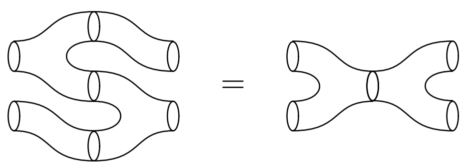
Joachim Kock

Frobenius algebras and 2D topological quantum field theories



Recife 2002

Joachim Kock
Laboratoire de Mathématiques J. A. Dieudonné
Université de Nice – Sophia Antipolis
Parc Valrose — 06108 Nice cédex 2
FRANCE

e-mail: kock@math.unice.fr

VERSION 31/03/2003

This text was written in `alpha`. It was typeset in LaTeX in standard `book` style, with `mathpazo` and `fancyheadings`. The figures were coded with the `texdraw` package, written by Peter Kabal. The diagrams were set using the `diagrams` package of Paul Taylor, except for the curved arrows which were coded by hand.

For fun

Preface

This text centres around notions of Frobenius structure which in recent years have drawn some attention in topology, physics, algebra, and computer science. In topology the structure arises in the category of 2-dimensional oriented cobordisms (and their linear representations, which are 2-dimensional topological quantum field theories) — this is the subject of the first chapter. The main result here (due to Abrams [1]) is a description in terms of generators and relations of the monoidal category ***2Cob***. In algebra, the structure manifests itself simply as Frobenius algebras, which are treated carefully in Chapter 2. The main result here is a characterisation of Frobenius algebras in terms of comultiplication which goes back to Lawvere [32] and was rediscovered by Quinn [43] and Abrams [1]. The main result of these notes is that these two categories are equivalent: the category of 2D topological quantum field theories and the category of commutative Frobenius algebras. This result is due to Dijkgraaf [16], further details of the proof having been provided by Quinn [43], Dubrovin [19], and Abrams [1]. The notions from category theory needed in order to express this rigorously (monoidal categories and their linear representations) are developed from an elementary level in Chapter 3. The categorical viewpoint allows us to extract the essence of what is going on in the two first chapters, and prove a natural generalisation of the theorem. To arrive at this insight, we carefully review the classical fact that the simplex category Δ is the free monoidal category on a monoid. (This means in particular that there is an equivalence of categories between the category of algebras and the category of 'linear representations' of Δ.) Now the notion of a Frobenius object in a monoidal category is introduced, and the promised generalisation of the theorem (main result of Chapter 3) states that ***2Cob*** is the free symmetric monoidal category on a commutative Frobenius object.

For more details on the mathematical content, see the Introduction.

[Version 31/03/2003]

The target. The book is based on notes prepared for an intensive two-week mini-course for advanced undergraduate students, given in the UFPE Summer School, Recife, Brazil, in January 2002. The prerequisites are modest: the students of the mini-course were expected to have followed these three standard courses taught at Brazilian universities: one on *Differential Topology*, one on *Algebraic Structures* (groups and rings) and one *Second Course in Linear Algebra*. From topology we need just some familiarity with the basic notions of differentiable manifolds; from algebra we need basic notions of rings and ideals, groups and algebras; and first and foremost the reader is expected to be familiar with tensor products and hom sets. Usually the course *Algebraic Structures* contains an introduction to categories and functors, but not enough to get acquainted with the categorical way of thinking and appreciate it; the exposition in this text is meant to take this into account. The basic definitions are given in an appendix, and the more specialised notions are introduced with patience and details, and with many examples — and hopefully the interplay between topology and algebra will provide the appreciation of the categorical viewpoint.

In a wider context these notes are targeted at undergraduate students with a similar background, as well as graduate students of all areas of mathematics. Experienced mathematicians and experts in the field will sometimes be bored by the amount of detail presented, but it is my hope the drawings will keep them awake.

The aim. At an immediate level, the aim of these notes is simply to expose some delightful and not very well-known mathematics where a lot of figures can be drawn: a quite elementary and very nice interaction between topology and algebra — and rather different in flavour from what one learns in a course in algebraic topology. On a deeper level, the aim is to convey an impression of unity in mathematics, an aspect which is often hidden from the students until later in their mathematical apprenticeship. Finally, perhaps the most important aim is to use this as motivation for category theory, and specifically to serve as an introduction to monoidal categories.

Admittedly, the main theorem is not a particularly useful tool that the students will draw upon again and again throughout their mathematical career, and one could argue that the time would be better spent on a course on group representations or distributions, for instance. But after all, this is a summer school (and this is Brazil!): maximising the throughput is not our main concern — the wonderful relaxed atmosphere I know from previous summer schools in

Recife is much more important — I hope the students when they go to the beach in the weekend will make drawings of 2D cobordisms in the sand! (I think they wouldn't take orthogonality relations or Fourier transforms with them to the beach. . .)

