

M_z -invariant subspaces in the Bloch space

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Abstract

We consider weak-star closed invariant subspaces of the shift operator in the classical Bloch space. We prove that any bounded analytic function decomposes into two factors, one which is cyclic and another one generating a proper invariant subspace, satisfying a certain permanence property, which in a certain way is opposite to cyclicity. Singular inner functions play the crucial role in this decomposition. We show in several different ways that the description of shift invariant subspaces generated by inner functions in the Bloch spaces deviates substantially from the corresponding description in the Bergman spaces, provided by the celebrated Korenblum and Roberts Theorem. Furthermore, the relationship between invertibility and cyclicity is also investigated and we provide an invertible function in the Bloch space which is not cyclic therein. Our results answer several open questions stated in the early nineties.

1 Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{B}$ be the classical Bloch space of analytic functions f in $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|)|f'(z)| < \infty.$$

Taking the closure of analytic polynomials in the norm above we obtain the so-called little Bloch space $\mathcal{B}_0$, which is a separable Banach subspace of $\mathcal{B}$ consisting of functions f satisfying

$$\lim_{|z| \rightarrow 1^-} (1 - |z|)|f'(z)| = 0.$$

Functions in the Bloch space are intimately related to conformal mappings, and as such they are regarded as a crucial objects in geometric function theory, see [21] for further details. Let H^∞ designate the Banach space of bounded analytic functions f in $\mathbb{D}$ equipped with the usual norm $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$, and recall that $H^\infty \subset \mathcal{B}$ by Schwarz lemma.

The Bloch space may be regarded as the limit when $p \rightarrow \infty$ of the Bergman spaces B_p of analytic functions f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where dA denotes the area measure on $\mathbb{D}$. For $1 \leq p < \infty$, we denote by W^p the Sobolev space of analytic functions g on $\mathbb{D}$ satisfying

$$\|g\|_{W^p}^p := |g(0)|^p + \int_{\mathbb{D}} |g'(z)|^p dA(z) < \infty.$$

Let dm denotes Lebesgue measure in $\partial\mathbb{D}$. It is well-known that the dual space of W^1 can be identified with $\mathcal{B}$ considered in the Cauchy-pairing. More precisely, for any $f \in \mathcal{B}$ and $g \in W^1$, the limit

$$\ell_f(g) := \lim_{r \rightarrow 1^-} \int_{\partial\mathbb{D}} f(r\zeta) \overline{g(r\zeta)} dm(\zeta)$$

exists and f induces a unique bounded linear functional on W^1 . Similarly, the dual of $\mathcal{B}_0$ can be identified with W^1 , see for instance [4]. Let M_z denote the multiplication operator by the independent variable z , that is, the linear operator $M_z f(z) = zf(z)$, $f \in \mathcal{B}$. Our main purpose is to investigate a certain class of M_z -invariant subspaces in $\mathcal{B}$, and to adjust for the fact that $\mathcal{B}$ is not separable, we shall instead consider weak-star closed subspaces in $\mathcal{B}$ which are M_z -invariant. Given a function $f \in \mathcal{B}$, we shall denote by $[f]_{\mathcal{B}}$ the smallest weak-star closed M_z -invariant subspace containing f , that is, $[f]_{\mathcal{B}}$ is the weak-star closure of polynomial multiples of f . If $f \in \mathcal{B}_0$, then $[f]_{\mathcal{B}}$ is actually the norm-closure of polynomial multiples of f by Mazur's Theorem. A function $f \in \mathcal{B}$ is called *cyclic* in $\mathcal{B}$ if $[f]_{\mathcal{B}} = \mathcal{B}$. Since the set of analytic polynomials is weak-star dense in $\mathcal{B}$, we have that f is cyclic in $\mathcal{B}$ if and only if $1 \in [f]_{\mathcal{B}}$. It is worth mentioning that $[f]_{\mathcal{B}}$ may be much larger than the subspace of weak-star limits of sequences of polynomial multiples of f , as we shall demonstrate in Theorem 1.10. A function $f \in H^\infty$ is declared to satisfy the *permanence property* (in $\mathcal{B}$) if the corresponding M_z -invariant subspace satisfies $[f]_{\mathcal{B}} \cap H^\infty = fH^\infty$. The permanence property should be understood as an antithesis to cyclicity, in the sense that any bounded analytic function in $[f]_{\mathcal{B}}$ is divisible by f .

In this note, we shall mainly be concerned with M_z -invariant subspaces in $\mathcal{B}$ generated by bounded analytic functions. Recall that any $f \in H^\infty$ can be factored as $f = FBS_\mu$, where F denotes the outer factor of f defined by

$$F(z) = \exp \left(\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \log |f(\zeta)| dm(\zeta) \right), \quad z \in \mathbb{D},$$

the function S_μ is the singular inner factor of f given by

$$S_\mu(z) = \exp \left(- \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right), \quad z \in \mathbb{D},$$

where μ is a positive finite Borel measure on $\partial\mathbb{D}$ which is singular with respect to dm , while B denotes the Blaschke product encoding the zeros of f on $\mathbb{D}$, that is,

$$B(z) = \prod_{a:f(a)=0} \frac{|a|}{a} \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D}.$$

Functions in H^∞ having radial limits of modulus one at almost every point of the unit circle are called *inner functions*, and they are always of the form $\Theta = BS_\mu$. Given $f \in H^\infty$ with inner-outer factorization $f = FBS_\mu$, we denote by ν_f its associated Herglotz-Nevanlinna measure defined by

$$d\nu_f = -\log |f(\zeta)| dm(\zeta) + d\mu(\zeta) + \sum_{z:f(z)=0} (1-|z|^2)\delta_z,$$

where δ_z denotes the Dirac point mass measure at $z \in \mathbb{D}$. Note that if $\|f\|_{H^\infty} \leq 1$, then ν_f is positive. It was established in [12] that classical outer functions belonging to $\mathcal{B}$ must be cyclic, and the existence of a cyclic singular inner function in $\mathcal{B}_0$ was proved in [3]. Our first result provides a structural theorem for M_z -invariant subspaces generated by bounded analytic functions, where singular inner functions play the decisive role.

Theorem 1.1. *Let $f = FS_\mu B$ be in H^∞ , where F is outer, S_μ is singular inner and B is a Blaschke product. Then there exists a unique (up to sets of μ -measure zero) decomposition $\mu = \mu_P + \mu_C$ of μ , where μ_P, μ_C are mutually singular positive measures giving rise to the following dichotomy.*

- (i) *The inner function $\Theta_0 := BS_{\mu_P}$ generates a proper M_z -invariant subspace on $\mathcal{B}$ satisfying the permanence property, that is, $[\Theta_0]_{\mathcal{B}} \cap H^\infty = \Theta_0 H^\infty$.*
- (ii) *FS_{μ_C} is cyclic in $\mathcal{B}$.*

Theorem 1.1 should be viewed as a Bloch space version of Theorem 1 in [28] by Roberts in the context of Bergman spaces. Our result shows that in order to understand M_z -invariant subspaces in $\mathcal{B}$ generated by functions in H^∞ , it suffices to study the permanence property and cyclicity of singular inner functions. It turns out that these two notions have natural interpretations in the context of model spaces. For $0 < p < \infty$ let H^p denote the classical Hardy space of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_p^p = \sup_{0 < r < 1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p dm(\zeta) < \infty.$$

We recall that given an inner function Θ , the associated model space K_Θ is defined as $K_\Theta = H^2 \ominus \Theta H^2$, that is, the orthogonal complement of ΘH^2 in the classical Hardy space H^2 . It follows from the celebrated Beurling Theorem that the model spaces are the only invariant subspaces for the adjoint $(M_z)^*$ viewed as an operator on H^2 . A deep result of Aleksandrov says that functions with continuous extensions to $\partial\mathbb{D}$ in any model space are

always dense therein, see [1] and also [2] and [25] for recent generalizations. Recall that a closed set $E \subset \partial\mathbb{D}$ of Lebesgue measure zero is said to be a Beurling-Carleson set if

$$\int_{\partial\mathbb{D}} \log \text{dist}(\zeta, E) dm(\zeta) > -\infty.$$

It was recently established that $\bigcap_{p>1} W^p \cap K_\Theta$ is dense in K_Θ if and only if $\Theta = BS_\mu$ and μ is carried on a countable union of Beurling-Carleson sets [26], while $\bigcup_{p>1} W^p \cap K_{S_\mu} = \{0\}$ if and only if μ does not charge any Beurling-Carleson set [17]. We shall now consider what happens when $p = 1$ in our next result. Recall that for an inner function Θ , one defines the associated model space $K_\Theta^1 := H^1 \cap \Theta z \overline{H^1}$, interpreted in the sense of boundary values on $\partial\mathbb{D}$. In other words, a function $f \in H^1$ belongs to K_Θ^1 if there exists a function $g \in H^1$ with $g(0) = 0$ such that $\Theta(\zeta)\overline{f(\zeta)} = g(\zeta)$ for m -a.e. $\zeta \in \partial\mathbb{D}$.

Theorem 1.2. *Let μ be a positive, finite, Borel, singular measure in $\partial\mathbb{D}$ and let S_μ be the corresponding singular inner function. Then*

- (i) S_μ is cyclic in $\mathcal{B}$ if and only if $K_{S_\mu}^1 \cap W^1 = \{0\}$.
- (ii) The permanence property $[S_\mu]_{\mathcal{B}} \cap H^\infty = S_\mu H^\infty$ holds if and only if $K_{S_\mu}^1 \cap W^1$ is dense in $K_{S_\mu}^1$.

Viewing $\mathcal{B}$ as a limiting case of the Bergman spaces, we shall now make a few comparisons. In the context of Bergman spaces, the celebrated Theorem of Korenblum and Roberts asserts that any singular measure μ uniquely decomposes as $\mu = \mu_{\mathcal{C}} + \mu_{\mathcal{K}}$, where $\mu_{\mathcal{C}}$ is concentrated on a countable union of Beurling-Carleson sets and gives rise to a proper M_z -invariant subspace generated by $S_{\mu_{\mathcal{C}}}$ satisfying the permanence property on the Bergman space, while $\mu_{\mathcal{K}}$ charges no Beurling-Carleson set and induces a cyclic vector $S_{\mu_{\mathcal{K}}}$ therein, see [23], [28]. We shall see that the situation in the Bloch space is very different and the results of Korenblum and Roberts do not carry over to this setting. Although we have not been able to find complete geometric descriptions of singular measures μ for which the corresponding singular inner functions S_μ are cyclic or satisfy the permanence property in $\mathcal{B}$, we are still able to provide several related conditions which allows us to answer various open questions and problems posed in [12], [3], [4] and [17].

We declare a continuous non-decreasing and sub-additive function w on $[0, 1]$ with $w(0) = 0$ to be a *majorant*. A closed set $E \subset \partial\mathbb{D}$ of Lebesgue measure zero is said to have finite w -entropy if

$$\int_{\partial\mathbb{D}} \log w(\text{dist}(\zeta, E)) dm(\zeta) > -\infty.$$

Of course, when $w(t) = t^\alpha$ for some $0 < \alpha < 1$, one retains the classical Beurling-Carleson sets. Various descriptions of sets having finite w -entropy have recently been treated in [22] and in [24]. Sets of finite w -entropy are precisely the boundary zero sets of analytic functions in $\mathbb{D}$ which extend continuously to $\partial\mathbb{D}$ whose modulus of continuity do not exceed w on $\overline{\mathbb{D}}$, see [13] for classical Beurling-Carleson sets and [30] for general majorants. As

mentioned above, a singular inner function is cyclic in the Bergman space if and only if its associated singular measure does not charge any Beurling-Carleson set. It turns out that if a singular inner function is cyclic in the Bloch space, its associated singular measure can not charge a far wider range of sets.

Theorem 1.3. *Let w be a majorant. Assume that there exists $0 < \gamma < 1$ such that $w(t)/t^\gamma$ is non-increasing and that w satisfies the Dini-condition*

$$\int_0^1 \frac{w(t)}{t} dt < \infty.$$

Assume μ is carried on a countable union of sets having finite w -entropy. Then the associated singular inner function S_μ satisfies the permanence property: $[S_\mu]_{\mathcal{B}} \cap H^\infty = S_\mu H^\infty$.

We shall in fact deduce Theorem 1.3 from a slightly stronger statement in Section 3 (see Theorem 3.1 therein), which essentially asserts that inner factors of certain weighted BMOA spaces gives rise to the permanence property. In order for S_μ to be cyclic in $\mathcal{B}$, Theorem 1.3 says that μ cannot charge any set of finite w -entropy. Note that the class of eligible majorants w for which our Theorem holds, goes far beyond classical Beurling-Carleson sets, as it also applies to slowly increasing majorants of type $w(t) = \log^{-\alpha}(e/t)$, with $\alpha > 1$. Since any function cyclic in $\mathcal{B}$ must necessarily be cyclic in the Bergman spaces, we have that S_μ is cyclic in $\mathcal{B}$ implies that μ cannot charge Beurling-Carleson sets. The question whether the converse is true was raised in 1991 by Brown and Shields [12]. Theorem 1.3 above shows that this is not the case, but we shall below illustrate that this actually fails in a very strong sense. Before stating our next result, note that if $S_\mu \in W^1$, then so are the reproducing kernels of K_{S_μ} , and since they span a dense subspace of $K_{S_\mu}^1$, part (ii) of Theorem 1.2 implies that the permanence property holds whenever $S_\mu \in W^1$.

Theorem 1.4. *Let w be a majorant with $w(t)/t \rightarrow \infty$ as $t \rightarrow 0^+$. Then there exists a singular probability measure $\mu = \mu(w)$ on $\partial\mathbb{D}$ satisfying the following properties:*

(i) *For any arc $I \subset \partial\mathbb{D}$, we have $\mu(I) \leq w(|I|)$.*

(ii) *$S_\mu \in W^1$, hence it satisfies the permanence property, that is, $[S_\mu]_{\mathcal{B}} \cap H^\infty = S_\mu H^\infty$.*

If we take $w(t) = t^\alpha$, $0 < \alpha < 1$, condition (i) implies that μ charges no Beurling-Carleson set [29] while (ii) shows that S_μ is not cyclic in $\mathcal{B}$. Hence not every cyclic singular inner function in the Bergman space is cyclic in $\mathcal{B}$, answering the problem left open in [12]. This brings to light a new remarkable discrepancy on M_z -invariant subspaces between the Hardy spaces and Bergman spaces. Although Beurling's Theorem on the Hardy spaces naturally carries over their corresponding limiting case $BMOA$ (see [1]), the Korenblum-Roberts Theorem on the Bergman spaces breaks down in $\mathcal{B}$. It turns out that the negative answer to the above mentioned question has an immediate consequence to another related problem in the context of model spaces, mentioned in [17]. As a consequence of Theorem 1.2 and Theorem 1.4 with $w(t) = t \log(e/t)$, we obtain the following conclusion.

