite angular derivative at ξ, it is customary to set $|f'(\xi)| := \infty$. With these notations it is well known that

$$|f'(\xi)| = \lim \frac{1-|f(z)|}{1-|z|}, \quad \xi \in \mathbb{T},$$

where the limit is taken as $z \in \mathbb{D}$ tends non-tangentially to ξ. See for instance [6] or Chapter IV of [18].

Let m be the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Inner functions are analytic self-mappings f of the unit disc such that their radial limits $\lim_{r\to 1} f(r\xi)$ have modulus one at m-almost every point $\xi \in \mathbb{T}$. Fix $0 < p < \infty$. An inner function f has finite $\mathcal{L}(p)$-entropy if its angular derivative $|f'(\xi)|$ is finite at m-almost every point $\xi \in \mathbb{T}$ and $\log |f'(\xi)| \in L^p(\mathbb{T})$. Note that when $p = 1$ these are the inner functions f with finite entropy, or equivalently, inner functions whose derivative is in the Nevanlinna class, which were first studied in [4] and have recently attracted some attention [9, 10, 12, 13].

Let f be an analytic self-mapping of the unit disc. Define the *Möbius distortion* of f as

$$\mu(f)(z) \;=\; 1 - D_h(f)(z), \qquad z \in \mathbb{D}.$$

The Möbius distortion $\mu(f)(z)$ measures how much f deviates from an automorphism of $\mathbb{D}$ near $z \in \mathbb{D}$. Several natural classes of inner functions can be described using the Möbius distortion. For instance, M. Heins proved in [8] that $\mu(f)(z) \to 0$ as $|z| \to 1$ if and only if f is a finite Blaschke product. Local versions of this result can be found in [15]. In [14] D. Kraus proved that f has a finite angular derivative at almost every point of the unit circle if and only if for almost every point $\xi \in \mathbb{T}$ we have $\mu(f)(z) \to 0$ as z tends non-tangentially to ξ.

Consider the accumulated Möbius distortion defined as

$$A(f)(\xi) = \int_0^1 \mu(f)(r\xi) \frac{2dr}{1-r^2}, \quad \xi \in \mathbb{T}. \tag{3}$$

The following pointwise estimate has been recently proved in [7] and [13] (see also [11]): for every analytic self-mapping f of the unit disc that fixes the origin, we have

$$A(f)(\xi) \le \log |f'(\xi)|, \quad \xi \in \mathbb{T}. \tag{4}$$

It is worth mentioning that a converse estimate of the form $\log |f'(\xi)| \le C_1 A(f)(\xi) + C_2$ where C_1, C_2 are absolute constants, can not hold. This can be seen by taking $f(z) = z(z-a)/(1-az)$ where $0 < a < 1$ for which $f'(1) = 2/(1-a)$ and $A(f)(1) = \log(1+a) + \log 2$. However recently, P. Gumenyuk, M. Kourou, A. Moucha and O. Roth and independently O. Ivrii and M. Urbanski, have proved that for any analytic mapping $f : \mathbb{D} \to \mathbb{D}$ and any point $\xi \in \mathbb{T}$, the angular derivative $|f'(\xi)|$ is finite if and only if $A(f)(\xi) < \infty$. See [7] and Theorem B.1 of [13]. The main purpose of this note is to show that even if $\log |f'|$ and $A(f)$ are not pointwise comparable, they still belong to the same $L^p(\mathbb{T})$ spaces, $0 < p < \infty$.

Theorem 1.1. *Fix $0 < p < \infty$. Let $f : \mathbb{D} \to \mathbb{D}$ be an inner function. Then f has finite $\mathcal{L}(p)$-entropy, that is $\log |f'| \in L^p(\mathbb{T})$, if and only if $A(f) \in L^p(\mathbb{T})$.*

When $p = 1$ the result follows easily from Green's Formula and the identity

$$\Delta\left(\log \frac{1-|f(z)|^2}{1-|z|^2}\right) = \frac{4(1-D_h(f)(z)^2)}{(1-|z|^2)^2}, \quad z \in \mathbb{D}, \tag{5}$$

which one can check by direct calculation. The general case $p \neq 1$ is harder and follows from the following good-λ inequality.

Theorem 1.2. *Given $0 < \eta < 1$ and $M > 2$ there exist constants $0 < \epsilon < 1$ and $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ and any inner function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$, we have that*

$$m(\{\xi \in \mathbb{T} : \log |f'(\xi)| \geq M\lambda \text{ and } A(f)(\xi) \leq \epsilon\lambda\}) \leq \eta\, m(\{\xi \in \mathbb{T} : \log |f'(\xi)| \geq \lambda\}).$$

Roughly speaking, Theorem 1.2 says in a quantitative way that, even though $\log |f'|$ and $A(f)$ are not pointwise comparable, the set of points $\xi \in \mathbb{T}$ where $\log |f'(\xi)|$ is large and $A(f)(\xi)$ is small, has a small measure. Our result is inspired by the classical good λ-inequalities of Fefferman and Stein which relate the size of the non-tangential maximal function of a harmonic function and the size of its Lusin area function. See [5]. These estimates have been complemented and extended to different contexts. See for instance [2]. The proof of Theorem 1.2 uses stopping time arguments, Green's Formula and the identity (5) and it is the most technical part of the paper.