What the lectures are meant to give the students are rather some techniques and viewpoints, and in the end this categorical perspective reduces the main theorem to a special case of general principles. A lot of emphasis is placed on universal properties, symmetry, distinction between structure and property, distinction between identity and natural isomorphism, the interplay between graphical and algebraic approaches to mathematics — as well as reflection on the nature of the most basic operations of mathematics: multiplication and addition. Getting acquainted with such categorical viewpoints in mathematics is certainly a good investment.

Finally, to be more concrete, the techniques learned in this course should constitute a good primer for going into quantum groups or knot theory.

The source — acknowledgements. The idea of these notes originated in a workshop I led at KTH, Stockholm, in 2000, whose first part was devoted to understanding the paper of Abrams [1] (corresponding more or less to Chapter 1 and 2 of this text). I am thankful for the contributions of the core participants of the workshop: ***Carel*** Faber, ***Helge*** Måkestad, ***Mats*** Boij, and ***Michael*** Shapiro, and in particular to ***Dan*** Laksov, for many fruitful discussions about Frobenius algebras.

The more categorical viewpoint of Chapter 3 was influenced by the people I work with here in Nice; I am indebted in particular to ***André*** Hirschowitz and ***Bertrand*** Toen. I have also benefited from discussions and e-mail correspondence with ***Arnfinn*** Laudal, ***Göran*** Fors, ***Jan*** Gorski, ***Jean-Louis*** Cathélineau, ***John*** Baez, and ***Pedro*** Ontaneda, all of whom are thanked. I am particularly indebted to ***Anders*** Kock, ***Peter*** Johnson, and ***Tom*** Leinster for many discussions and helpful e-mails, and for carefully reading preliminary versions of the manuscript, pointing out grim errors, annoying inaccuracies, and misprints.

Israel Vainsencher, ***Joaquim*** Roé, ***Ramón*** Mendoza, and ***Sérgio*** Santa Cruz also picked up some misprints — thanks. My big sorrow about these notes is that I don't understand the physics behind it all, in spite of a great effort by ***José*** Mourão to explain it to me — I am grateful to him for his patience.

During the redaction of these notes I have reminisced about math classes in primary school, and some of the figures are copied from my very first math books. Let me take the opportunity to thank ***Marion*** Kuhlmann and ***Jørgen*** Skaftved for the math they taught me when I was a child.

During my work with this subject and specifically with these notes, I have been supported by ***The National Science Research Council of Denmark***, The Nordic Science Research Training Academy ***NorFA***, and (currently) a Marie Curie Fellowship from ***The European Commission***. In neither case was I supposed to spend so much time with Frobenius algebras and topological quantum field theories — it is my hope that these notes, as a concrete outcome of the time spent, do it justice to some extent.

I am indebted to my wife ***Andrea*** for her patience and support.

Last but not least, I wish to thank the organisers of the Summer School in Recife — in particular ***Letterio*** Gatto — for inviting me to give this mini-course, which in addition to being a very dear opportunity to come back to Recife — *Voltei, Recife! foi a saudade que me trouxe pelo braço* — has also been a welcome incentive to work out the details of this material and learn a lot of mathematics.

Feedback is most welcome. Please point out mathematical errors or misunderstandings, misleading viewpoints, unnecessary pedantry, or things that should be better explained; typos, mispellings, bad English, TEX-related issues. I intend to keep a list of errata on my web site.

Recife, January 2002 — Nice, January 2003

JOACHIM KOCK
kock@math.unice.fr

Contents

General conventions

We consistently write composition of functions (or arrows) from the left to the right: given functions (or arrows)

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

we denote the composite fg. Similarly, we put the symbol of a function to the right of its argument, writing for example

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto xf. \end{aligned}$$

Introduction

In this Introduction we briefly explain the words of the title of these notes, give a sketch of what we are going to do with these notions, and outline the viewpoint we will take in order to understand the structures. In the course of this introduction a lot of other words will be used which are probably no more familiar than those they are meant to explain — but don't worry: in the main text, all these words are properly defined and carefully explained. . .

0.0.1 Frobenius algebras. A Frobenius algebra is a finite-dimensional algebra equipped with a nondegenerate bilinear form compatible with the multiplication. (Chapter 2 is all about Frobenius algebras.) Examples are matrix rings, group rings, the ring of characters of a representation, and artinian Gorenstein rings (which in turn include cohomology rings, local rings of isolated hypersurface singularities. . .)

In algebra and representation theory such algebras have been studied for a century, along with various related notions — see Curtis-Reiner [15].

0.0.2 Frobenius structures. During the past decade, Frobenius algebras have shown up in a variety of topological contexts, in theoretical physics and in computer science. In physics, the main scenery for Frobenius algebras is that of topological quantum field theory, which in its axiomatisation amounts to a precise mathematical theory. In computer science, Frobenius algebras arise in the study of flowcharts, proof nets, circuit diagrams. . .

In any case, the reason Frobenius algebras show up is that it is essentially a topological structure: it turns out the axioms for a Frobenius algebra can be given completely in terms of graphs — or as we shall do, in terms of topological surfaces.