Corollary 1.5. *There exists a singular inner function Θ for which the following distinctive phenomenons occur:*

$$(i) \quad K_{\Theta} \cap \bigcup_{p>1} W^p = \{0\}.$$

(ii) *The reproducing kernels of K_{Θ} belong to W^1 and hence bounded analytic functions in $K_{\Theta} \cap W^1$ form a dense subset in K_{Θ} .*

It was also asked in [3] whether the condition $|f(z)| \gtrsim \log^{-1}(e/(1-|z|))$ for any $z \in \mathbb{D}$ of a Bloch function f , ensures that f is cyclic in $\mathcal{B}$. Our Theorem 1.4 shows that this is not the case, answering Problem 1 in [3] in the negative. In fact, given any majorant w with $w(t)/t \rightarrow \infty$ as $t \rightarrow 0^+$, Theorem 1.4 provides a positive singular measure μ such that S_{μ} is not cyclic in $\mathcal{B}$ while condition (i) readily translates to the estimate

$$|S_{\mu}(z)| \gtrsim \exp \left(-c \frac{w(1-|z|)}{1-|z|} \right), \quad z \in \mathbb{D},$$

for some numerical constant $c > 0$. We conclude that there cannot be any bound from below of a Bloch function which ensures it to be cyclic in $\mathcal{B}$. This pathological behavior is very different from the context of Bergman-type spaces, see [7]. As previously mentioned, S_{μ} satisfies the permanence property in Bergman spaces if and only if μ is concentrated on a countable union of Beurling-Carleson sets. Our next result says that the situation in the Bloch space is completely different and no condition on the support of μ alone can describe singular inner functions satisfying the permanence property in $\mathcal{B}$.

Theorem 1.6. *Let $E \subset \partial\mathbb{D}$ be a closed set of Lebesgue measure zero. Then there exists a singular probability measure μ supported on E , such that*

$$(i) \quad \mu(I) \geq |I|, \text{ for any dyadic arc } I \text{ with } I \cap E \neq \emptyset,$$

$$(ii) \quad S_{\mu} \in W^1, \text{ hence it satisfies the permanence property, that is, } [S_{\mu}]_{\mathcal{B}} \cap H^{\infty} = S_{\mu}H^{\infty}.$$

In order to construct a cyclic singular inner function in $\mathcal{B}$, the authors in [3] provided a sufficient condition for functions in $\mathcal{B}$ to be cyclic. Our next result gives a different sufficient condition which in a certain sense generalizes the above mentioned result.

Theorem 1.7. *Let $f \in H^{\infty}$ without zeros in $\mathbb{D}$, $\|f\|_{H^{\infty}} \leq 1$. Let $\nu = \nu_f$ denote its associated Herglotz-Nevanlinna measure. Assume there exists a constant $C = C(f) > 0$, such that for any pair of contiguous arcs $I, I' \subset \partial\mathbb{D}$ of same length $|I| = |I'|$, we have*

$$\left| \frac{\nu(I)}{|I|} - \frac{\nu(I')}{|I'|} \right| \leq C \inf \exp \left(-\frac{\nu(J)}{|J|} \right), \quad (1)$$

where the infimum is taken over all arcs $J \subset \partial\mathbb{D}$ with $J \supset I \cup I'$. Then f is cyclic in $\mathcal{B}$.

Note that condition (1) clearly implies that ν is a Zygmund measure on $\partial\mathbb{D}$, that is, the expression in the left hand side of (1) is uniformly bounded over the collection of pairs of contiguous arcs of the same length. However, condition (1) is considerably stronger as it asserts that if the density of the measure over an arc is large, then the oscillation of the densities of the measure on subarcs must be substantially smaller. It is worth mentioning that the construction in [3] of a singular measure μ for which S_μ is cyclic in $\mathcal{B}$ actually implies that μ satisfies the stronger condition

$$\sup_{|I| \leq \delta} \left| \frac{\mu(I)}{|I|} - \frac{\mu(I')}{|I'|} \right| \leq C \inf_{|J| \geq \delta} \exp \left(-\frac{\mu(J)}{|J|} \right),$$

which readily implies (1). It turns out that if one removes the infimum in condition (1), then one retains a description of bounded analytic functions which are invertible in $\mathcal{B}$.

Theorem 1.8. *Let $f \in H^\infty$ without zeros in $\mathbb{D}$, $\|f\|_{H^\infty} \leq 1$. Let $\nu = \nu_f$ denote its associated Herglotz-Nevanlinna measure. Then $1/f \in \mathcal{B}$ if and only if there exists a constant $C = C(f) > 0$ such that for any pair of contiguous arcs $I, I' \subset \partial\mathbb{D}$ of the same length, we have*

$$\left| \frac{\nu(I)}{|I|} - \frac{\nu(I')}{|I'|} \right| \leq C \exp \left(-\frac{\nu(I)}{|I|} \right). \quad (2)$$

For a description of elements in $H^\infty \cap \mathcal{B}_0$ of similar flavour, we refer the reader to the work of Bishop in [6]. We now turn our attention to the problem of determining the relationship between invertibility and cyclicity, which is a vastly investigated topic in various spaces of analytic functions. It is well-known that in the setting of commutative Banach algebras of analytic functions with units, the notions of cyclicity and invertibility are equivalent, while moving towards classes of analytic function for which the algebra property fails, such as the classical Dirichlet spaces and the Hardy spaces, invertibility is strictly stronger than cyclicity, see [11]. However, the situation for Bergman spaces attracted considerable amount of attention and remained open for quite some time until it was resolved by Borichev and Hedenmalm in [8]. In their deep work, they constructed invertible functions in the Bergman Spaces, which are not cyclic therein, showing that the two notions are quite different in that setting. Our next result asserts that a similar phenomenon occurs in the Bloch space. This answers Problem 2 from [3] in the negative.

Theorem 1.9. *There exists $f \in \mathcal{B}$ with $1/f \in \mathcal{B}$ such that f is not cyclic in $\mathcal{B}$.*

The crucial property of a function f as above is that $|f'(z)|$ is as huge as possible for a considerably large set of points $z \in \mathbb{D}$. This aligns well with the principal philosophy surrounding the work in [8], that an impediment to cyclicity of a Bergman space function is that the function enjoys maximal growth at a "massive" set in $\mathbb{D}$.

Our final result should be viewed as a pre-cautionary warning when considering weak-star closures of M_z -invariant subspaces generated by Bloch functions. In fact it may be viewed as a Bloch space version of Theorem 1.2 by Borichev and Hedenmalm in [9]. It asserts that sequential limits are not enough in order to capture the behavior of M_z -invariant subspaces in $\mathcal{B}$. Recall that classical outer functions in $\mathcal{B}$ are always cyclic therein.

Theorem 1.10. *There exists an outer function $f \in \mathcal{B}$ with $1/f \in H^\infty$, such that the set of all weak-star sequential limits in $\mathcal{B}$ of functions of the type $f(z)Q(z)$, where Q is an analytic polynomial, is a proper subspace of $\mathcal{B}$.*

The paper is organized as follows. Section 2 is devoted to establish Theorem 1.1 and Theorem 1.2 in the more general framework of so-called regular spaces. In Section 3, we deduce Theorem 1.3 on the permanence property induced by certain boundary zero sets, while Section 4 contains the proofs of Theorem 1.4 and Theorem 1.6, which illustrate that the results of Korenblum and Roberts in Bergman spaces do not extend to the Bloch space. Finally, Section 5 is devoted to establishing our results on the theme of invertibility versus cyclicity, containing Theorem 1.7, Theorem 1.8, Theorem 1.9 and Theorem 1.10.

2 Beurling-type theorems and model spaces

In this section, we shall keep a broad point of view and illustrate a close relationship between Beurling-type theorems on M_z -invariant subspaces generated by inner functions Θ and the ample of functions with sufficiently regular boundary values belonging the model space K_Θ .

2.1 Regular spaces

A Banach space X of analytic functions in $\mathbb{D}$ will be referred to as a *regular space* if the following three properties hold:

- (i) $X \subseteq H^1$.
- (ii) The set of analytic polynomials and the set of functions analytic in a neighborhood of $\overline{\mathbb{D}}$ are dense in X .
- (iii) For any ℓ in the dual space X^* , we have $\limsup_{n \rightarrow \infty} |\ell(z^n)|^{1/n} \leq 1$.

Assumption (i) ensures that the boundary values of functions in X are integrable on $\partial\mathbb{D}$ and by the closed graph theorem, there exists a constant $C > 0$ such that $\|f\|_1 \leq C\|f\|_X$ for any $f \in X$. Assumption (ii) implies that X is a separable Banach space, thus Helly's selection theorem ensures that any closed and bounded set in the norm of its Banach space dual X^* is sequentially compact wrt to the weak-star topology. In conjunction with (iii), we may substitute the abstract Banach space dual-pairing between X, X^* with the more practical *Cauchy dual* X' , considered via the H^2 -pairing

$$\lim_{r \rightarrow 1^-} \int_{\partial\mathbb{D}} f(r\zeta) \overline{g(r\zeta)} dm(\zeta), \quad f \in X, g \in X'.$$

Note that condition (i) also gives the containment $H^\infty \subset X'$. For our purposes, we shall equip X' with the weak-star topology. Let us briefly verify that $X = W^1$ is indeed a

regular space. Note that if $g \in W^1$, then

$$\int_{\partial\mathbb{D}} |g(r\zeta) - g(0)| dm(\zeta) \leq \int_{\partial\mathbb{D}} \int_0^r |g'(s\zeta)| ds dm(\zeta) \leq \|g\|_{W^1}, \quad 0 < r < 1.$$

Hence $W^1 \subset H^1$ as required. It is clear that analytic polynomials are dense in W^1 . Finally since the Cauchy-dual relation $(W^1)' \cong \mathcal{B}$ holds, condition (iii) is also fulfilled as the Taylor coefficients of Bloch functions are well-known to be bounded, see [4]. We thus conclude that $X = W^1$ indeed is a regular space. More generally, classical examples of regular spaces are provided by the analytic Sobolev spaces W^p with $p \geq 1$ and their counter-parts involving multiple derivatives.

2.2 M_z -invariant subspaces in duals of Regular Spaces

Let X be a regular space and let X' denote its Cauchy dual. In this subsection we collect several general properties of M_z -invariant subspaces in X' . Given a function $f \in X'$, we denote by $[f]_{X'}$ the weak-star closure in X' of the smallest M_z -invariant subspace containing f . We declare $f \in X'$ to be cyclic in X' if $[f]_{X'} = X'$. Similarly, given an inner function Θ , we say that Θ satisfies the *permanence property* in X' if

$$[\Theta]_{X'} \cap H^\infty \subseteq \Theta H^\infty.$$

The permanence property was recently introduced in the papers [25] and [27], in more specific context than ours. We also mention that it implicitly appeared in the earlier work of J. Roberts [28], in the context of Bergman spaces. Let X be a regular space, let Θ be an inner function and let $\mathcal{P}$ be the set of analytic polynomials. Observe that $X \cap K_\Theta^1$ regarded as a subset of X is precisely the pre-annihilator of $\Theta\mathcal{P} \subset X'$, denoted by $\Theta\mathcal{P}_\perp$. A standard argument involving the Hahn-Banach separation theorem gives the following equality of sets

$$(X \cap K_\Theta^1)^\perp = (\Theta\mathcal{P}_\perp)^\perp = [\Theta]_{X'}. \quad (3)$$

Since the Cauchy reproducing kernels are in X , weak-star convergence in X' implies convergence on compact subsets of $\mathbb{D}$. Hence the permanence property always holds for Blaschke products. In what follows, we shall thus restrict our attention to the permanence property for singular inner functions. Recall that H^∞ is the Cauchy dual of the Banach space $L^1(dm)/H_0^1$, where H_0^1 denotes the subspace of functions in H^1 vanishing at the origin. Our first simple observation is that for regular spaces X , the weak-star topology in X' is coarser than the weak-star topology in H^∞ .

Lemma 2.1. *Let X be a regular space. Then $H^\infty \subset X'$ holds, and whenever $\{f_n\}_n$ converges to f in weak-star of H^∞ , then f_n converges to f in weak-star of X' .*

Proof. Both these statements follow immediately from the assumption $X \subseteq H^1$ of regular spaces. $\square$

Our next observation allows us to substitute multiples of analytic polynomials by multiples of H^∞ -functions when considering (weak-star) cyclicity of a bounded analytic function in X' .

Proposition 2.2. *Let $f \in H^\infty$. Then the weak-star closure in X' of $fH^\infty := \{fh : h \in H^\infty\}$ equals $[f]_{X'}$.*

Proof. It suffices to verify that $fh \in [f]_{X'}$ for any $h \in H^\infty$. Let $\{Q_n\}_n$ be a sequence of polynomials which converge weak-star in H^∞ to h . Then since $f \in H^\infty$, Q_nf converges weak-star in H^∞ to fh . The conclusion now follows from Lemma 2.1. $\square$

Our next results illustrates a certain ordering for M_z -invariant subspaces generated by bounded analytic functions.

Proposition 2.3 (Division principle). *Let $f, g \in H^\infty$ with the property that $f/g \in H^\infty$. Then $[f]_{X'} \subseteq [g]_{X'}$. In particular, whenever f is weak-star cyclic in X' , then g is weak-star cyclic in X' .*

Proof. Set $h = f/g \in H^\infty$ and note that it suffices to show that $f = hg \in [g]_{X'}$. This however follows from the same argument as in Proposition 2.2. $\square$

For singular inner functions S_μ , the division principle implies that whenever S_μ is cyclic in X' , then so is any divisor S_{μ_0} of S_μ , where $0 \leq \mu_0 \leq \mu$. In the opposite direction, a monotonicity principle for the permanence property of singular inner functions in X' holds.