Recall that any inner function f factors as $f = BS_\mu$ where B is a Blaschke product and S_μ is a singular inner function defined as

$$S_\mu(z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)\right), \quad z \in \mathbb{D}, \tag{6}$$

where μ is a finite positive Borel measure on $\mathbb{T}$ singular with respect to the Lebesgue measure m. Given a closed set $E \subset \mathbb{T}$ let $\operatorname{dist}(\xi, E)$ denote the distance from ξ to E. Fix $0 < p < \infty$. A closed subset $E \subset \mathbb{T}$ of Lebesgue measure zero will be called a $\mathcal{C}(p)$-Beurling–Carleson set if

$$\int_{\mathbb{T}} |\log \operatorname{dist}(\xi, E)|^p\, dm(\xi) < \infty.$$

Note that $\mathcal{C}(1)$-Beurling–Carleson sets are the classical Beurling–Carleson sets which appear in many problems in function theory. See for instance the references in [12]. Our next result relates the support of a singular measure μ with the $\mathcal{L}(p)$-entropy of S_μ, extending the corresponding result for $p = 1$ proved in [9]. See also [12, Theorem 2].

Theorem 1.3. *Fix $0 < p < \infty$. Let μ be a finite positive Borel measure on $\mathbb{T}$ which is singular with respect m and let S_μ be the corresponding singular inner function defined in* (6)*. Consider the following conditions:*

1. *The measure μ is supported in a $\mathcal{C}(p)$-Beurling–Carleson set.*
2. *S_μ has finite $\mathcal{L}(p)$-entropy, that is, $\log |S'_\mu| \in L^p(\mathbb{T})$.*
3. *For every $0 < c < 1$, the integral*
$$\int_{\{z \in \mathbb{D} : |S_\mu(z)| < c\}} \frac{|\log(1 - |z|)|^{p-1}}{1 - |z|}\, dA(z)$$
converges.
4. *There exists $0 < c < 1$ such that the integral*
$$\int_{\{z \in \mathbb{D} : |S_\mu(z)| < c\}} \frac{|\log(1 - |z|)|^{p-1}}{1 - |z|}\, dA(z)$$
converges.

(5) The measure μ is concentrated in a countable union of $\mathcal{C}(p)$-Beurling–Carleson sets.

Then, the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ *hold.*

Fix $0 < p < \infty$. A positive finite Borel measure μ on $\mathbb{T}$ which is singular with respect m will be called $\mathcal{L}(p)$-invisible if for every non-trivial positive measure $\nu \leq \mu$, the corresponding singular inner function S_ν satisfies $\log |S'_\nu| \notin L^p(\mathbb{T})$. The case $p = 1$ was already considered in [9]. As a direct consequence of Theorem 1.3, we have the following description of $\mathcal{L}(p)$-invisible measures which in the case $p = 1$ was already proved in [9].

Corollary 1.4. *A positive finite Borel measure μ on $\mathbb{T}$ which is singular with respect to the Lebesgue measure m, is $\mathcal{L}(p)$-invisible if and only if $\mu(E) = 0$ for any $\mathcal{C}(p)$-Beurling–Carleson set E.*

This paper is organised in three further sections. The next section is devoted to some auxiliary results which will be used in the proofs of Theorem 1.1 and Theorem 1.2 which are given in Section 3. Lastly, in Section 4 we prove Theorem 1.3.

We will use the letter C to denote different absolute constants whose values may change from line to line.

2. Preliminaries

Given an analytic mapping $f : \mathbb{D} \to \mathbb{D}$ consider

$$G(f)(z) = \log \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}. \tag{7}$$

For a point $z \in \mathbb{D} \setminus \{0\}$ we will denote by $I(z)$ the subarc of $\mathbb{T}$, centered at $z/|z|$ with $m(I(z)) = 1 - |z|$. We start with some elementary properties of $G(f)$ which will be used in the proof of Theorem 1.2.

Lemma 2.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic mapping and consider $G(f)$ as defined in* (7)*. Then,*

(a) There exists a universal constant $C > 0$ such that $|G(f)(z) - G(f)(w)| \leq C d_h(z, w)$ for any $z, w \in \mathbb{D}$.

(b) There exists a universal constant $C > 0$ such that $G(f)(w) \geq G(f)(z) - C$ for any $z, w \in \mathbb{D}$ with $I(w) \subset I(z)$.

(c) The inequality $|\log |z|| |\nabla G(f)(z)| \leq 4$ holds for any $z \in \mathbb{D}$ with $\frac{1}{2} < |z| < 1$.

Proof. (a) Note that

$$G(f)(z) - G(f)(w) = \log \frac{1 - |f(z)|^2}{1 - |f(w)|^2} + \log \frac{1 - |w|^2}{1 - |z|^2}, \quad z, w \in \mathbb{D}.$$

Since there exists an absolute constant $C_1 > 0$ such that

$$\left| \log \frac{1 - |w|^2}{1 - |z|^2} \right| \leq C_1 d_h(z, w), \quad z, w \in \mathbb{D}, \tag{8}$$

Schwarz's Lemma gives that $|G(f)(z) - G(f)(w)| \leq 2C_1 d_h(z, w)$, $z, w \in \mathbb{D}$, which gives the estimate in (a).

(b) Note that

$$G(f)(w) - G(f)(z) = \log \frac{1 - |z|^2}{1 - |w|^2} + d_h(0, f(z)) - d_h(0, f(w)) + 2 \log \frac{1 + |f(w)|}{1 + |f(z)|},$$

for any $z, w \in \mathbb{D}$. Observe that there exists an absolute constant $C_1 > 0$ such that for any $z, w \in \mathbb{D}$ with $I(w) \subset I(z)$, we have

$$\log \frac{1-|z|^2}{1-|w|^2} \geq d_h(z,w) - C_1.$$

The triangular inequality gives that

$$G(f)(w) - G(f)(z) \geq d_h(z,w) - C_1 - d_h(f(z), f(w)) - 2\log 2$$

and Schwarz's Lemma finishes the proof.

(c) Given a differentiable function $u : \mathbb{D} \to \mathbb{R}$ we use the notation $\nabla u(z) = u_x(z) + iu_y(z)$, $z \in \mathbb{D}$. A simple computation shows that $\nabla \log(1-|f(z)|^2) = -2\overline{f'(z)}f(z)/(1-|f(z)|^2)$, $z \in \mathbb{D}$. Thus

$$\nabla G(f)(z) = 2\frac{z}{1-|z|^2} - 2\frac{\overline{f'(z)}f(z)}{1-|f(z)|^2}, \quad z \in \mathbb{D}.$$

Since $-\log|z| \leq 1-|z|^2$ for any $\frac{1}{2} < |z| < 1$, applying Schwarz-Pick inequality we get that

$$|\log|z|||\nabla G(f)(z)| \leq 2\Big|z - \overline{f'(z)}f(z)\frac{1-|z|^2}{1-|f(z)|^2}\Big| \leq 2+2D_h(f)(z) \leq 4, \quad \frac{1}{2} < |z| < 1.$$

□

For $1 \leq p < \infty$, let $\mathcal{N}_p$ be the class of analytic functions g in $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_{\mathbb{T}} (\log^+ |g(r\xi)|)^p dm(\xi) < \infty.$$

These classes were first considered by I. Privalov. Note that $\mathcal{N}_1$ is the Nevanlinna class. For $p > 1$, Privalov proved that $g \in \mathcal{N}_p$ if and only if g factors as $g = IE$, where I is an inner function and E is an outer function whose boundary values satisfy $\log|E| \in L^p(\mathbb{T})$. See [17, pag. 93]. Our next auxiliary result relates inner functions with finite $\mathcal{L}(p)$-entropy with the corresponding Privalov class.

Lemma 2.2. *Let f be an inner function and $1 \leq p < \infty$. Then f has finite $\mathcal{L}(p)$-entropy if and only if $f' \in \mathcal{N}_p$.*

Proof. Since $G(f)(r\xi) \to \log|f'(\xi)|$ as $r \to 1$ for any $\xi \in \mathbb{T}$, Schwarz-Pick inequality and part (b) of Lemma 2.1 provide an absolute constant $C > 0$ such that

$$|f'(r\xi)| \leq \frac{1-|f(r\xi)|^2}{1-r^2} \leq C|f'(\xi)|, \quad 0 < r < 1, \xi \in \mathbb{T}.$$

Hence if f has finite $\mathcal{L}(p)$-entropy, then $f' \in \mathcal{N}_p$. Conversely assume that $f' \in \mathcal{N}_p$. Then f' has radial limits at m-almost every point of the unit circle and thus, Fatou's lemma finishes the proof.

□

In the proof of Theorem 1.2 we will also use the following technical result.

Lemma 2.3. *There exists an absolute constant $C > 0$ such that for any analytic mapping $f : \mathbb{D} \to \mathbb{D}$, for any $0 < \ell < 1$ and for any pair of points $z = |z|\xi_1, w = |w|\xi_2 \in \mathbb{D}$ with $|z| > \ell$ and $|w| > \ell$, we have*

$$\Big|\int_\ell^{|z|} \mu(f)(s\xi_1)\frac{2ds}{1-s^2} - \int_\ell^{|w|} \mu(f)(s\xi_2)\frac{2ds}{1-s^2}\Big| \leq C d_h(z,w). \tag{9}$$

Proof. Since $d_h(|z|,|w|) \leq d_h(z,w)$ for any $z,w \in \mathbb{D}$, it is sufficient to show that there exists a universal constant $C>0$ such that

$$|\int_\ell^{|z|} D_h(f)(s\xi_1)\frac{2ds}{1-s^2} - \int_\ell^{|w|} D_h(f)(s\xi_2)\frac{2ds}{1-s^2}| \leq C d_h(z,w),\ z,w \in \mathbb{D}. \tag{10}$$

It is well known that

$$d_h(D_h(f)(z), D_h(f)(w)) \leq 2d_h(z,w), \quad z,w \in \mathbb{D},$$

(see [3, Corollary 3.7]). Using the elementary estimate $2x \leq \log(1+x) - \log(1-x)$ for $0 \leq x < 1$, we deduce that

$$\rho(D_h(f)(s\xi_1), D_h(f)(s\xi_2)) \leq d_h(s\xi_1, s\xi_2), \quad 0 \leq s < 1.$$

Hence

$$|D_h(f)(s\xi_1) - D_h(f)(s\xi_2)| \leq 2d_h(s\xi_1, s\xi_2), \quad 0 \leq s < 1.$$

We first assume that $|z| = |w|$ and $d_h(z,w) \leq 1$. Then there exists a universal constant $C_1 > 0$ such that $d_h(s\xi_1, s\xi_2) \leq C_1|z-w|/(1-s^2)$ for any $0 < s < |z|$. Hence the left hand side of (10) is bounded by

$$4C_1 \int_\ell^{|z|} \frac{|z-w|}{(1-s^2)^2} ds \leq 4C_1 \frac{|z-w|}{1-|z|} \leq C d_h(z,w),$$

where $C>0$ is an absolute constant.

Assume now that $d_h(z,w) \leq 1$ but $|z| \neq |w|$. We can assume $|z| < |w|$. Let I denote the left hand side of (10) and consider the point $z^* = |z|\xi_2$. The previous argument shows that

$$I \leq C d_h(z, z^*) + \int_{|z|}^{|w|} D_h(f)(s\xi_2)\frac{2ds}{1-s^2}.$$

Since the last integral is bounded by $d_h(z^*, w)$ we deduce that $I \leq C d_h(z,z^*) + d_h(z^*, w)$. Now the estimate $d_h(z,z^*) + d_h(z^*,w) \leq C_2 d_h(z,w)$ finishes the proof in this case.