Proposition 2.4 (Monotonicity principle). *Let $\{\mu_n\}_n$ be a sequence of positive, finite, Borel, singular measures with $\mu_n \leq \mu_{n+1}$, $n \geq 1$, such that $\mu_n \rightarrow \mu$ weak-star in the space of complex finite Borel measures on $\partial\mathbb{D}$. Then if for any $n \geq 1$, the associated singular inner functions S_{μ_n} satisfies the permanence property in X' , then so does S_μ .*

Proof. Note that by monotonicity each S_{μ_n} is a divisor of S_μ hence $K_{S_{\mu_n}}^1 \subseteq K_{S_\mu}^1$ for each n . Since S_{μ_n} satisfies the permanence property for any $n \geq 1$, then according to Proposition 2.3, we obtain

$$[S_\mu]_{X'} \cap H^\infty \subseteq [S_{\mu_n}]_{X'} \cap H^\infty \subseteq S_{\mu_n} H^\infty, \quad n \geq 1.$$

Let $f \in [S_\mu]_{X'} \cap H^\infty$ be arbitrary and note that by the above containment, we can for any n find a bounded analytic function h_n such that $f = S_{\mu_n} h_n$. Now since $\|h_n\|_\infty = \|f\|_\infty$, for $n \geq 1$, we may extract a subsequence $\{h_{n_k}\}_k$ which converges pointwise in $\mathbb{D}$ to some $h \in H^\infty$. Since S_{μ_n} converges pointwise in $\mathbb{D}$ to S_μ , we conclude that $f = S_\mu h$. Thus S_μ satisfies the permanence property in X' as desired. $\square$

Our final lemma asserts that M_z -invariant subspaces generated by bounded analytic functions are invariant under the multiplication of outer functions in H^∞ . In particular, outer functions in H^∞ always cyclic in X' .

Lemma 2.5. *Let $F \in H^\infty$ be an outer function and $g \in H^\infty$. Then $[Fg]_{X'} = [g]_{X'}$.*

Proof. The containment $Fg \in [g]_{X'}$ follows from the proof of Proposition 2.2 and gives the inclusion $\subseteq$. For the converse inclusion, we apply Theorem 7.4 in [20] which yields a sequence of bounded analytic functions $\{F_n\}_n$ satisfying the following properties

- (i) $|F_n(z)F(z)| \leq 1, \quad z \in \mathbb{D},$
- (ii) $F_n(\zeta)F(\zeta)$ converges pointwise to 1 for m -a.e $\zeta \in \partial\mathbb{D}$.

Then FgF_n converges weak star in H^∞ to g . The result now follows from Lemma 2.1 and Proposition 2.2. $\square$

2.3 Proof of Theorem 1.2 in the context of Regular Spaces

Our main purpose is to establish the connection between the containment of a regular space X in the model spaces K_Θ to M_z -invariant subspaces generated by Θ in X' in the weak-star topology. However, the assumption $X \subseteq H^1$ actually makes the model spaces K_Θ^1 more appropriate in this regard. We recall that the reproducing kernels of K_Θ^1 are explicitly given by

$$\kappa_\Theta(z, \lambda) := \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}$$

and their linear span forms a dense subset in K_Θ^1 . Our first observation is related to cyclicity of inner functions Θ in X' and the existence of non-trivial functions in $X \cap K_\Theta^1$. We remark that similar results were known for a wide range of analytic Sobolev spaces X in [17] and more recently, it has appeared implicitly in a more general setting similar to ours in [25].

Proposition 2.6. *Let X be a regular space. An inner function Θ is weak-star cyclic in X' if and only if $X \cap K_\Theta^1$ contains no non-trivial function.*

Proof. Any $g \in X \cap K_\Theta^1$ is annihilated by polynomial multiples of Θ , hence if we assume that Θ is weak-star cyclic in X' , then $g \equiv 0$. Conversely, if Θ is not weak-star cyclic in X' , then there exists a non-trivial $g \in X$ such that

$$\int_{\partial\mathbb{D}} \overline{g(\zeta)}\Theta(\zeta)\zeta^n dm(\zeta) = 0, \quad n \geq 0.$$

The F. and M. Riesz Theorem implies that there exists $g_0 \in H^1$ with $g_0(0) = 0$ such that $g = \Theta\overline{g_0}$ at almost every point of $\partial\mathbb{D}$. Hence $g \in H^1 \cap \Theta\overline{H^1} = K_\Theta^1$. $\square$

Next we shall relate the density of regular spaces in the model spaces K_Θ^1 to the *permanence property* of Θ in X' . The following result is very similar to Theorem 1.3 in [25], but we shall need a more general version for our setting.

Proposition 2.7. *Let X be a regular space and let Θ be an inner function. Then $X \cap K_\Theta^1$ is dense in K_Θ^1 if and only if Θ satisfies the permanence property in X' .*

Proof. Suppose that $X \cap K_\Theta^1$ is dense in K_Θ^1 and let $f \in [\Theta]_{X'} \cap H^\infty$. According to (3), f must necessarily annihilate the family of reproducing kernels $\{k_\Theta(\cdot, \lambda)\}_{\lambda \in \mathbb{D}}$ of K_Θ^1 , that is

$$f(\lambda) = \Theta(\lambda) \int_{\partial\mathbb{D}} \frac{\overline{\Theta(\zeta)}f(\zeta)}{1 - \bar{\zeta}\lambda} dm(\zeta), \quad \lambda \in \mathbb{D}.$$

Hence $f \in \Theta H^2$, but since $f \in H^\infty$, we get that $f \in \Theta H^\infty$ as desired. Conversely, if the permanence property holds, then in fact we verify that the following stronger condition holds:

$$[\Theta]_{X'} \cap H^2 \subseteq \Theta H^2. \quad (4)$$

To see this, note that if $f \in [\Theta]_{X'} \cap H^2$, then $[f]_{X'} \subseteq [\Theta]_{X'}$. Let $f = F\Phi$, where F is outer and Φ inner. We want to show that Θ divides Φ in H^2 . For $n \geq 1$, consider the outer function F_n defined as $|F_n| = \min(n, |F|)$ on $\partial\mathbb{D}$ and note that if we set $f_n := F_n\Phi$, then $f_n/f \in H^\infty$ and hence by the division principle Proposition 2.3, we get

$$[f_n]_{X'} \subseteq [f]_{X'} \subseteq [\Theta]_{X'}, \quad n \geq 1.$$

However since $f_n = F_n\Phi$ is bounded for each $n \geq 1$, Lemma 2.5 implies that $[f_n]_{X'} = [\Phi]_{X'}$ and thus we conclude that $\Phi \in [\Theta]_{X'} \cap H^\infty \subseteq \Theta H^\infty$, proving the claim. With this at hand, note that (4) in conjunction with (3) may be rephrased as

$$(X \cap K_\Theta^1)^\perp \cap H^2 \subseteq \Theta H^2.$$

Now since the Cauchy dual $(K_\Theta^1)'$ is contained in K_Θ^2 (see [19]), we see that any annihilator of $X \cap K_\Theta^1$ is contained in $K_\Theta^2 \cap \Theta H^2 = \{0\}$. This is enough to conclude that $X \cap K_\Theta^1$ is dense in K_Θ^1 . $\square$

Proof of Theorem 1.2. Taking $X = W^1$, part (a) follows from Proposition 2.6, while (b) follows from Proposition 2.7. $\square$

2.4 An abstract decomposition for singular measures

Inspired from the Korenblum-Roberts Theorem on M_z -invariant subspaces generated by inner functions, this section is devoted to extending their result to the context of Regular Spaces. Here we shall follow ideas developed by Roberts in [28], with certain required adaptations in order to deal with our general framework.

Theorem 2.8 (Abstract decomposition). *Let X be a regular space and let S_μ be a singular inner function. Then there exists a unique decomposition of $\mu = \mu_P + \mu_C$ (up to sets of μ -measure zero) into mutually singular parts μ_P, μ_C satisfying the following properties:*

- (i) S_{μ_P} has the permanence property in X' , that is, $[S_{\mu_P}]_{X'} \cap H^\infty \subseteq S_{\mu_P} H^\infty$,
- (ii) S_{μ_C} is weak-star cyclic in X' , that is, $[S_{\mu_C}]_{X'} = X'$.

Proof. Let $\mu|_E$ denote the restriction of μ to the Borel set E . Consider the the following collection of Borel subsets of $\partial\mathbb{D}$

$$\mathcal{M}(X', \mu) := \{E \subset \partial\mathbb{D} : E \text{ Borel set, } \mu(E) > 0, [S_{\mu|_E}]_{X'} \cap H^\infty \subseteq S_{\mu|_E} H^\infty\}.$$

If the set $\mathcal{M}(X', \mu)$ is empty, then we simply take $\mu_C = \mu$. For the being moment, we shall assume that the collection $\mathcal{M}(X', \mu)$ is non-empty, and our goal is to primarily establish that $\mathcal{M}(X', \mu)$ is closed under the formation of unions. To this end, pick two sets $E, F \in \mathcal{M}(X', \mu)$ which we may assume are not contained in one or another. Set $\nu := \mu|_{E \cup F}$ and note that the containment of $K_{S_{\mu|_E}}^1 \subset K_{S_\nu}^1$ in conjunction with (3) implies

$$[S_\nu]_{X'} \cap H^\infty = (X \cap K_{S_\nu}^1)^\perp \cap H^\infty \subseteq (X \cap K_{S_{\mu|_E}}^1)^\perp \cap H^\infty = [S_{\mu|_E}]_{X'} \cap H^\infty \subseteq S_{\mu|_E} H^\infty.$$

Therefore, any $f \in [S_\nu]_{X'} \cap H^\infty$ is divisible by $S_{\mu|_E}$, and by switching the roles of E and F , we deduce in a similar way that f is also divisible by $S_{\mu|_F}$. Altogether, we obtain $f/S_\nu \in H^\infty$ and therefore $E \cup F \in \mathcal{M}(X', \mu)$. Now consider the quantity

$$\gamma(X', \mu) := \sup \{\mu(E) : E \in \mathcal{M}(X', \mu)\},$$

and observe that since $\mathcal{M}(X', \mu)$ is closed under finite unions, we can find a sequence of Borel sets $\{E_n\}_n$ with $E_n \in \mathcal{M}(X', \mu)$ and $E_n \subset E_{n+1}$ for any n , such that $\mu(E_n) \rightarrow \gamma(X', \mu)$ as $n \rightarrow \infty$. Set $E := \cup_n E_n$ and we claim that our candidates are given by $\mu_P = \mu|_E$ and $\mu_C = \mu|_{\partial\mathbb{D} \setminus E}$. We first check that $E \in \mathcal{M}(X', \mu)$. To this end, let $f \in [S_{\mu|_E}]_{X'} \cap H^\infty$ and note that since each $E_n \in \mathcal{M}(X', \mu)$, an application of (3) implies

$$[S_{\mu|_E}]_{X'} \cap H^\infty = (X \cap K_{S_{\mu|_E}}^1)^\perp \cap H^\infty \subseteq (X \cap K_{S_{\mu|_{E_n}}}^1)^\perp \cap H^\infty = [S_{\mu|_{E_n}}]_{X'} \cap H^\infty \subseteq S_{\mu|_{E_n}} H^\infty,$$

for all n . Hence for each n , there exists $\{h_n\}_n \subset H^\infty$ such that $f = S_{\mu|_{E_n}} h_n$. But since $\|h_n\|_\infty = \|f\|_\infty$, we may extract a subsequence $\{h_{n_k}\}_k$ that converges pointwise in $\mathbb{D}$ to a certain function $h \in H^\infty$. Now since the sequence of sets $\{E_n\}_n$ is increasing, we actually obtain $f = S_{\mu|_E} h$. This proves that $E \in \mathcal{M}(X', \mu)$ and consequently we have established claim (i) with $\mu_P = \mu|_E$.

We now proceed to establish claim (ii) by showing that if $\mu_C := \mu|_{\partial\mathbb{D} \setminus E}$ then S_{μ_C} is weak-star cyclic in X' . To this end, suppose that S_{μ_C} is not weak-star cyclic in X' , then there exists a non-trivial $g \in X$ such that

$$\int_{\partial\mathbb{D}} \overline{g(\zeta)} S_{\mu_C}(\zeta) \zeta^n dm(\zeta) = 0, \quad n \geq 0.$$

By the F. and M. Riesz Theorem, there exists a function $h_1 \in H^1$ with $h_1(0) = 0$ such that

$$\overline{g(\zeta)} = \frac{h_1(\zeta)}{S_{\mu_C}(\zeta)} = \frac{h_2(\zeta)}{S_\nu(\zeta)}, \quad m - \text{a.e. } \zeta \in \partial\mathbb{D}.$$

Here $h_2 \in H^1$, $h_2(0) = 0$, $0 \leq \nu \leq \mu_C$ and the quotient h_1/h_2 is a singular inner function dividing S_{μ_C} (possibly a unimodular constant). Now if $\nu \equiv 0$, then $g \in H^1 \cap \overline{H^1}$ on $\partial\mathbb{D}$

with $g(0) = 0$, which implies $g \equiv 0$, thus contradicting the assumption that g is non-trivial. Hence we may assume that $\nu \neq 0$, and our aim is to establish that S_ν satisfies the permanence property, which will contradict the maximality of S_{μ_P} . To this end, note that

$$\int_{\partial\mathbb{D}} \overline{g(\zeta)} S_\nu(\zeta) \zeta^n dm(\zeta) = \int_{\partial\mathbb{D}} h_2(\zeta) \zeta^n dm(\zeta) = 0, \quad n \geq 0,$$

hence by the F. and M. Riesz Theorem, we have $g \in K_{S_\nu}^1 := H^1 \cap S_\nu \overline{H_0^1}$. Now if $u \in [S_\nu]_{X'} \cap H^\infty$, then according to (3) we actually get

$$\int_{\partial\mathbb{D}} \overline{g(\zeta)} u(\zeta) \zeta^n dm(\zeta) = 0, \quad n \geq 0.$$

Applying the F. and M. Riesz Theorem once again, we can find $h_3 \in H^1$ with $h_3(0) = 0$ such that

$$\overline{g(\zeta)} = \frac{h_3(\zeta)}{u(\zeta)} = \frac{h_2(\zeta)}{S_\nu(\zeta)}, \quad m - \text{a.e. } \zeta \in \partial\mathbb{D}$$

which implies that $u \in S_\nu H^\infty$. Hence S_ν satisfies the permanence property in X' . We have thus obtained the desired contradiction and it follows that S_{μ_C} is weak-star cyclic in X' . Obviously the same argument works in case the collection $\mathcal{M}(X', \mu)$ is void. The claim regarding uniqueness of the above decomposition of μ follows from maximality of μ_P with respect to the permanence property in X' . □

Proof of Theorem 1.1. The following proof of Theorem 1.1 actually holds for any regular space X . Note that statement (i) in Theorem 2.8 regarding the permanence property is easily shown to remain true if S_{μ_P} is multiplied by a Blaschke product. The claim on cyclicity in (ii) of Theorem 1.1 follows from the cyclicity of S_{μ_C} in conjunction with Lemma 2.5. This completes the proof. □

As a consequence of Theorem 2.8 in conjunction with the results in Section 2.3, we conclude this section with the following description regular spaces in model spaces.