Finally assume that $d_h(z,w) > 1$. Let N be the positive integer satisfying $N < d_h(z,w) \leq N+1$. Pick points $z_0 = z, z_1, \ldots, z_N = w$ with $d_h(z_k, z_{k+1}) \leq 1$, $k = 0, 1, \ldots N-1$. The previous argument gives that

$$|\int_\ell^{|z_{k+1}|} D_h(f)(s\frac{z_{k+1}}{|z_{k+1}|})\frac{2ds}{1-s^2} - \int_\ell^{|z_k|} D_h(f)(s\frac{z_k}{|z_k|})\frac{2ds}{1-s^2}| \leq C.$$

Adding over $k = 0, \ldots, N-1$ one finishes the proof. □

Let $\Gamma(\xi,\alpha) = \{z \in \mathbb{D} : |z-\xi| < \alpha(1-|z|)\}$ be the Stölz angle with vertex at $\xi \in \mathbb{T}$ and aperture $\alpha > 1$. Given an analytic self-mapping f of the unit disc, consider its conical accumulated Möbius distortion defined as

$$B_\alpha(f)(\xi) = \int_{\Gamma(\xi,\alpha)} \mu(f)(z)\frac{dA(z)}{(1-|z|^2)^2}, \quad \xi \in \mathbb{T}. \tag{11}$$

It turns out that $B_\alpha(f)$ and $A(f)$ are pointwise comparable.

Lemma 2.4. *Given $\alpha > 1$ there exists a constant $C(\alpha) > 0$ such that for any analytic self-mapping f of the unit disc and any point $\xi \in \mathbb{T}$ one has*

$$C(\alpha)^{-1} A(f)(\xi) \leq B_\alpha(f)(\xi) \leq C(\alpha) A(f)(\xi).$$

Proof. By [3, Corollary 3.7], we have that $d_h(D_h(f)(z), D_h(f)(w)) \leq 2d_h(z,w)$ for any pair of points $z, w \in \mathbb{D}$. Thus, for any $C_1 > 0$ there exists a positive constant $C_2 > 1$ such that

$$C_2^{-1}\mu(f)(w) \leq \mu(f)(z) \leq C_2\mu(f)(w),$$

for any pair of points $z, w \in \mathbb{D}$ with $d_h(z,w) \leq C_1$. Hence fixed $\alpha > 1$, there exists a constant $C = C(\alpha) > 1$ such that

$$C^{-1}\mu(f)(r\xi) \leq \mu(f)(z) \leq C\mu(f)(r\xi)$$

for any pair of points $z, r\xi \in \mathbb{D}$ with $z \in \Gamma(\xi, \alpha)$ and $(1-r)/2 < 1 - |z| < 2(1-r)$. We deduce that

$$C^{-1}A(f)(\xi) \leq B_\alpha(f)(\xi) \leq CA(f)(\xi).$$

□

3. Proof of Theorem 1.1

We start with the pointwise estimate mentioned in the Introduction which was already proved in [7] and [13]. For the sake of completeness we give its short proof.

Lemma 3.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with $f(0) = 0$. Then $A(f)(\xi) \leq \log|f'(\xi)|$ for any $\xi \in \mathbb{T}$.*

Proof. Fix $\xi \in \mathbb{T}$. We can assume $|f'(\xi)| < \infty$ and hence f has radial limit of modulus 1 along the radius ending at ξ. Since

$$d_h(f(R\xi), 0) \leq \int_0^R D_h(f)(r\xi)\frac{2dr}{1-r^2}, \quad 0 < R < 1,$$

we have

$$\int_0^R (1 - D_h(f)(r\xi))\frac{2dr}{1-r^2} \leq d_h(0, R\xi) - d_h(f(R\xi), 0) =$$
$$= \log\frac{1 - |f(R\xi)|}{1-R} + \log\frac{1+R}{1+|f(R\xi)|}, \quad 0 < R < 1.$$

The estimate follows by taking $R \to 1$.

□

We now introduce some notation. Given an arc $I \subset \mathbb{T}$ we consider the Carleson box $Q = Q(I) = \{r\xi : \xi \in I, 0 < 1 - r \leq m(I)\}$. If $Q = Q(I)$ is a Carleson box, it is costumary to denote $\ell(Q) = m(I)$. Given a Carleson box Q denote by $I(Q) = \{z/|z| : z \in Q\}$ its radial projection on the unit circle and $T(Q) = \{z \in Q : 1 - |z| = \ell(Q)\}$ its top side. Note that $I(Q(I)) = I$. Also $\xi(I)$ denotes the center of the arc I and $z(Q) = (1 - \ell(Q))\xi(I(Q))$ the center of $T(Q)$. The dyadic decomposition of an arc $I \subset \mathbb{T}$ into dyadic subarcs $\{I_{j,k}\}$ gives the corresponding dyadic decomposition of the Carleson box $Q(I)$ into dyadic subboxes $\{Q(I_{j,k})\}$.

Given a Carleson box Q we consider the local accumulated Möbius distortion defined as

$$A_Q(f)(\xi) = \int_{1-\ell(Q)}^1 \mu(f)(r\xi)\frac{2dr}{1-r^2}, \quad \xi \in I(Q). \tag{12}$$

Next auxiliary result is the main step in the proof of Theorem 1.2.