Corollary 2.9. *Let X be a regular space, let $\Theta = BS_\mu$ where B is a Blaschke product and S_μ is a singular inner function. Consider the decomposition $\mu = \mu_P + \mu_C$ given in Theorem 2.8. Then the following holds:*

Abundance of regular functions: $X \cap K_{BS_{\mu_P}}^1$ is a dense subspace of $K_{BS_{\mu_P}}^1$,

Absence of regular functions: $K_{S_{\mu_C}}^1 \cap X = \{0\}$.

3 The permanence property and boundary zero sets

This section is devoted to showing that inner factors of certain weighted BMOA spaces give rise to the permanence property in $\mathcal{B}$, for which Theorem 1.3 will be derived as a corollary.

3.1 Inner factors in weighted BMOA

Let w be a majorant and denote by $BMO_w(\partial\mathbb{D})$ the space of Lebesgue integrable functions h on $\partial\mathbb{D}$ equipped with the semi-norm

$$\|h\|_{BMO_w} := \sup_I \frac{1}{w(|I|)} \frac{1}{|I|} \int_I |h - h_I| dm < \infty.$$

We denote by $C_w(\partial\mathbb{D})$ the space of continuous functions h on $\partial\mathbb{D}$ equipped with the semi-norm

$$\|h\|_{C_w} := \sup_{\zeta \neq \xi} \frac{|h(\zeta) - h(\xi)|}{w(|\zeta - \xi|)} < \infty.$$

It is straightforward to verify the containment $C_w(\partial\mathbb{D}) \subseteq BMO_w(\partial\mathbb{D})$ and it turns out that these spaces share very intimate features. If the majorant w satisfies the Dini-condition

$$\int_0^1 \frac{w(t)}{t} dt < \infty, \quad (5)$$

then functions in $BMO_w(\partial\mathbb{D})$ extend continuously to $\partial\mathbb{D}$ with modulus of continuity not exceeding a constant multiple of $\int_0^\delta \frac{w(t)}{t} dt$. Conversely, if the Dini-condition (5) on w fails then $BMO_w(\partial\mathbb{D})$ contains discontinuous and unbounded functions. See [31] for details on these matters. Analytic counter-parts of the above defined spaces will be important, namely we consider the spaces $BMOA_w := H^2 \cap BMO_w(\partial\mathbb{D})$ and $A_w := H^\infty \cap C_w(\partial\mathbb{D})$, hence the containment $A_w \subseteq BMOA_w$ holds. Moving forward, we shall now restrict our attention to majorants w satisfying the following condition. There exists a constant $0 < \gamma < 1$ such that

$$w(t)/t^\gamma \text{ is non-increasing on } [0, 1]. \quad (6)$$

We now state our main result in this section.

Theorem 3.1. *Let w be a majorant satisfying condition (6) and the Dini condition (5). Then any non-trivial singular inner factor S_μ of a function in $BMOA_w$ satisfies the permanence property in $\mathcal{B}$, that is, $[S_\mu]_{\mathcal{B}} \cap H^\infty = S_\mu H^\infty$. In particular, no non-trivial singular inner divisor of a function in $BMOA_w$ can be cyclic in $\mathcal{B}$.*

An important class of singular inner factors of $BMOA_w$ -functions are in fact provided by singular inner factors of A_w -functions, which turn out to be intimately connected to sets of finite w -entropy due to a deep result by Shirokov in [30]. For the moment being, we shall primarily establish Theorem 3.1 and then illustrate how Theorem 1.3 can be derived as a corollary. In the pursuit towards proving Theorem 3.1, we shall need a couple of preparatory results. The first lemma can be found in a slight greater generality in [18] (see Proposition 2.6 therein for further details).

Lemma 3.2. *Let w be a majorant satisfying condition (6). Then the BMO_w semi-norm is equivalent to the Garsia-type semi-norm, that is,*

$$\|h\|_{BMO_w} \asymp \sup_{z \in \mathbb{D}} \frac{1}{w(1 - |z|)} \int_{\partial\mathbb{D}} |h(\zeta) - P(h)(z)| P_z(\zeta) dm(\zeta)$$

where $P(h)$ denotes the Poisson extension of h on $\mathbb{D}$ and $P_z(\zeta) = (1 - |z|^2)|1 - \bar{z}\zeta|^{-2}$, $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$, is the Poisson kernel on $\mathbb{D}$.

With Lemma 3.2 at hand, we may derive the following result on the Cauchy projection.

Lemma 3.3. *Let w be a majorant satisfying condition (6) and the Dini condition (5). Then the Cauchy projection P_+ maps $BMO_w(\partial\mathbb{D})$ continuously into W^1 .*

Proof. Fix an arbitrary $h \in BMO_w(\partial\mathbb{D})$ and observe that

$$P_+(h)'(z) = \int_{\partial\mathbb{D}} \frac{\bar{\zeta}h(\zeta)}{(1 - \bar{\zeta}z)^2} dm(\zeta) = \int_{\partial\mathbb{D}} \frac{\bar{\zeta}(h(\zeta) - P(h)(z))}{(1 - \bar{\zeta}z)^2} dm(\zeta), \quad z \in \mathbb{D}.$$

According to Lemma 3.2, we have

$$|P_+(h)'(z)| \leq \int_{\partial\mathbb{D}} \frac{|h(\zeta) - P(h)(z)|}{|1 - \bar{\zeta}z|^2} dm(\zeta) \lesssim \|h\|_{BMO_w} \frac{w(1 - |z|)}{1 - |z|}, \quad z \in \mathbb{D}.$$

It now readily follows that

$$\int_{\mathbb{D}} |P_+(h)'(z)| dA(z) \lesssim \|h\|_{BMO_w} \int_0^1 \frac{w(t)}{t} dt,$$

which shows that $P_+ : BMO_w(\partial\mathbb{D}) \rightarrow W^1$ continuously. $\square$

The following result on division and multiplication by inner functions on weighted $BMOA$ spaces due to K. Dyakonov in [18] plays a crucial role in our developments.

Theorem 3.4 (Dyakonov). *Let w be a majorant satisfying condition (6). Let $g \in BMOA_w$ and let Θ be an inner function. Then $g\bar{\Theta}$ belongs to $BMO_w(\partial\mathbb{D})$ if and only if $g\Theta$ belongs to $BMOA_w$.*

It follows from the work in [18] that if w satisfies condition (6), then $BMOA_w$ enjoys the factorization property. More specifically, whenever $g \in BMOA_w$ and Θ is an inner function with $g/\Theta \in H^\infty$, then in fact $g/\Theta \in BMOA_w$.

Proof of Theorem 3.1. Let $\Theta := S_\mu$ be a singular inner factor of a function $g \in BMOA_w$. By means of applying the factorization property of $BMOA_w$, we may without loss of generality assume that $g = G\Theta$, where G is an outer function in $BMOA_w$. Now applying Theorem 3.4 to the function G , we conclude that $G\bar{\Theta}$ belongs to $BMO_w(\partial\mathbb{D})$ and thus so does $\bar{G}\Theta$. Hence if k_λ denotes the Cauchy kernel at the point $\lambda \in \mathbb{D}$, then the function $g_\lambda := \bar{G}\Theta k_\lambda$ also remains in $BMO_w(\partial\mathbb{D})$. Applying Lemma 3.3, we conclude that $P_+(g_\lambda) \in W^1$. Given $H \in L^\infty(\partial\mathbb{D}, dm)$, we denote by T_H the Toeplitz operator with symbol H defined as $T_H(f) := P_+(Hf)$. If $\kappa_\Theta(\cdot, \lambda)$ denotes the reproducing kernel of the model space K_Θ at the point $\lambda \in \mathbb{D}$, we have that

$$T_{\bar{G}}(\kappa_\Theta(\cdot, \lambda))(z) = T_{\bar{G}}(k_\lambda)(z) - \overline{\Theta(\lambda)} P_+(g_\lambda)(z), \quad z \in \mathbb{D},$$

belongs to $W^1 \cap K_\Theta$. This follows from the following properties of Toeplitz operators with co-analytic symbols:

- (a) $T_{\overline{H}} : W^p \rightarrow W^p$ for any $H \in H^\infty$ and $1 < p < \infty$.
- (b) $T_{\overline{H}} : K_\Phi \rightarrow K_\Phi$ for any $H \in H^\infty$ and any inner function Φ .

For instance, see [14]. Property (a) is actually more than what is needed here, but it certainly guarantees that $T_{\overline{f}}(k_\lambda)$ belongs to W^1 for each $\lambda \in \mathbb{D}$, while property (b) shows that $T_{\overline{G}}(\kappa_\Theta(\cdot, \lambda))$ belongs to K_Θ . Consequently, we conclude that $T_{\overline{G}}(\kappa_\Theta(\cdot, \lambda))$ belong to $W^1 \cap K_\Theta$. We now proceed by verifying that the linear span $\mathcal{M}$ of the set $\{T_{\overline{G}}(\kappa_{S_\Theta}(\cdot, \lambda))\}_{\lambda \in \mathbb{D}}$ is dense in K_Θ . To this end, let $f \in K_\Theta$ be an element which annihilates $\mathcal{M}$. Then

$$0 = \int_{\partial\mathbb{D}} T_{\overline{G}}(\kappa_\Theta(\cdot, \lambda))(\zeta) \overline{f(\zeta)} dm(\zeta) = \int_{\partial\mathbb{D}} \kappa_\Theta(\zeta, \lambda) \overline{G(\zeta)} \overline{f(\zeta)} dm(\zeta), \quad \lambda \in \mathbb{D}.$$

Since the linear span of the reproducing kernels are dense in K_Θ , we conclude that there exists a function $h \in H^2$ such that $Gf = \Theta h$. Now recalling that G was outer, we conclude that $f \in \Theta H^2 \cap K_\Theta = \{0\}$, and thus $\mathcal{M}$ is dense in K_Θ . Now since K_Θ is dense in K_Θ^1 , we also conclude that $\mathcal{M}$ is dense in K_Θ^1 . Recalling that W^1 is the Cauchy pre-dual of $\mathcal{B}$, it now follows from Proposition 2.7 that $[\Theta]_{\mathcal{B}} \cap H^\infty \subseteq \Theta H^\infty$, hence the proof is complete. $\square$

We now turn our attention to the proof of Theorem 1.3.

Proof of Theorem 1.3. We shall primarily assume that the positive finite singular measure μ on $\partial\mathbb{D}$ is supported on a single (closed) set E of finite w -entropy. According to Shirokov's Theorem in [30], there exists an outer function $f \in A_w$ such that the product fS_μ belongs to $A_w \subset BMOA_w$. Hence by Theorem 3.1 S_μ satisfies the permanence property in $\mathcal{B}$. Now assume that μ is concentrated on countable union of sets $\{E_n\}_n$ having finite w -entropy, which we may assume are increasing: $E_n \subseteq E_{n+1}$ for any n . Consider $\mu_n := \mu|_{E_n}$ and observe that μ_n converges to μ in the weak-star topology of finite complex Borel measures on $\partial\mathbb{D}$. Since each S_{μ_n} satisfies the permanence property in $\mathcal{B}$, so does S_μ by the monotonicity principle in Proposition 2.4. The proof is now complete. $\square$

4 The permanence property and W^1

4.1 The permanence property and invisibility

Let μ be a positive finite Borel singular measure on $\partial\mathbb{D}$. It was already mentioned in the introduction that the permanence property holds for singular inner functions in W^1 . Our first observation in this section provides a simple way for inducing the permanence property in $\mathcal{B}$ of singular inner functions. Let μ be a positive finite singular Borel measure on $\partial\mathbb{D}$ and assume

$$\sup \{ \mu(E) : S_{\mu|_E} \in W^1 \} > 0. \quad (7)$$

We may pick a sequence of Borel sets $\{E_N\}_N$ with the property that each $S_{\mu|_{E_N}} \in W^1$ and such that $\mu(E_N)$ converges to the supremum above. Observe that whenever ϕ, ψ are bounded functions in W^1 , so is their product $\phi\psi$, and thus we may without loss of generality, assume that the $E_N \subseteq E_{N+1}$ for any N . Now let $E := \cup_N E_N$ and set $\mu_0 := \mu - \mu|_E$ and observe that by maximality, there exists no non-trivial $\nu_0 \leq \mu_0$ with $S_{\nu_0} \in W^1$. Measures satisfying the above condition are declared to be W^1 invisible. As previously observed in the introduction, each $S_{\mu|_{E_N}}$ satisfies the permanence property in $\mathcal{B}$ and thus according to the monotonicity principle in Proposition 2.4, so does $S_{\mu|_E}$. Our discussion can thus be summarized in the following result.

Proposition 4.1. *Let μ be a positive finite singular Borel measure on $\partial\mathbb{D}$ and $\mu|_E$ denote the corresponding piece in the decomposition of μ appearing in the previous paragraph. Then $S_{\mu|_E}$ satisfies the permanence property in $\mathcal{B}$, that is, $[S_{\mu|_E}]_{\mathcal{B}} \cap H^\infty \subseteq S_{\mu|_E} H^\infty$. As a consequence, if S_μ is cyclic in $\mathcal{B}$, then μ is W^1 invisible.*

4.2 Singular inner functions in W^1

This section is devoted to study singular inner functions in W^1 . For any arc $I \subseteq \partial\mathbb{D}$, we denote its associated Carleson square by $Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1 - |z| \leq |I|\}$, and we let $T_I = \{z \in Q_I : 1 - |z| \geq |I|/2\}$ be the top-half of Q_I .

Lemma 4.2. *There exists a universal constant $C > 0$, such that for any singular measure μ on $\partial\mathbb{D}$ and any arc $I \subset \partial\mathbb{D}$ with $\mu(I) = 0$, the associated singular inner function S_μ satisfies*

$$\int_{Q_I} |S'_\mu(z)| dA(z) \leq C|I|.$$

Proof. Let $\{I_k\}_{k \in \mathbb{Z}}$ denote the Whitney decomposition of I satisfying

$$\text{dist}(I_k, \partial I) = |I_k| = \frac{1}{3 \cdot 2^{|k|}} |I|, \quad k \in \mathbb{Z}.$$

We first observe that there exists an absolute constant $C > 0$ such that

$$\int_{Q_I \setminus \cup_k Q_{I_k}} |S'_\mu(z)| dA(z) \leq \int_{Q_I \setminus \cup_k Q_{I_k}} \frac{dA(z)}{1 - |z|^2} \leq C|I|. \quad (8)$$

It remains to estimate the integral of $|S'_\mu|$ on $\cup_k Q_{I_k}$. To this end, for each Whitney arc I_k , we denote by ξ_k its center and let $z_k = (1 - |I_k|)\xi_k$. Note that there exists an absolute constant $c > 0$ such that $c^{-1}|\zeta - z_k| \leq |\zeta - z| \leq c|\zeta - z_k|$ for any $z \in Q_{I_k}$ and any $\zeta \in \partial\mathbb{D} \setminus I$. Since $\mu(I) = 0$ the Poisson extension $P(\mu)$ of μ satisfies

$$P(\mu)(z) = \frac{1 - |z|^2}{1 - |z_k|^2} \int_{\partial\mathbb{D} \setminus I} \frac{1 - |z_k|^2}{|\zeta - z|^2} d\mu(\zeta) \asymp (1 - |z|) \frac{P(\mu)(z_k)}{|I_k|}, \quad z \in Q_{I_k}.$$

Now using this in conjunction with the obvious estimate

$$(1 - |z|^2)|S'_\mu(z)| \leq 2P(\mu)(z) \exp(-P(\mu)(z)), \quad z \in \mathbb{D},$$

it follows that there exists a universal constant $C > 0$ such that

$$\int_{Q_{I_k}} |S'_\mu(z)| dA(z) \lesssim \frac{P(\mu)(z_k)}{|I_k|} \int_{Q_{I_k}} \exp\left(-C(1 - |z|) \frac{P(\mu)(z_k)}{|I_k|}\right) dA(z).$$

A straightforward computation of the integral above gives

$$\int_{Q_{I_k}} |S'_\mu(z)| dA(z) \lesssim |I_k|.$$

Summing over k completes the proof. $\square$

We shall now locate the critical domain of integration in order for S_μ to belong to W^1 , which will be convenient for our further developments.

Corollary 4.3. *Let μ be a positive singular Borel measure supported on a closed set $E \subset \partial\mathbb{D}$. Then S_μ belongs to W^1 if and only if*

$$\sum_{\substack{I \in \mathcal{D} \\ I \cap E \neq \emptyset}} \int_{T_I} |S'_\mu(z)| dA(z) < \infty,$$

where $\mathcal{D}$ denotes the collection of dyadic arcs on $\partial\mathbb{D}$.

Proof. It is sufficient to show that the above condition implies that $S_\mu \in W^1$. To this end, let $\mathcal{G}$ denote the collection of maximal dyadic arcs which do not intersect E . Observe that we can write

$$\int_{\mathbb{D}} |S'_\mu(z)| dA(z) = \sum_{\substack{I \in \mathcal{D} \\ I \cap E \neq \emptyset}} \int_{T_I} |S'_\mu(z)| dA(z) + \sum_{I \in \mathcal{G}} \int_{Q_I} |S'_\mu(z)| dA(z)$$

Note that Lemma 4.2 gives

$$\sum_{I \in \mathcal{G}} \int_{Q_I} |S'_\mu(z)| dA(z) \leq C \sum_{I \in \mathcal{G}} |I| < \infty,$$

which is enough to prove the desired claim. $\square$

As a consequence, we obtain the following sufficient condition for membership of S_μ in W^1 .

Corollary 4.4. *Let μ be a positive finite singular Borel measure supported on a Beurling Carleson set of the unit circle. Then S_μ belongs to W^1 .*

Proof. It is well known that a closed set $E \subset \partial\mathbb{D}$ of Lebesgue measure zero is a Beurling Carleson set if and only if

$$\sum_{\substack{I \in \mathcal{D} \\ I \cap E \neq \emptyset}} |I| < \infty,$$

see [10]. The claim now follows from Schwarz Lemma and Corollary 4.3. $\square$

4.3 No estimate from below implies cyclicity in $\mathcal{B}$

This subsection is devoted to Theorem 1.4, which essentially asserts that no condition on the modulus of continuity on μ alone is an impediment for S_μ to be a member in W^1 . According to the discussion in the introduction, this implies that there cannot be any estimate from below of a singular inner function which ensures it to be cyclic in $\mathcal{B}$.

Proof of Theorem 1.4. We shall divide the proof in three different steps.

Step 1: Construction of μ

Let $\{n_l\}_l$ be an increasing sequence of positive integers to be specified later according to the majorant w . Set $\mu(\partial\mathbb{D}) = 1$ and consider the subcollection $G_1 = \{I_j^{(1)}\}_j$ of every other dyadic arc of generation n_1 . In other words $|I_j^{(1)}| = 2^{-n_1}$ and $\text{dist}(I_j^{(1)}, I_{j+1}^{(1)}) = 2^{-n_1}$ for any j . On these arcs, we set $\mu(I_j^{(1)}) = 2|I_j^{(1)}|$ for each j , and note that

$$\sum_j \mu(I_j^{(1)}) = 2 \sum_j |I_j^{(1)}| = 1.$$

Hence μ spreads its mass precisely on the collection G_1 . Next we consider the subcollection $G_2 = \{I_j^{(2)}\}_j$ of dyadic arcs of generation n_2 contained in $\cup I_j^{(1)}$, in such a way that inside each $I_k^{(1)}$, we pick every other dyadic arc of generation n_2 to include in our subcollection. Thus $|I_j^{(2)}| = 2^{-n_2}$ and $\text{dist}(I_j^{(2)}, I_{j+1}^{(2)}) \geq 2^{-n_2}$ for each j . With this at hand, we now set $\mu(I_j^{(2)}) = 2^2|I_j^{(2)}|$ for each j and observe that for any arc $I_k^{(1)} \in G_1$, we have

$$\sum_{I_j^{(2)} \subset I_k^{(1)}} \mu(I_j^{(2)}) = 2^2 \sum_{I_j^{(2)} \subset I_k^{(1)}} |I_j^{(2)}| = 2|I_k^{(1)}| = \mu(I_k^{(1)}).$$

Hence inside each $I_k^{(1)}$, we again evenly distribute the mass of μ on every other dyadic arc of generation n_2 , and denote this joint collection by $G_2 = \{I_j^{(2)}\}$. We proceed by induction. Assume the first $l-1$ collections of dyadic arcs have been constructed. Then we consider the collection $G_l = \{I_k^{(l)}\}$ of every other dyadic arc contained in an arc of G_{l-1} and set $\mu(I_j^{(l)}) = 2^l|I_j^{(l)}|$, for each j . Hence for each $I_k^{(l-1)} \in G_{l-1}$ we have

$$\sum_{I_j^{(l)} \subset I_k^{(l-1)}} \mu(I_j^{(l)}) = 2^l \sum_{I_j^{(l)} \subset I_k^{(l-1)}} |I_j^{(l)}| = 2^{l-1}|I_k^{(l-1)}| = \mu(I_k^{(l-1)}).$$

Note also that for each $l \geq 1$, we have

$$\sum_k |I_k^{(l)}| = 2^{-l}. \tag{9}$$

Now extend μ to a Borel measure on $\partial\mathbb{D}$. Note that by construction, μ is supported inside $E := \cap_l \cup_j I_j^{(l)}$, which has Lebesgue measure zero and thus μ is indeed a positive finite singular measure.

Step 2: Verifying condition (i)

We shall now specify a choice for the sequence of positive integers $\{n_l\}_l$ so that condition (i) holds. To this end, notice that since $w(t)/t \rightarrow \infty$ as $t \rightarrow 0^+$, for any non-negative integer l , we can pick n_l in such a way that

$$\frac{w(2^{-n_l})}{2^{-n_l}} \geq 2^l 6.$$

With this choice, we have

$$\mu(I_j^{(l)}) = 2^l |I_j^{(l)}| = 2^l 2^{-n_l} \leq w(2^{-n_l})/6 = w(|I_j^{(l)}|)/6.$$

We first check that condition (i) holds for dyadic arcs I . We fix an arbitrary dyadic arc I of length 2^{-n} and pick a positive integer l such that $n_{l-1} \leq n < n_l$. We have

$$\mu(I) = \sum_{j: I_j^l \subset I} \mu(I_j^{(l)}) = \sum_{j: I_j^l \subset I} 2^l |I_j^{(l)}| \leq 2^{l-1} |I| \leq \frac{w(2^{-n_{l-1}})}{2^{-n_{l-1}}} \frac{|I|}{3} \leq \frac{w(|I|)}{3},$$

where in the last step we have used the property that $w(s)/s \leq 2w(t)/t$ whenever $s < t$, which easily follows from the sub-additivity of majorants. Let $I \subset \partial\mathbb{D}$ be an arbitrary arc and pick $n > 0$ such that $2^{-n} < b - a \leq 2^{-(n-1)}$. Since I is contained in at most three dyadic arcs of length 2^{-n} , the estimate (i) holds.

Step 3: Verifying condition (ii)

To this end, for each pair of positive integers (k, l) we consider the set

$$\Omega_k^{(l)} := \left\{ z \in \mathbb{D} : 2^{-n_{l+1}} < 1 - |z| \leq 2^{-n_l}, z/|z| \in I_k^{(l)} \right\}$$

and note that according to (iii) of Corollary 4.3 it suffices to prove that

$$\sum_{k,l} \int_{\Omega_k^{(l)}} |S'_\mu(z)| dA(z) < \infty. \quad (10)$$

For $z \in \mathbb{D} \setminus \{0\}$ let I_z denote the arc centered at $z/|z|$ of length $1 - |z|$. Observe that there exists a positive constant $c > 0$ such that for $z \in \Omega_k^{(l)}$, we have that $P(\mu)(z) \geq c\mu(I_z)/|I_z| \geq c2^l$. Using the formula $S'_\mu(z) = S_\mu(z)H(\mu)'(z)$, $z \in \mathbb{D}$, where $H(\mu)$ denotes the Herglotz transform of μ , we get

$$\sum_{k,l} \int_{\Omega_k^{(l)}} |S'_\mu(z)| dA(z) \leq \sum_{k,l} e^{-c2^l} \int_{\Omega_k^{(l)}} |H(\mu)'(z)| dA(z). \quad (11)$$

The rest of the proof shall be devoted to establishing the following estimate

$$\int_{\Omega_k^{(l)}} |H(\mu)'(z)| dA(z) \lesssim 2^l |I_k^{(l)}| = \mu(I_k^{(l)}), \quad k, l \geq 1. \quad (12)$$

Indeed, once (12) is established we simply use the fact that $\sum_k \mu(I_k^{(l)}) = 1$ and apply the estimate (11) to deduce that

$$\sum_{k,l} \int_{\Omega_k^{(l)}} |S'_\mu(z)| dA(z) \lesssim \sum_l e^{-c2^l} \sum_k \mu(I_k^{(l)}) < \infty,$$

which gives (10) and finishes the proof. So it only remains to show that estimate (12) holds. For $z \in \Omega_k^{(l)}$ we write

$$\begin{aligned} H(\mu)'(z) &= \sum_j \int_{I_j^{(l+1)}} \frac{2\zeta d\mu(\zeta)}{(\zeta - z)^2} = \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} \int_{I_j^{(l+1)}} \frac{2\zeta d\mu(\zeta)}{(\zeta - z)^2} + \sum_{j: I_j^{(l+1)} \cap I_k^{(l)} = \emptyset} \int_{I_j^{(l+1)}} \frac{2\zeta d\mu(\zeta)}{(\zeta - z)^2} \\ &= (I) + (II). \end{aligned}$$

We shall first treat the term (II). Note that if $z \in \Omega_k^{(l)}$ and $\zeta \in I_j^{(l+1)}$ with $I_j^{(l+1)} \cap I_k^{(l)} = \emptyset$, then $|\zeta - z| \gtrsim |I_k^{(l)}|$. Then

$$|(II)| \lesssim \sum_{j: I_j^{(l+1)} \cap I_k^{(l)} = \emptyset} \frac{\mu(I_j^{(l+1)})}{\text{dist}(I_j^{(l+1)}, I_k^{(l)})^2} = 2^{l+1} \sum_{j: I_j^{(l+1)} \cap I_k^{(l)} = \emptyset} \frac{|I_j^{(l+1)}|}{\text{dist}(I_j^{(l+1)}, I_k^{(l)})^2} \lesssim \frac{2^l}{|I_k^{(l)}|}.$$

This implies

$$\int_{\Omega_k^{(l)}} |(II)| dA \lesssim 2^l |I_k^{(l)}|,$$

and hence we may now devote our attention to estimating (I). To this end, we shall decompose it into two pieces by rewriting

$$\begin{aligned} (I) &= \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} \int_{I_j^{(l+1)}} \frac{2\zeta}{(\zeta - z)^2} 2^{l+1} d\mu(\zeta) + \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} \int_{I_j^{(l+1)}} \frac{2\zeta}{(\zeta - z)^2} (d\mu(\zeta) - 2^{l+1} d\mu(\zeta)) \\ &= A + B. \end{aligned}$$

Denote by $\xi_j^{(l+1)}$ the center of $I_j^{(l+1)}$ and observe that since $\mu(I_j^{(l+1)}) = 2^{l+1} |I_j^{(l+1)}|$, we can add a cancellative term to B and write

$$B = \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} \int_{I_j^{(l+1)}} \left(\frac{2\zeta}{(\zeta - z)^2} - \frac{2\xi_j^{(l+1)}}{(\xi_j^{(l+1)} - z)^2} \right) (d\mu(\zeta) - 2^{l+1} d\mu(\zeta)).$$