Lemma 3.2. *There exists a universal constant $C > 0$ such that the following statement holds. Let f be an inner function with $f(0) = 0$. Let $a, b, c > 0$ be positive constants and let $Q \subset \{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$ be a Carleson box such that $G(f)(z(Q)) = a$. Consider the set*

$$E(b,c) = \{\xi \in I(Q) : \log|f'(\xi)| \geq b \text{ and } A_Q(f)(\xi) \leq c\}.$$

Then

$$(b-a)m(E(b,c)) \leq C(c+1)m(I(Q)).$$

Proof. Given a Carleson box $R \subset Q$ we denote by $L(R) = [0, z(R)] \cap Q$ the piece of the radial segment joining the origin and the point $z(R)$, contained in Q. By Lemma 2.3 there exists a universal constant $C_0 > 0$ such that for any Carleson box $R \subset Q$ and any point $w \in T(R)$ we have

$$\left| \int_{L(R)} \mu(f)(z) \frac{2|dz|}{1-|z|^2} - \int_{[0,w]\cap Q} \mu(f)(z) \frac{2|dz|}{1-|z|^2} \right| \leq C_0.$$

Now we use the following stopping time argument. Let $\{R_j\}$ be the collection of maximal dyadic Carleson sub-boxes of Q such that

$$\int_{L(R_j)} \mu(f)(z) \frac{2|dz|}{1-|z|^2} > c + C_0.$$

Note that the maximality and Lemma 2.3 give that there exists a universal constant $C > 0$ such that

$$c + C_0 < \int_{L(R_j)} \mu(f)(z) \frac{2|dz|}{1-|z|^2} < c + C_0 + C. \tag{13}$$

The choice of C_0 gives that

$$\int_{[0,w]\cap Q} \mu(f)(z) \frac{2|dz|}{1-|z|^2} > c, \tag{14}$$

for any $w \in T(R_j)$ and any $R_j \in \{R_j\}$. Let us define $B = Q \setminus \cup_j R_j$ and $B_n = B \cap \{z \in \mathbb{D} : |z| \leq r_n\}$ where $r_n = 1 - 2^{-n}\ell(Q)$. Applying Green's formula to the functions $G(f)$ defined in (7) and $v(z) = -\log|z|$, we have that

$$\int_{\partial B_n} \left(v(z) \frac{\partial G(f)(z)}{\partial n} - G(f)(z) \frac{\partial v(z)}{\partial n} \right) d\sigma(z) = \int_{B_n} v(z) \Delta(G(f)(z)) dA(z), \tag{15}$$

where dA denotes the area measure and $d\sigma$ denotes the linear measure on ∂B_n. Let us name these integrals as

$$I_1 = \int_{\partial B_n} v(z) \frac{\partial G(f)(z)}{\partial n} d\sigma(z),$$

$$I_2 = \int_{\partial B_n} G(f)(z) \frac{\partial v(z)}{\partial n} d\sigma(z),$$

and

$$I_3 = \int_{B_n} v(z) \Delta(G(f)(z)) dA(z).$$

First we estimate I_1. By Lemma 2.1, part (c), we have

$$|I_1| = |\int_{\partial B} v(z) \frac{\partial G(f)(z)}{\partial n} d\sigma| \leq 4\sigma(\partial B) \leq Cm(I(Q)). \tag{16}$$

For I_2, note that one can decompose $\partial B_n = C_n \cup D_n$ with

$$C_n = T(Q) \cup (\cup_{\mathcal{A}_n} T(R_j)) \cup J_n,$$

where $\mathcal{A}_n$ is the subfamily of those R_j with $\ell(R_j) \geq 1 - r_n$ and $J_n \subset \{z \in \mathbb{D} : |z| = r_n\}$, while D_n is contained in a finite union of radius emanating from the origin. Roughly speaking, ∂B_n is decomposed in a circular part C_n and a radial part D_n. Since v is radial, its normal derivative vanishes on D_n. On the other hand

$$\frac{\partial v(z)}{\partial n} = \frac{-1}{|z|}, z \in T(Q); \quad \frac{\partial v(z)}{\partial n} = \frac{1}{|z|}, z \in C_n \setminus T(Q).$$

Hence

$$I_2 = \sum_{\mathcal{A}_n} \frac{1}{|z(R_j)|} \int_{T(R_j)} G(f) d\sigma + \frac{1}{r_n} \int_{J_n} G(f) d\sigma - \frac{1}{|z(Q)|} \int_{T(Q)} G(f) d\sigma. \tag{17}$$

Since $G(f)(z_Q) = a$ and $m(I(Q)) = \sigma(T(Q))/|z(Q)|$, part (a) of Lemma 2.1 gives that there exists an absolute constant $C > 0$ such that

$$\left| \frac{1}{|z(Q)|} \int_{T(Q)} G(f) d\sigma - a m(I(Q)) \right| \leq C m(I(Q)) \tag{18}$$

Since

$$m(I(Q)) = \sum_{\mathcal{A}_n} \ell(R_j) + m(r_n^{-1} J_n),$$

from (17) and (18), we deduce that

$$I_2 = \sum_{\mathcal{A}_n} \frac{1}{|z(R_j)|} \int_{T(R_j)} (G(f) - a) d\sigma + \frac{1}{r_n} \int_{J_n} (G(f) - a) d\sigma + K, \tag{19}$$

where $|K| \leq C m(I(Q))$.

Let us now turn our attention to I_3. By identity (5), we have

$$I_3 = \int_{B_n} v(z) \Delta(G(f)(z)) dA(z) \leq 8 \int_{B_n} |\log |z|| \mu(f)(z) \frac{dA(z)}{(1 - |z^2|)^2}.$$

Let $L(\xi) = [0, \xi] \cap B_n$. Now Fubini's theorem, estimate (13) and the choice of C_0 give that

$$I_3 \leq C \int_{I(Q)} \int_{L(\xi)} \mu(f)(r\xi) \frac{2dr}{1 - r^2} d\xi \leq C(C + 2C_0 + c) m(I(Q)). \tag{20}$$

Combining the above estimates (16), (19), (20), from Green's formula in (15) we derive that there exists a universal constant $C > 0$ such that

$$\frac{1}{r_n} \int_{J_n} (G(f) - a) d\sigma + \sum_{\mathcal{A}_n} \frac{1}{|z(R_j)|} \int_{T(R_j)} (G(f) - a) d\sigma \leq C(1 + c) m(I(Q)).$$