Now using the mean value Theorem together and the fact that $|\zeta - z| \gtrsim |\xi_j^{(l+1)} - z|$ whenever $\zeta \in I_j^{(l+1)}$ and $z \in \Omega_k^{(l)}$, we get

$$|B| \lesssim \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} \frac{\mu(I_j^{(l+1)}) + 2^{l+1} |I_j^{(l+1)}|}{|\xi_j^{(l+1)} - z|^3} |I_j^{(l+1)}| = \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} \frac{2^{l+2} |I_j^{(l+1)}|^2}{|\xi_j^{(l+1)} - z|^3}. \quad (13)$$

A straightforward calculation shows that

$$\int_{\Omega_k^{(l)}} \frac{dA(z)}{|\xi_j^{(l+1)} - z|^3} \lesssim \frac{1}{|I_j^{(l+1)}|}. \quad (14)$$

Combining this with (13), we conclude

$$\int_{\Omega_k^{(l)}} |B| dA \lesssim 2^l \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} |I_j^{(l+1)}| \leq 2^l |I_k^{(l)}|.$$

Hence it only remains to estimate the quantity A . Denote by $I + |I|$ the arc I rotated by $|I|$ -units and observe that by construction of the dyadic arcs $I_j^{(l+1)}$, we have

$$\bigcup_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} I_j^{(l+1)} \cup \left(I_j^{(l+1)} + |I_j^{(l+1)}| \right) = I_k^{(l)}.$$

Applying the mean value Theorem as before, we get

$$\left| \int_{I_j^{(l+1)}} \frac{\zeta dm(\zeta)}{(\zeta - z)^2} - \int_{I_j^{(l+1)} + |I_j^{(l+1)}|} \frac{\zeta dm(\zeta)}{(\zeta - z)^2} \right| \lesssim \frac{|I_j^{(l+1)}|^2}{|z - \xi_j^{(l+1)}|^3}, \quad z \in \Omega_k^{(l)}, \quad j, l, k \geq 1.$$

Hence

$$\begin{aligned} \int_{\Omega_k^{(l)}} \left| A - 2^l \int_{I_k^{(l)}} \frac{2\zeta dm(\zeta)}{(\zeta - z)^2} \right| dA(z) &\lesssim \\ \int_{\Omega_k^{(l)}} 2^l \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} \left| \int_{I_j^{(l+1)}} \frac{\zeta dm(\zeta)}{(\zeta - z)^2} - \int_{I_j^{(l+1)} + |I_j^{(l+1)}|} \frac{\zeta dm(\zeta)}{(\zeta - z)^2} \right| dA(z) &\lesssim \\ 2^l \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} |I_j^{(l+1)}|^2 \int_{\Omega_k^{(l)}} \frac{dA(z)}{|\xi_j^{(l+1)} - z|^3} &\lesssim 2^l \sum_{j: I_j^{(l+1)} \subseteq I_k^{(l)}} |I_j^{(l+1)}| \leq 2^l |I_k^{(l)}|. \end{aligned}$$

In the penultimate step, we have used (14). In order to finish the proof, it just remains to verify that

$$\int_{\Omega_k^{(l)}} \left| \int_{I_k^{(l)}} \frac{\zeta dm(\zeta)}{(\zeta - z)^2} \right| dA(z) \lesssim |I_k^{(l)}|. \quad (15)$$

To this end, let $I_k^{(l)} = (a_k^{(l)}, b_k^{(l)})$ and observe that fixed $z \in \mathbb{D}$, the primitive of $\zeta/(\zeta - z)^2$ is explicitly given by $\log(\zeta - z) - z/(\zeta - z)$, which allows us to compute

$$\int_{I_k^{(l)}} \frac{\zeta dm(\zeta)}{(\zeta - z)^2} = \log \left(\frac{b_k^{(l)} - z}{a_k^{(l)} - z} \right) + \frac{z |I_k^{(l)}|}{(b_k^{(l)} - z)(a_k^{(l)} - z)}.$$

Then

$$\left| \int_{I_k^{(l)}} \frac{\zeta dm(\zeta)}{(\zeta - z)^2} \right| \lesssim \frac{|I_k^{(l)}|}{|b_k^{(l)} - z||a_k^{(l)} - z|}, \quad z \in \Omega_k^{(l)},$$

and thus

$$\int_{\Omega_k^{(l)}} \left| \int_{I_k^{(l)}} \frac{\zeta dm(\zeta)}{(\zeta - z)^2} \right| dA(z) \lesssim |I_k^{(l)}| \int_{\Omega_k^{(l)}} \frac{dA(z)}{|b_k^{(l)} - z||a_k^{(l)} - z|}.$$

Since either $|z - b_k^{(l)}| \geq |I_k^{(l)}|/2$ or $|z - a_k^{(l)}| \geq |I_k^{(l)}|/2$ for any $z \in \Omega_k^{(l)}$, we deduce

$$\int_{\Omega_k^{(l)}} \frac{dA(z)}{|b_k^{(l)} - z||a_k^{(l)} - z|} \lesssim 1.$$

This gives (15) and finishes the proof. $\square$

4.4 The permanence property discriminates no compact set

Here we devote our attention to the proof of Theorem 1.6, which roughly asserts that any compact set on the unit circle of Lebesgue measure zero can support a singular measure μ for which the associated singular inner function is a member of W^1 . This implies that no condition on the carrier set of a singular measure μ alone can describe the permanence property of S_μ in the Bloch space.

Proof of Theorem 1.6. The proof is divided in three steps.

Step 1: Constructing a covering of E

We first construct by induction an appropriate sequence of coverings of E by dyadic arcs. Set $\mathcal{G}_0 := \{\partial\mathbb{D}\}$ and assume that a covering $\mathcal{G}_k = \{I_j^{(k)} : j \geq 1\}$ of E by dyadic arcs has already been constructed and $E \cap I_j^{(k)} \neq \emptyset$ for any $j \geq 1$. For any $I \in \mathcal{G}_k$ we denote by $\tilde{\mathcal{G}}(I)$ the collection of maximal dyadic subarcs $\tilde{J}$ of I such that at least one of the two dyadic children of $\tilde{J}$ does not intersect E . Note that $\tilde{\mathcal{G}}(I) = \{I\}$ if one of the two dyadic children of $I \in \mathcal{G}_k$ does not intersect E . Moreover, by maximality of the collection $\tilde{\mathcal{G}}(I)$, for each $\tilde{J} \in \tilde{\mathcal{G}}(I)$ there exists exactly one dyadic child J of $\tilde{J}$ with $J \cap E \neq \emptyset$. Now consider

$$\mathcal{G}(I) := \left\{ J : J \text{ dyadic child of some } \tilde{J} \in \tilde{\mathcal{G}}(I), \text{ with } J \cap E \neq \emptyset \right\}.$$

The set $\mathcal{G}_{k+1} := \bigcup_{I \in \mathcal{G}_k} \mathcal{G}(I)$ is again a covering of E that will be denoted by $\mathcal{G}_{k+1} = \{I_j^{(k+1)} : j \geq 1\}$. Since E is a closed set of Lebesgue measure zero, for any arc $I \in \mathcal{G}_k$, the union of the arcs of the collection $\tilde{\mathcal{G}}(I)$ covers almost every point of I . Hence the collection $\{\mathcal{G}_k\}_k$ satisfies the following packing condition: for any $I_l^{(k)} \in \mathcal{G}_k$ we have

$$\sum_{\substack{J \in \mathcal{G}_{k+1} \\ J \subset I_l^{(k)}}} |J| = \sum_{\substack{j \geq 1 \\ I_j^{(k+1)} \subseteq I_l^{(k)}}} |I_j^{(k+1)}| = \frac{|I_l^{(k)}|}{2}, \quad k, l \geq 1. \quad (16)$$

Observe that by construction $E = \cap_k \cup_{J \in \mathcal{G}_k} J$. Indeed, the inclusion $\subseteq$ follows from the fact that each $\mathcal{G}_k$ is a covering of E , while if $\zeta_0 \notin E$, then there exists a dyadic arc I_0 containing ζ_0 with $I_0 \cap E = \emptyset$. Hence for every sufficiently large k , we must have $\zeta_0 \in I_0 \subset \cap_{J \in \mathcal{G}_k} \partial \mathbb{D} \setminus J$, which establishes the reverse inclusion $\supseteq$.

Step 2: Constructing the measure μ :

We now construct μ by declaring its mass on dyadic arcs. Initially, we set $\mu(\partial \mathbb{D}) = 1$. Now let $I \subset \partial \mathbb{D}$ be a dyadic arc and assume, by means of induction, that $\mu(I)$ has already been defined in such a way that $\mu(I) = 0$ whenever $I \cap E = \emptyset$. Denote by I_+, I_- the dyadic children of I . If both I_+, I_- happens to meet E , then we distribute the mass evenly by declaring

$$\frac{\mu(I_+)}{|I_+|} = \frac{\mu(I_-)}{|I_-|} = \frac{\mu(I)}{|I|}.$$

If not, say $I_+ \cap E = \emptyset$, then we distribute all the mass to I_- by setting

$$\frac{\mu(I_-)}{|I_-|} = 2 \frac{\mu(I)}{|I|}, \quad \mu(I_+) = 0.$$

Indeed, this is consistent since

$$\frac{\mu(I)}{|I|} = \frac{1}{2} \frac{\mu(I_+)}{|I_+|} + \frac{1}{2} \frac{\mu(I_-)}{|I_-|}.$$

Observe that by construction, the support of μ is equal to $E = \cap_k \cup_{J \in \mathcal{G}_k} J$ which has Lebesgue measure zero. Hence μ is a non-trivial singular measure. Moreover, it is clear that for any dyadic arc I with $I \cap E \neq \emptyset$, we have $\mu(I) \geq |I|$. Observe also that by construction, for any $I_j^{(k)} \in \mathcal{G}_k$, we have

$$\frac{\mu(I_j^{(k)})}{|I_j^{(k)}|} \geq 2^k, \quad j, k \geq 1. \quad (17)$$

Step 3: Verifying condition (ii)

It now remains to prove that the associated singular inner function S_μ belongs to W^1 . To this end, for each $k \geq 1$ and $I_j^{(k)} \in \mathcal{G}_k$ consider the region

$$\Omega_j^{(k)} := Q_{I_j^{(k)}} \setminus \bigcup_{\tilde{J} \in \tilde{\mathcal{G}}(I_j^{(k)})} Q_{\tilde{J}}.$$

According to (iii) of Corollary 4.3, it is sufficient to establish that

$$\sum_{k,j} \int_{\Omega_j^{(k)}} |S'_\mu| dA < \infty.$$

Next we shall make yet another reduction. Let $I = I_j^{(k)}$ and denote by μ_I the restriction of μ to I . According to Lemma 4.2, there exists an absolute constant $C > 0$ such that

$$\int_{Q_I} |S'_{\mu_{\partial\mathbb{D}\setminus I}}(z)| dA(z) \leq C|I|.$$

Hence using (16), we get

$$\sum_{k,j} \int_{\Omega_j^{(k)}} |S'_{\mu_{\partial\mathbb{D}\setminus I_j^{(k)}}}(z)| dA(z) \leq C \sum_{k,j} |I_j^{(k)}| = \sum_{k \geq 1} 2^{-k} = C.$$

With this observation at hand, it then suffices to show that

$$\sum_{k,j} \int_{\Omega_j^{(k)}} |S'_{\mu_{I_j^{(k)}}}(z)| dA(z) < \infty. \quad (18)$$

Using (17) in conjunction with standard estimates of Poisson kernels, there exists a universal constant $c > 0$ such that

$$|S'_{\mu_{I_j^{(k)}}}(z)| \leq e^{-c2^k} |H(\mu_{I_j^{(k)}})'(z)|, \quad z \in \Omega_j^{(k)}, k, j \geq 1. \quad (19)$$

Thus (18) follows from the estimate

$$\int_{\Omega_j^{(k)}} |H(\mu_{I_j^{(k)}})'(z)| dA(z) \lesssim \mu(I_j^{(k)}) \quad j, k \geq 1.$$

whose proof is quite similar to the proof of estimate (12). We omit the details. □

5 Cyclicity and invertibility

5.1 A sufficient condition for cyclicity

This subsection is devoted to proving Theorem 1.7. A finite complex Borel measure ν on $\partial\mathbb{D}$ (not necessarily singular) is said to be a *Zygmund measure* if

$$\|\nu\|_* := \sup \left| \frac{\nu(I)}{|I|} - \frac{\nu(I')}{|I'|} \right| < \infty,$$

where the supremum is taken over all pairs of contiguous arcs I, I' of the same length. For the sake of future references, we shall state the following result due to Duren, Shapiro and Shields in [16] below, see also [15].

Lemma 5.1. *A finite complex Borel measure ν on $\partial\mathbb{D}$ is a Zygmund measure if and only if its Herglotz transform defined as*

$$H(\nu)(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta), \quad z \in \mathbb{D},$$

belongs to $\mathcal{B}$. In this case, there exists a universal constant $C > 0$ such that

$$|H(\nu)(z) - H(\nu)(w)| \leq C \|\nu\|_* \beta(z, w), \quad z, w \in \mathbb{D},$$

where $\beta(z, w)$ denotes the hyperbolic distance on $\mathbb{D}$ between the points z and w . Moreover, the following asymptotic relation holds:

$$P(\nu)(z) = \frac{\nu(I_z)}{|I_z|} + \mathcal{O}(1) \|\nu\|_*, \quad z \in \mathbb{D}. \quad (20)$$

where $I_z \subset \partial\mathbb{D}$ denotes the arc centered at $z/|z|$ of length $1 - |z|$.

We will also use an auxiliary result that has appeared in the setting of $\mathbb{R}^d$ in [15]. We shall need it in the context of the unit circle where its proof is an exercise left to the reader.

Lemma 5.2. *Let ν be a finite Borel measure on $\partial\mathbb{D}$. Then there exists an absolute constant $C > 0$, such that*

$$(1 - |z|)|H(\nu)'(z)| \leq C \int_0^{2\pi} \frac{(1 - |z|)}{|e^{it} - |z||^3} |\nu(I(z, t)) - \nu(I(z, -t))| dt, \quad z \in \mathbb{D},$$

where $I(z, t)$ denotes the smallest arc joining $z/|z|$ to $ze^{it}/|z|$ if $z \neq 0$ and $I(0, t)$ is the arc joining 1 and e^{it} .