By part (b) of Lemma 2.1, there exists a universal constant $C_1 > 0$ such that $G(f)(z) - a + C_1 \geq 0$ for any $z \in \cup_j R_j \cup J_n$. We deduce that there exist a universal constant $C > 0$ such that

$$\frac{1}{r_n} \int_{J_n} (G(f) - a) d\sigma \leq C(c + 1) m(I(Q)). \tag{21}$$

Consider $E_n(b,c) = \{z \in Q : |z| = r_n, G(f)(z) \geq b \text{ and } A_Q(f)(z/|z|) \leq c\}$. Note that by (14) we have $J_n \supset E_n(b,c)$. Thus, by (21)

$$C(1+c)m(I(Q)) \geq \int_{E_n(b,c)} (G(f) - a)d\sigma \geq (b-a)m(E_n(b,c)).$$

Therefore it suffices to notice that $\limsup m(E_n(b,c)) \geq m(E(b,c))$ which follows from the observation that $E(b,c) \subset \liminf r_n^{-1} E_n(b,c)$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $C_0 > 0$ be the maximum of the universal constants C appearing in Lemma 2.1, and let $\lambda_0 > 3C_0$ be a constant to be fixed later. Fix $\lambda > \lambda_0$. Let $\mathcal{A} = \{Q_j\}$ be the collection of maximal dyadic Carleson boxes Q such that

$$\sup\{G(f)(z) : z \in T(Q)\} > \lambda + \lambda_0.$$

Since $G(f)(0) = 0$, part (a) of Lemma 2.1 gives that $\ell(Q_j) < 1/2$ for any $Q_j \in \mathcal{A}$. Fix $Q_j \in \mathcal{A}$ and let Q_j^* be the dyadic Carleson box which contains Q_j with $\ell(Q_j^*) = 2\ell(Q_j)$. The maximality gives that $\sup\{G(f)(z) : z \in T(Q_j^*)\} < \lambda + \lambda_0$. Since part (a) of Lemma 2.1 gives that $|G(f)(z) - G(f)(w)| \leq 2C_0$ for any pair of points $z \in T(Q_j)$, $w \in T(Q_j^*)$, we deduce that

$$\lambda \leq G(f)(z(Q_j)) \leq \lambda + \lambda_0 + 2C_0. \tag{22}$$

By part (b) of Lemma 2.1 we have

$$\cup_j I(Q_j) \subset \{\xi \in \mathbb{T} : \log|f'(\xi)| > \lambda\}. \tag{23}$$

By construction we have $\log|f'(\xi)| \leq \lambda + \lambda_0$ for any $\xi \in \mathbb{T} \setminus \cup I(Q_j)$. Fix $M > 2$. Since $\lambda > \lambda_0$ we have

$$\{\xi \in \mathbb{T} : \log|f'(\xi)| > M\lambda\} \subset \cup I(Q_j).$$

Fix $Q_j \in \mathcal{A}$. Apply Lemma 3.2 with $a = G(f)(z(Q_j))$, $b = M\lambda$ and $c = \epsilon\lambda$ to obtain

$$m(\{\xi \in I(Q_j) : \log|f'(\xi)| > M\lambda, A(f)(\xi) < \epsilon\lambda\}) \leq C\frac{\epsilon\lambda + 1}{M\lambda - G(f)(z(Q_j))}\ell(Q_j),$$

where $C > 0$ is an absolute constant. By (22) we have

$$\frac{\epsilon\lambda + 1}{M\lambda - G(f)(z(Q_j))} \leq \frac{\epsilon\lambda + 1}{(M-1)\lambda - \lambda_0 - 2C_0} \leq \frac{\epsilon + 1/\lambda_0}{(M-2) - 2C_0/\lambda_0}.$$

Given $0 < \eta < 1$, taking $0 < \epsilon < 1$ sufficiently small and $\lambda_0 > 0$ sufficiently large, we deduce

$$C\frac{\epsilon + 1/\lambda_0}{(M-2) - 2C_0/\lambda_0} < \eta.$$

Hence

$$m(\{\xi \in I(Q_j) : \log|f'(\xi)| > M\lambda, A(f)(\xi) < \epsilon\lambda\}) \leq \eta\,\ell(Q_j).$$

Summing over $j = 1, 2, \ldots$ and applying (23), the proof is completed. □

Theorem 1.1 follows from Theorem 1.2 by standard methods.

Proof of Theorem 1.1. Composing f with an automorphism of the unit disc, if necessary, we can assume that $f(0) = 0$. By Lemma 3.1 we only need to show that $\log|f'| \in L^p(\mathbb{T})$ if $A(f) \in L^p(\mathbb{T})$. Since $f(0) = 0$, Schwarz lemma gives that $|f'(\xi)| \geq 1$ for any $\xi \in \mathbb{T}$. We use the notation $E(\lambda) = \{\xi \in \mathbb{T} : \log|f'(\xi)| \geq \lambda\}$

for $\lambda > 0$. Fix $M > 2$ and apply Theorem 1.2 with $\eta = 2^{-1}M^{-p}$ to find constants $0 < \epsilon < 1$ and $\lambda_0 > 1$ such that

$$m(\{\xi \in E(M\lambda) : A(f)(\xi) \leq \epsilon\lambda\}) \leq \frac{1}{2M^p} m(E(\lambda)),$$

for any $\lambda > \lambda_0$. Then

$$\begin{aligned}
&\int_{E(M\lambda_0)} (\log|f'|)^p = pM^p \int_{\lambda_0}^{\infty} \lambda^{p-1} m(E(M\lambda)) d\lambda \\
&\leq \frac{p}{2}\int_{\lambda_0}^{\infty} \lambda^{p-1} m(E(\lambda)) d\lambda + pM^p \int_{\lambda_0}^{\infty} \lambda^{p-1} m(\{\xi \in \mathbb{T} : A(f)(\xi) \geq \epsilon\lambda\}) d\lambda \\
&\leq \frac{1}{2}\|\log|f'|\|_{L^p}^p + \left(\frac{M}{\epsilon}\right)^p \|A(f)\|_{L^p}^p.
\end{aligned}$$