Given an arc $I \subset \partial\mathbb{D}$ and $C > 0$ let CI denote the arc with the same center as I and length $C|I|$. We will also need the following technical result.

Lemma 5.3. *Let ν be a positive finite Borel measure on $\partial\mathbb{D}$. Assume there exists a constant $C = C(\nu) > 0$ such that*

$$\left| \frac{\nu(I)}{|I|} - \frac{\nu(I')}{|I'|} \right| \leq C e^{-\nu(I)/|I|}, \quad (21)$$

for any pair of contiguous arcs $I, I' \subset \partial\mathbb{D}$ of the same length. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any integer n with $|n| \leq \delta e^{\nu(I)/|I|}$ we have

$$\left| \frac{\nu(2^n I)}{|2^n I|} - \frac{\nu(I)}{|I|} \right| \leq \min\{\varepsilon n, 1\}. \quad (22)$$

Proof. Note that the assumption gives that ν is a Zygmund measure. Moreover (21) gives that there exists a constant $C_1 > 0$ such that

$$\left| \frac{\nu(2I)}{2I} - \frac{\nu(I)}{|I|} \right| \leq C_1 e^{-\max\{\nu(2I)/|2I|, \nu(I)/|I|\}},$$

for any arc $I \subset \partial\mathbb{D}$. Using the elementary estimate $|e^x - e^y| \leq e^{\max\{x, y\}}|x - y|$, $x, y \in \mathbb{R}$, we deduce

$$|e^{\nu(2I)/|2I|} - e^{\nu(I)/|I|}| \leq C_1,$$

for any arc $I \subset \partial\mathbb{D}$. Hence for any integer k we have

$$\left| e^{\nu(2^k I)/|2^k I|} - e^{\nu(I)/|I|} \right| \leq C_1 |k|. \quad (23)$$

Now if $|n| \leq (2C_1)^{-1} e^{\nu(I)/|I|}$, we obtain

$$|e^{\nu(2^n I)/|2^n I|} - e^{\nu(I)/|I|}| \leq \frac{1}{2} e^{\nu(I)/|I|}. \quad (24)$$

Hence

$$\left| \frac{\nu(2^n I)}{|2^n I|} - \frac{\nu(I)}{|I|} \right| \leq 1.$$

Note that to check (22) we can assume that $\nu(I)/|I|$ is large. Fix $\varepsilon > 0$. Let $M = M(\varepsilon) > 0$ be a large number to be fixed later. Use (23) to pick $\delta > 0$ such that $\nu(2^k I)/|2^k I| > M$ if k is an integer with $|k| < \delta e^{\nu(I)/|I|}$. Then assumption (21) gives that

$$\left| \frac{\nu(2^k I)}{|2^k I|} - \frac{\nu(2^{k-1} I)}{|2^{k-1} I|} \right| \lesssim e^{-M}, \quad \text{if } |k| \leq \delta e^{\nu(I)/|I|}.$$

We deduce that

$$\left| \frac{\nu(2^n I)}{|2^n I|} - \frac{\nu(I)}{|I|} \right| \leq \sum_{k=1}^n \left| \frac{\nu(2^k I)}{|2^k I|} - \frac{\nu(2^{k-1} I)}{|2^{k-1} I|} \right| \lesssim e^{-M} n, \quad \text{if } |n| \leq \delta e^{\nu(I)/|I|}.$$

and taking $M \lesssim \log(1/\varepsilon)$, estimate (22) follows. $\square$

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. It is sufficient to find a sequence of bounded analytic functions h_n such that:

- (i) $\sup_n \|h_n f\|_{\mathcal{B}} < \infty$,
- (ii) $h_n(z) f(z) \rightarrow 1$ for any $z \in \mathbb{D}$.

Set $r_n := 1 - 2^{-n}$ and consider $h_n(z) := 1/f(r_n z)$, $z \in \mathbb{D}$. Note that condition (ii) is trivially satisfied, hence it remains to verify (i). Observe that we can write

$$(1 - |z|) \left(\frac{f(z)}{f(r_n z)} \right)' = (1 - |z|) \left(\frac{f'(z)}{f(r_n z)} \right) - (1 - |z|) \left(\frac{r_n f(z) f'(r_n z)}{f^2(r_n z)} \right) =: A + B.$$

We proceed by estimating both terms separately.

Step 1: Estimation of B

Set $f = \exp(-H(\nu))$ where $H(\nu)$ is the Herglotz transform of the Herglotz-Nevanlinna measure ν of f . Observing that ν is necessarily a Zygmund measure, we may apply Lemma 5.1 to deduce

$$(1 - |r_n z|) \left| \frac{r_n f'(r_n z)}{f(r_n z)} \right| \leq \|H(\nu)\|_{\mathcal{B}} \lesssim \|\mu\|_*, \quad z \in \mathbb{D},$$

and thus

$$|B| \lesssim \frac{1 - |z|}{1 - |r_n z|} \frac{|f(z)|}{|f(r_n z)|} = \frac{1 - |z|}{1 - |r_n z|} \exp(P(\nu)(r_n z) - P(\nu)(z)), \quad z \in \mathbb{D}, \quad (25)$$

where $P(\nu)$ denotes the Poisson extension of ν . Now for $|z| \leq r_n$, we clearly have $\beta(z, r_n z) \lesssim 1$. An application of Lemma 5.1 then gives $|P(\nu)(r_n z) - P(\nu)(z)| \lesssim 1$ if $|z| \leq r_n$. and hence

$$\sup_{|z| \leq r_n} |B| \lesssim 1.$$

We may thus fix $z \in \mathbb{D}$ with $|z| \geq r_n$. Note that in order to estimate the right hand side of (25), we may without loss of generality assume that $P(\nu)(r_n z)$ is large, which according to Lemma 5.1 is equivalent to $\nu(I_{r_n z})/|I_{r_n z}|$ being large. Doing so, we pick a large number $R > 0$ for which $\nu(I_{r_n z})/|I_{r_n z}| > R$, to be specified momentarily. Observe that condition (1), then implies that

$$\left| \frac{\nu(I)}{|I|} - \frac{\nu(I/2)}{|I/2|} \right| \leq C e^{-R} \quad (26)$$

for any arc $I \subseteq I_{r_n z}$. Let $N := N(n, z)$ denote the largest positive integer satisfying

$$N \leq \log \left(\frac{|I_{r_n z}|}{|I_z|} \right) = \log \left(\frac{1 - |r_n z|}{1 - |z|} \right).$$

By means of iterating the estimate (26) N times, we obtain

$$\left| \frac{\nu(I_z)}{|I_z|} - \frac{\nu(I_{r_n z})}{|I_{r_n z}|} \right| \leq C N e^{-R} \leq C e^{-R} \log \left(\frac{1 - |r_n z|}{1 - |z|} \right).$$

Yet another application of Lemma 5.1 allows us to express

$$P(\nu)(r_n z) - P(\nu)(z) = \frac{\nu(I_{r_n z})}{|I_{r_n z}|} - \frac{\nu(I_z)}{|I_z|} + \mathcal{O}(\|\nu\|_*)$$

and thus we get

$$|P(\nu)(r_n z) - P(\nu)(z)| \leq C e^{-R} \log \left(\frac{1 - |r_n z|}{1 - |z|} \right) + \mathcal{O}(\|\nu\|_*).$$

Now it is just a matter of choosing $R > 0$ sufficiently large so that $Ce^{-R} < 1$, which ultimately yields

$$|B| \lesssim \left(\frac{1 - |z|}{1 - |r_n z|} \right)^{1 - Ce^{-R}} \lesssim 1.$$

Step 2: Estimation of A

By definition, we readily have

$$|A| \lesssim (1 - |z|) |H(\nu)'(z)| \exp(P(\nu)(r_n z) - P(\nu)(z)), \quad z \in \mathbb{D}. \quad (27)$$

As observed previously $\beta(z, r_n z) \lesssim 1$ whenever $|z| \leq r_n$. Hence using that ν is a Zygmund measure, we have $(1 - |z|) |H(\mu)'(z)| \lesssim 1$ and $|P(\nu)(r_n z) - P(\nu)(z)| \lesssim 1$ if $|z| \leq r_n$. This gives

$$\sup_{|z| \leq r_n} |A| \lesssim 1.$$

In what follows, we shall thus fix $z \in \mathbb{D}$ with $|z| > r_n$ and moreover, we may also assume that $P(\nu)(r_n z)$ is sufficiently large. Denote by $I(z, t)$ the smallest arc on $\partial\mathbb{D}$ joining $z/|z|$ with $ze^{it}/|z|$. An application of Lemma 5.2 gives

$$(1 - |z|) |H(\nu)'(z)| \lesssim \int_0^{2\pi} \frac{(1 - |z|)}{|e^{it} - |z||^3} |\nu(I(z, t)) - \nu(I(z, -t))| dt.$$

Decomposing the integral according to the intervals $I(k) = \{t \in [0, 2\pi] : |t| \leq 2^k(1 - |z|)\}$, we get

$$\begin{aligned} (1 - |z|) |H(\nu)'(z)| &\lesssim (1 - |z|)^{-2} \int_{I(1)} |\nu(I(z, t)) - \nu(I(z, -t))| dt \\ &\quad + \sum_{k > 0} 2^{-3k} (1 - |z|)^{-2} \int_{I(k+1) \setminus I(k)} |\nu(I(z, t)) - \nu(I(z, -t))| dt. \end{aligned}$$

Let $T = T(n, z)$ denote the integer part of $\log_2(2^{-n}/(1 - |z|))$. Observe that if $t \in I(k)$ with $k \leq T$, then $I(z, t)$ is contained in a fixed multiple of $I_{r_n z}$. This in conjunction with the assumption (1) yields

$$|\nu(I(z, t)) - \nu(I(z, -t))| \lesssim |t| \exp\left(-\frac{\nu(I_{r_n z})}{|I_{r_n z}|}\right).$$

Hence we get

$$\sum_{0 \leq k < T} 2^{-3k} (1 - |z|)^{-2} \int_{I(k)} |\nu(I(z, t)) - \nu(I(z, -t))| dt \lesssim \exp\left(-\frac{\nu(I_{r_n z})}{|I_{r_n z}|}\right).$$

Fix $t \in I(k+1) \setminus I(k)$ with $k > T$. Note that the arc $I_{r_n z}$ is contained in a fixed multiple of $I(z, t) \cup I(z, -t)$. According to Lemma 5.3, there exists $\delta > 0$ such that

$$\left| \frac{\nu(I(z, t))}{|I(z, t)|} - \frac{\nu(I_{r_n z})}{|I_{r_n z}|} \right| \lesssim 1, \quad \text{if } \log_2\left(\frac{2^k(1 - |z|)}{2^{-n}}\right) \leq \delta e^{\frac{\nu(I_{r_n z})}{|I_{r_n z}|}}. \quad (28)$$

The assumption (1) implies in this case

$$\left| \frac{\nu(I(z, t))}{|I(z, t)|} - \frac{\nu(I(z, -t))}{|I(z, -t)|} \right| \lesssim \exp \left(-\frac{\nu(I(z, t))}{|I(z, t)|} \right).$$

Using estimate (28) in the right hand side term, we obtain

$$\left| \frac{\nu(I(z, t))}{|I(z, t)|} - \frac{\nu(I(z, -t))}{|I(z, -t)|} \right| \lesssim \exp \left(-\frac{\nu(I_{r_n z})}{|I_{r_n z}|} \right), \quad \text{if } \log_2 \left(\frac{2^k(1-|z|)}{2^{-n}} \right) \leq \delta e^{\frac{\nu(I_{r_n z})}{|I_{r_n z}|}}.$$

Let T^* be the largest integer such that $T^* + \log_2 \left(\frac{1-|z|}{2^{-n}} \right) \leq \delta e^{\nu(I_{r_n z})/|I_{r_n z}|}$. With this choice, we get

$$\sum_{k=T}^{T^*} 2^{-3k} (1-|z|)^{-2} \int_{I(k+1)} |\nu(I(z, t)) - \nu(I(z, -t))| dt \lesssim \exp \left(-\frac{\nu(I_{r_n z})}{|I_{r_n z}|} \right).$$

On the other hand, using that ν is a Zygmund measure we arrive at

$$\sum_{k>T^*} 2^{-3k} (1-|z|)^{-2} \int_{I(k+1)} |\nu(I(z, t)) - \nu(I(z, -t))| dt \lesssim \exp \left(-\frac{\nu(I_{r_n z})}{|I_{r_n z}|} \right).$$

Finally, combining all terms in the sum, we obtain

$$(1-|z|)|H'(\nu)(z)| \lesssim \exp \left(-\frac{\nu(I_{r_n z})}{|I_{r_n z}|} \right), \quad |z| > r_n.$$

With these estimates at hand, we may now return back to (27), deducing that $|A| \lesssim \exp(-P(\nu)(z)) \leq 1$. This completes the proof. $\square$

5.2 A characterization of invertibility in Bloch

Proof of Theorem 1.8. In order to prove the sufficiency, note that by means of writing $(1/f)' = H(\nu)' / f$, it suffices to show that $(1-|z|^2)|H(\nu)'(z)| \lesssim |f(z)|$, $z \in \mathbb{D}$. Again, condition (2) readily implies that ν is a Zygmund measure. In view of the estimate (20) in Lemma 5.1, it suffices to prove that

$$(1-|z|^2)|H(\nu)'(z)| \lesssim e^{-\nu(I_z)/|I_z|}, \quad z \in \mathbb{D}. \quad (29)$$

For $z \in \mathbb{D}$ and $n \in \mathbb{Z}$, consider the intervals $I_n = \{t \in [0, 2\pi] : 2^{n-1}(1-|z|) \leq t < 2^n(1-|z|)\}$. According to Lemma 5.2 we have

$$(1-|z|^2)|H(\nu)'(z)| \lesssim \sum_{n \in \mathbb{Z}} \int_{I_n} \frac{1-|z|}{|e^{it} - |z||^3} |\nu(I(z, t)) - \nu(I(z, -t))| dt, \quad z \in \mathbb{D}.$$

We now estimate each term in the sum. Assumption (2) implies

$$|\nu(I(z, t)) - \nu(I(z, -t))| \leq C|t|e^{-\nu(I_t(z))/|I_t(z)|}, \quad (30)$$

where as before, $I_t(z)$ denotes the arc $I_t(z) = I(z, t) \cup I(z, -t)$. Let $\varepsilon > 0$ be a small number to be fixed later. Applying Lemma 5.3, we obtain a number $\delta > 0$ such that

$$\left| \frac{\nu(I_t(z))}{|I_t(z)|} - \frac{\nu(I_z)}{|I_z|} \right| \leq \varepsilon n, \quad t \in I_n, \quad |n| \leq T := \delta e^{\nu(I_z)/|I_z|}. \quad (31)$$

It thus follows from (30) and (31) that

$$\left| \frac{\nu(I(z, t))}{t} - \frac{\nu(I(z, -t))}{t} \right| \lesssim e^{\varepsilon n} e^{-\nu(I_z)/|I_z|}, \quad t \in I_n, \quad |n| \leq T.$$

Now if $0 < \varepsilon < \log 2$, we deduce that

$$\sum_{n: |n| \leq T} \int_{I_n} \frac{1 - |z|}{|e^{it} - |z||^3} |\nu(I(z, t)) - \nu(I(z, -t))| dt \lesssim e^{-\nu(I_z)/|I_z|}.$$

The tail of the sum is estimated using the fact that ν is a Zygmund measure. In fact, one obtains

$$\sum_{n: |n| > T} \int_{I_n} \frac{1 - |z|}{|e^{it} - |z||^3} |\nu(I(z, t)) - \nu(I(z, -t))| dt \lesssim 2^{-T} \leq C(\delta) e^{-\nu(I_z)/|I_z|},$$

where $C(\delta) > 0$ is a constant depending on δ . This proves the estimate (29) and therefore completes the proof of the sufficiency. We now turn our attention to the necessity part of the proof. Rewrite $f' = H(\nu)'f$, we see that our assumption reads

$$(1 - |z|^2) |H(\nu)'(z)| \lesssim e^{-P(\nu)(z)}, \quad z \in \mathbb{D}. \quad (32)$$

It turns out that condition (32) implies that

$$P(\nu)(z) = \frac{\nu(I_z)}{|I_z|} + \mathcal{O}(1) e^{-\nu(I_z)/|I_z|}, \quad z \in \mathbb{D}. \quad (33)$$

This is completely analogous to the estimate (20) stated in Lemma 5.1 which holds for Zygmund measures. Actually one can prove (33) mimicking the proof of (20) given in [15]. The details are omitted. Note that (32) gives that $H(\nu) \in \mathcal{B}$. According to Lemma 5.1, ν is a Zygmund measure. Using this, we see that the estimate (32) implies

$$|P(\nu)(z_I) - P(\nu)(z_{I'})| \leq |H(\nu)(z_I) - H(\nu)(z_{I'})| \lesssim e^{-\nu(I)/|I|},$$

for any pair of contiguous arcs $I, I' \subset \partial\mathbb{D}$ of the same length. This in conjunction with (33) then shows that condition (2) holds, which finishes the proof. $\square$

5.3 Invertibility does not imply cyclicity in $\mathcal{B}$

This section is devoted to a self-contained proof of Theorem 1.9.

Proof of Theorem 1.9. Let E be a sequence of complex numbers containing the origin such that both $\mathbb{C} \setminus E$ and $\mathbb{C} \setminus E^{-1}$ are open sets which do not contain arbitrarily large discs, that is, there exists $R_0 > 0$ such that no disc of radius R_0 is contained in either $\mathbb{C} \setminus E$ or $\mathbb{C} \setminus E^{-1}$. Here $E^{-1} = \{w \in \mathbb{C} : 1/w \in E\}$. Let $f : \mathbb{D} \rightarrow \mathbb{C} \setminus E$ be an analytic universal covering map. According to Bloch's Theorem, we have that both f and $1/f$ belong to $\mathcal{B}$. Next we shall show that f is not cyclic in $\mathcal{B}$.

Assume f has non-tangential limit, say L , at a point of the unit circle. Since f is a covering map, L must be either infinity or a point in E . Since E is countable, Privalov's Theorem gives that f can only have non-tangential limit at a set of Lebesgue measure zero of points of the unit circle. By Plessner's Theorem (for instance, see p. 205 in [21]), its non-tangential cluster set at almost every point of the unit circle is the whole extended plane, that is,

$$\overline{\cap_{0 < r < 1} f(\Gamma_r(\xi))} = \mathbb{C} \cup \{\infty\}, \quad m\text{- a.e. } \xi \in \partial\mathbb{D}. \quad (34)$$

Here $\Gamma_r(\xi) = \{z \in \mathbb{D} : |z - \xi| < M(1 - |z|), |z| > r\}$ denotes the truncated Stolz angle of some fixed aperture $M > 1$ with vertex at the point $\xi \in \partial\mathbb{D}$.

Next we will show that $\mathcal{A}(f) = \{fh \in \mathcal{B} : h \in H^\infty\}$ is weak-star closed in $\mathcal{B}$. Since $\mathcal{A}(f)$ is convex, it suffices by the Krein-Smulian Theorem to show that it is weak-star sequentially closed. To this end, let $h_n \in H^\infty$ such that fh_n tends to F weak-star in $\mathcal{B}$. Then there exists a constant $C > 0$ such that

$$\sup_{z \in \mathbb{D}} |(1 - |z|^2)(f'(z)h_n(z) + f(z)h'_n(z))| \leq C. \quad (35)$$

Fix $a \in \mathbb{C} \setminus E$ and use (34) to show that for almost every $\xi \in \partial\mathbb{D}$ there exist points $z_k(\xi) \in \mathbb{D}$ tending non-tangentially to ξ such that $f(z_k(\xi)) \rightarrow a$. Since $(1 - |z|^2)|f'(z)| \geq \text{dist}(f(z), E)$, $z \in \mathbb{D}$ (see [5]), we deduce that there exists a constant $c > 0$ such that

$$\liminf_{k \rightarrow \infty} (1 - |z_k(\xi)|^2)|f'(z_k(\xi))| > c \quad (36)$$

Note that Bloch's Theorem shows that if an analytic function F in the unit disc has non-tangential limit at a given point $\zeta \in \partial\mathbb{D}$, then $(1 - |z|)|F'(z)| \rightarrow 0$ as $z \in \mathbb{D}$ tends non-tangentially to ζ . Since h_n has non-tangential limit along almost every radius, for each integer n we have

$$\lim_{z \rightarrow \xi} (1 - |z|)|h'_n(z)| = 0, \quad m\text{- a.e. } \xi \in \partial\mathbb{D},$$

where the limit is taken as $z \in \mathbb{D}$ tends to ξ non-tangentially. Applying (35) and (36) we deduce that

$$\limsup_{k \rightarrow \infty} |h_n(z_k(\xi))| \leq C, \quad m\text{- a.e. } \xi \in \partial\mathbb{D}, \quad n = 1, 2, \dots$$

Hence $\|h_n\|_\infty \leq C$, $n = 1, 2, \dots$ and we may extract a subsequence of $\{h_n\}$ which converges pointwise in $\mathbb{D}$ to a function $h \in H^\infty$ and thus $F = fh$. We conclude that $\mathcal{A}(f)$ is weak-star

closed in $\mathcal{B}$. Now $[f]_{\mathcal{B}} \subseteq \mathcal{A}(f)$ and it is clear that $\mathcal{A}(f)$ is a proper subspace of $\mathcal{B}$. We conclude that f is not cyclic in $\mathcal{B}$. $\square$

5.4 A problem of weak-star sequential closure in $\mathcal{B}$

In the proof of Theorem 1.10 we will need to construct functions in $\mathcal{B}$ which enjoy maximal radial growth at considerable large subset of $\mathbb{D}$.

Proposition 5.4. *There exists $f \in BMOA \subset \mathcal{B}$ satisfying the following properties:*

- (i) *The function $1/f$ belongs to H^∞ .*
- (ii) *For some $\delta > 0$, the set $E(f) := \left\{ z \in \mathbb{D} : |f(z)| \geq \delta \log\left(\frac{1}{1-|z|}\right) \right\}$ accumulates everywhere on the unit circle $\partial\mathbb{D}$, that is, $\partial\mathbb{D} \subset \overline{E(f)}$.*

Observe that such an f must necessarily be outer, hence it is cyclic in $\mathcal{B}$. The set of points for which the Bloch function f attains its maximal growth are located in the set $E(f)$. In the context of cyclic vectors in growth spaces, Hedenmalm and Borichev observed, roughly speaking that cyclic functions therein cannot grow "maximally" on too "massive" sets. See [9]. Our construction below is deeply inspired from this idea. Taking Proposition 5.4 for granted, we first deduce Theorem 1.10.

Proof of Theorem 1.10. Let $f \in BMOA$ be given by Proposition 5.4. Let $\{Q_n\}_n$ be a sequence of analytic polynomials with $\sup_n \|fQ_n\|_{\mathcal{B}} < \infty$. It follows that

$$\sup_n \sup_{z \in \mathbb{D}} \log^{-1} \left(\frac{e}{1-|z|} \right) |f(z)Q_n(z)| < \infty.$$

However, this implies

$$\sup_n \sup_{z \in E(f)} |Q_n(z)| < \infty.$$

Now the assumption on $E(f)$ in conjunction with the maximum principle implies that $\sup_n \|Q_n\|_{H^\infty} < \infty$. By Helly's selection theorem, we can extract a subsequence $\{Q_{n_k}\}_k$ which converges pointwise on $\mathbb{D}$ to some function $h \in H^\infty$. Hence any weak star sequential limit of $\{fQ_n\}$ is of the form fh with $h \in H^\infty$. But functions of that form have finite radial limits at m -a.e on $\partial\mathbb{D}$, hence the set $\{fh \in \mathcal{B} : h \in H^\infty\}$ is a proper subspace of $\mathcal{B}$, which completes the proof. $\square$

The proof of Proposition 5.4 is based on the following construction.

Lemma 5.5. *Let $0 < \delta < 1$. Then there exists a positive finite Borel measure ν on $\partial\mathbb{D}$ satisfying the following property: For any arc $I \subset \partial\mathbb{D}$ there exists a dyadic subarc $J \subset I$, such that $\nu(J) \geq |J|^{1-\delta}$.*

Proof. Fix $0 < \eta < 1$ to be specified later. We shall define the measure ν by declaring its mass on dyadic arcs. First we set $\nu(\partial\mathbb{D}) = 1$. Inductively, assume that $\nu(J)$ has already been defined for some dyadic arc $J \subset \partial\mathbb{D}$. We denote by J_+, J_- the right and left dyadic children of J , respectively. We shall assign a larger portion of the available mass to the right child than to the left by declaring

$$\frac{\nu(J_\pm)}{|J_\pm|} = (1 \pm \eta) \frac{\nu(J)}{|J|}.$$

Now it is a standard fact in measure theory that this construction gives rise to a uniquely defined positive finite Borel measure ν on $\partial\mathbb{D}$. Hence it suffices to verify that ν has the required property. We may without loss of generality assume that I is a dyadic arc. Now let $J_n \subset I$ be the rightmost dyadic subarc of I with $|J_n| = 2^{-n}|I|$. Then by construction, we have

$$\frac{\nu(J_n)}{|J_n|} = (1 + \eta)^n \frac{\nu(I)}{|I|} \geq \frac{1}{|J_n|^\delta} = \frac{2^{\delta n}}{|I|^\delta}$$

whenever $(1 + \eta)^n / 2^{\delta n} \geq |I|^{1-\delta} / \nu(I)$. However, choosing $0 < \eta < 1$ so that $2^\delta < 1 + \eta$, the above inequality will certainly hold for sufficiently large n . This finishes the proof. $\square$

Proof of Proposition 5.4. Let ν be the positive finite Borel measure given by Lemma 5.5. Consider an analytic selfmap $b = b_\nu$ of $\mathbb{D}$ defined via the Herglotz transform of ν :

$$\frac{1 + b(z)}{1 - b(z)} = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta), \quad z \in \mathbb{D}.$$

In other words, ν is the so-called Aleksandrov-Clark measure for b . Consider the function

$$f(z) = \log \left(\frac{e}{1 - b(z)} \right), \quad z \in \mathbb{D}.$$

Note that since f has bounded imaginary part, it follows that f belongs to $\text{BMOA} \subset \mathcal{B}$. Moreover, the trivial fact that $|1 - b(z)| < 2$ for $z \in \mathbb{D}$, implies that $1/f \in H^\infty$. It remains to show that the closure of the corresponding set $E(f)$ contains the entire unit circle. Indeed, the reason for our choice of f stems from the observation that the set $E(f)$ contains

$$\tilde{E}(b) := \left\{ z \in \mathbb{D} : \operatorname{Re} \left(\frac{1 + b(z)}{1 - b(z)} \right) \geq (1 - |z|)^{-\delta} \right\},$$

thus it suffices to check that the closure of the smaller set $\tilde{E}(b)$ contains $\partial\mathbb{D}$. To this end, fix a small $\varepsilon > 0$ and let $\zeta \in \partial\mathbb{D}$ be an arbitrary point. If $0 < \delta' < 1$ is fixed, then an application of Lemma 5.5 shows that for any arc $I \subset \partial\mathbb{D}$ containing ζ with length at most $\varepsilon/2$, there exists a small dyadic arc $J \subset I$ (not necessarily containing ζ), such that $\nu(J) \geq |J|^{1-\delta'}$. Let ξ_J denote the center of the arc J and $z_J := (1 - |J|)\xi_J$, and observe that a trivial estimate of the Poisson kernel implies that

$$\operatorname{Re} \left(\frac{1 + b(z_J)}{1 - b(z_J)} \right) = P(\nu)(z_J) \geq c \frac{\nu(J)}{|J|} \geq \frac{c}{|J|^{\delta'}} = c(1 - |z_J|)^{-\delta'} \geq (1 - |z_J|)^{-\delta},$$

where $c > 0$ is an absolute constant and $0 < \delta < \delta'$ sufficiently small. But this shows that $z_J \in \widetilde{E}(b)$ with $|z_J - \zeta| \leq 2|I| < \varepsilon$. Hence it follows that for sufficiently small $\delta > 0$, the corresponding set $E(f)$ accumulates to any point $\zeta \in \partial\mathbb{D}$, thus we conclude the proof of Proposition 5.4. $\square$

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