Hence

$$\|\log|f'|\|_{L^p}^p \leq \frac{1}{2}\|\log|f'|\|_{L^p}^p + M^p\epsilon^{-p}\|A(f)\|_{L^p}^p + (M\lambda_0)^p,$$

which finishes the proof. □

Note that the previous proof shows that given $0 < p < \infty$ there exist constants $C_1, C_2 > 0$ depending on p, such that for any analytic mapping f of the unit disc with $f(0) = 0$, we have

$$\|A(f)\|_{L^p}^p \leq \|\log|f'|\|_{L^p}^p \leq C_1\|A(f)\|_{L^p}^p + C_2.$$

The authors do not know if one can take $C_2 = 0$ in the estimate above.

4. Singular Inner Functions and $\mathcal{C}(p)$-Beurling–Carleson sets

We will use the following auxiliary result from [12, Lemma 3.1].

Lemma 4.1. *Let $E \subset \mathbb{T}$ be a closed set of Lebesgue measure zero. Denote by $\{I_j\}$ the collection of its complementary arcs, that is, $\mathbb{T} \setminus E = \cup I_j$. Fix $0 < p < \infty$. The following conditions are equivalent:*

(a) E is a $\mathcal{C}(p)$-Beurling–Carleson set.

(b) $\sum |I_j||\log|I_j||^p < \infty$.

(c) $\sum |I||\log|I||^{p-1} < \infty$, where the sum is taken over all dyadic arcs of $\mathbb{T}$ that meet E.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3.

(1) ⇒ (2) Assume μ is supported in a $\mathcal{C}(p)$-Beurling–Carleson set $E \subset \mathbb{T}$. Since

$$\left|\frac{S_\mu'(z)}{S_\mu(z)}\right| = \left|\int_{\mathbb{T}} \frac{2\xi}{(\xi - z)^2} d\mu(\xi)\right| \leq 2\mu(\mathbb{T})\,\mathrm{dist}(z, E)^{-2}, z \in \mathbb{T} \setminus E,$$

it suffices to show that the integral

$$\int_{\mathbb{T}\setminus E} |\log \mathrm{dist}(z, E)|^p dm(z),$$

converges. Consider the complementary arcs $\{I_j\}$ of E, that is, $\mathbb{T} \setminus E = \cup_j I_j$ to write the above integral as

$$\sum_j \int_{I_j} |\log \mathrm{dist}(z, E)|^p dm(z).$$

Since there exists a constant $C > 0$ such that

$$\int_{I_j} |\log \operatorname{dist}(z, E)|^p dm(z) \leq C|I_j||\log |I_j||^p, \quad j = 1, 2, \dots,$$

Lemma 4.1 finishes the proof.

(2) $\Rightarrow$ (3) This implication holds for any inner function f. Recall that Ahern proved the estimate $|f'(r\xi)| \leq 4|f'(\xi)|$ for any $0 \leq r < 1$, any $\xi \in \mathbb{T}$ and any inner function f. See Lemma 6.1 of [1] or [16]. Then

$$1 - |f(r\xi)| \leq \int_r^1 |f'(s\xi)| ds \leq 4|f'(\xi)|(1 - r), \quad 0 < r < 1, \xi \in \mathbb{T}.$$

Fix $0 < c < 1$. Using the notation

$$I = \int_{\{z \in \mathbb{D} : |f(z)| < c\}} \frac{|\log(1 - |z|)|^{p-1}}{1 - |z|} dA(z),$$

we have that

$$\begin{aligned} I &= \int_0^1 \frac{r|\log(1-r)|^{p-1}}{1-r} m(\{\xi \in \mathbb{T} : |f(r\xi)| < c\}) dr \\ &\leq \int_0^1 \frac{|\log(1-r)|^{p-1}}{1-r} m(\{\xi \in \mathbb{T} : \log|f'(\xi)| > \log \frac{1-c}{4(1-r)}\}) dr. \end{aligned}$$

Using the change of variables $x = -\log(1 - r)$, we obtain

$$I \leq \int_0^\infty x^{p-1} m(\{\xi \in \mathbb{T} : \log |f'(\xi)| > x + \log(1 - c) - 2\log 2\}) dx$$

which is finite whenever $\log |f'| \in L^p(\mathbb{T})$.

(3) $\Rightarrow$ (4) This implication is obvious.

(4) $\Rightarrow$ (5) In order to prove this implication we will make use of the heavy-light decomposition from [12, Section 4.1] which we now briefly recall. Let μ be a positive finite Borel singular measure on $\mathbb{T}$ and fix $M > 0$. Let $\{I_j^{(1)}\}$ be the family of maximal dyadic arcs of $\mathbb{T}$ such that

$$\frac{\mu(I_j^{(1)})}{m(I_j^{(1)})} \geq M.$$

In each arc $I_j^{(1)}$, consider the family $\{J_k^{(1)}\}$ of maximal dyadic subarcs of $I_j^{(1)}$ such that

$$\frac{\mu(J_k^{(1)})}{m(J_k^{(1)})} \leq \frac{M}{100}.$$

In each $J_k^{(1)}$ we again find the maximal dyadic arcs $\{I_j^{(2)}\}$ contained in $J_k^{(1)}$ such that

$$\frac{\mu(I_j^{(2)})}{m(I_j^{(2)})} \geq M.$$

Continuing this construction we are left with two families of arcs $\{I_j^{(l)}\}$ and $\{J_k^{(l)}\}$ which satisfy the following properties:

(1) For any j, l we have that

$$\sum_{k:J_k^{(l)}\subset I_j^{(l)}} m(J_k^{(l)}) = m(I_j^{(l)}).$$

(2) For any j, l we have that

$$\sum_{k:I_k^{(l+1)}\subset J_j^{(l)}} m(I_k^{(l+1)}) \leq \frac{1}{M}\mu(J_j^{(l)}) \leq \frac{m(J_j^{(l)})}{100}.$$

(3) The measure μ is concentrated on

$$\bigcup_{j,l}\left(\overline{I_j^{(l)}} \setminus \bigcup_{k:J_k^{(l)}\subset I_j^{(l)}} (J_k^{(l)})^\circ\right),$$

where $(J_k^{(l)})^\circ$ denotes the interior of $J_k^{(l)}$

The arcs $J_k^{(l)}$ are called light and $I_j^{(l)}$ are called heavy arcs of the measure μ. Now let us return to the proof of (4) $\Rightarrow$ (5). Let $\{I_j^{(i)}\}, \{J_k^{(l)}\}$ be the heavy-light decomposition of μ. Then it suffices to show that $E_{j,l} = \overline{I}_j^{(l)} \setminus \cup_k J_k^{(l)}$ is a $\mathcal{C}(p)$-Beurling–Carleson set for any j, l. Fix $E_{j,l}$. Let $\mathcal{B} = \mathcal{B}(j, l)$ be the collection of dyadic arcs I of $\mathbb{T}$ with $m(I) < m(I_j^l)$ that meet $E_{j,l}$. Let $I \in \mathcal{B}$ and write $T_I = \{z \in \mathbb{D} : z/|z| \in I, \frac{|I|}{2} < 1 - |z| < |I|\}$. Then by construction, $\mu(I) > Mm(I)/100$. Hence there exists an absolute constant $C > 0$ such that $P[\mu](z) \geq CM$ for all $z \in T_I$. Here $P[\mu]$ denotes the Poisson integral of the measure μ. Since there exists an absolute constant $C_1 > 0$ such that

$$\int_{T_I} \frac{|\log(1-|z|)|^{p-1}}{1-|z|} dA(z) \geq C_1|I||\log|I||^{p-1},$$

we get that

$$C_1 \sum_{I\in\mathcal{B}} |I||\log|I||^{p-1} \leq \int_{\{z\in\mathbb{D}:P[\mu](z)\geq CM\}} \frac{|\log(1-|z|)|^{p-1}}{1-|z|} dA(z).$$

Choosing an appropriate constant M, Lemma 4.1 finishes the proof . $\square$

References

[1] Patrick Ahern. The mean modulus and the derivative of an inner function. *Indiana Univ. Math. J.*, 28(2):311–347, 1979.

[2] R. Bañuelos and C. N. Moore. *Probabilistic behavior of harmonic functions*, volume 175 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[3] A. F. Beardon and D. Minda. A multi-point Schwarz-Pick lemma. *J. Anal. Math.*, 92:81–104, 2004.

[4] M. Craizer. Entropy of inner functions. *Israel J. Math.*, 74(2-3):129–168, 1991.

[5] C. Fefferman and E. M. Stein. H^p spaces of several variables. *Acta Math.*, 129(3-4):137–193, 1972.

[6] J. B. Garnett. *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, first edition, 2007.

[7] P. Gumenyuk, M. Kourou, A. Moucha, and O. Roth. Hyperbolic distortion and conformality at the boundary, 2024. arXiv:2410.13965.

[8] M. Heins. On a class of conformal metrics. *Nagoya Math. J.*, 21:1–60, 1962.

[9] O. Ivrii. Prescribing inner parts of derivatives of inner functions. *J. Anal. Math.*, 139(2):495–519, 2019.
[10] O. Ivrii and U. Kreitner. Critical values of inner functions. *Adv. Math.*, 452:Paper No. 109815, 43, 2024.
[11] O. Ivrii and A. Nicolau. Analytic mappings of the unit disk which almost preserve hyperbolic area. *Proc. London Math. Soc.*, 129(5):e70001, 2024.
[12] O. Ivrii and A. Nicolau. Beurling-Carleson sets, inner functions and a semilinear equation. *Anal. PDE*, 17(7):2585–2618, 2024.
[13] O. Ivrii and M. Urbański. Inner functions and laminations, 2024. arXiv:2409.11963.
[14] D. Kraus. Critical sets of bounded analytic functions, zero sets of Bergman spaces and non-positive curvature. *Proc. Lond. Math. Soc. (3)*, 106(4):931–956, 2013.
[15] D. Kraus, O. Roth, and S. Ruscheweyh. A boundary version of Ahlfors' lemma, locally complete conformal metrics and conformally invariant reflection principles for analytic maps. *J. Anal. Math.*, 101:219–256, 2007.
[16] J. Mashreghi. *Derivatives of inner functions*, volume 31 of *Fields Institute Monographs*. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2013.
[17] I. I. Privalov. *Graničnye svoĭstva analitičeskih funkciĭ.* Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 2d ed.].
[18] J. H. Shapiro. *Composition operators and classical function theory.* Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

Department of Mathematical Sciences, Norwegian University of Sciences and Technology (NTNU), NO-7491 Trondheim, Norway
Email address: konstantinos.bampouras@ntnu.no

Departament de Matemàtiques, Universitat Autònoma de Barcelona, and Centre de Recerca Matemàtica, 08193, Barcelona, Spain
Email address: artur.nicolau@uab.cat