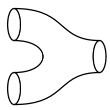
Frobenius algebras are just algebraic representations of this structure — the goal of these notes is to make all this precise. We will focus on topological quantum field theories — and in particular on dimension 2. This is by far the best picture of the Frobenius structures since the topology is explicit, and since there is no additional structure to complicate things. In fact, the main theorem of these notes states that there is an equivalence of categories between that of 2D TQFTs and that of commutative Frobenius algebras.

(There will be no further mention of computer science in these notes.)

0.0.3 Topological quantum field theories. In the axiomatic formulation (due to M. Atiyah [5]), an n-dimensional topological quantum field theory is a rule $\mathscr{A}$ which to each closed oriented manifold Σ (of dimension $n-1$) associates a vector space $\Sigma\mathscr{A}$, and to each oriented n-manifold whose boundary is Σ associates a vector in $\Sigma\mathscr{A}$. This rule is subject to a collection of axioms which express that topologically equivalent manifolds have isomorphic associated vector spaces, and that disjoint unions of manifolds go to tensor products of vector spaces, etc.

0.0.4 Cobordisms. The clearest formulation is in categorical terms: first one defines a category of cobordisms ***nCob***: the objects are closed oriented $(n-1)$-manifolds, and an arrow from Σ to Σ' is an oriented n-manifold M whose 'in-boundary' is Σ and whose 'out-boundary' is Σ'. (The cobordism M is defined up to diffeomorphism rel the boundary.) The simplest example of a cobordism is the cylinder $\Sigma \times I$ over a closed manifold Σ — say a circle. It is a cobordism from one copy of Σ to another.

Here is a drawing of a cobordism from the union of two circles to one circle



Composition of cobordisms is defined by gluing together the underlying manifolds along common boundary components; the cylinder $\Sigma \times I$ is the identity arrow on Σ. The operation of taking disjoint union of manifolds and cobordisms gives this category *monoidal structure* — more about monoidal categories later. On the other hand, the category ***Vect***$_{\Bbbk}$ of vector spaces is monoidal under tensor products.

Now the axioms amount to saying that a TQFT is a (symmetric) monoidal functor from ***nCob*** to ***Vect***$_{\Bbbk}$. This is also called a linear representation of ***nCob***.

So what does this have to do with Frobenius algebras? Before we come to the relation between Frobenius algebras and 2D TQFTs, let us make a couple of remarks on the motivation for TQFTs.

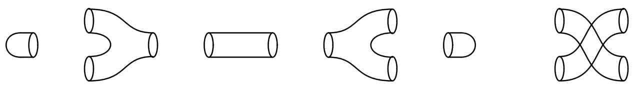
0.0.5 Physical interest in TQFTs comes mainly from the observation that TQFTs possess certain features one expects from a theory of quantum gravity. It serves as a baby model in which one can do calculations and gain experience before embarking on the quest for the full-fledged theory, which is expected to be much more complicated. Roughly, the closed manifolds represent *space*, while the cobordisms represent *space-time*. The associated vector spaces are then the *state spaces*, and an operator associated to a space-time is the time-evolution operator (also called transition amplitude, or Feynman path integral). That the theory is topological means that the transition amplitudes do not depend on any additional structure on space-time (like riemannian metric or curvature), but only on the topology. In particular there is no time-evolution along cylindrical space-time. That disjoint union goes to tensor product expresses the common principle in quantum mechanics that the state space of two independent systems is the tensor product of the two state spaces.

(No further explanation of the relation to physics will be given — the author of these notes recognises he knows nearly nothing of this aspect. The reader is referred to Dijkgraaf [17] or Barrett [11], for example.)

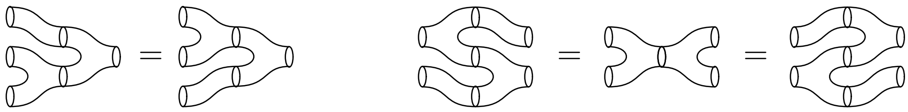
0.0.6 Mathematical interest in TQFTs stems from the observation that they produce invariants of closed manifolds: an n-manifold without boundary is a cobordism from the empty $(n-1)$-manifold to itself, and its image under $\mathscr{A}$ is therefore a linear map $\Bbbk \to \Bbbk$, i.e., a scalar. It was shown by E. Witten how TQFT in dimension 3 is related to invariants of knots and the Jones polynomial — see Atiyah [6].

The viewpoint of these notes is different however: instead of developing TQFTs in order to describe and classify manifolds, we work in dimension 2 where a complete classification of surfaces already exists; we then use this classification to describe TQFTs!

0.0.7 Cobordisms in dimension 2. In dimension 2, 'everything is known': since surfaces are completely classified, one can also describe the cobordism category completely. Every cobordism is obtained by composing the following basic building blocks (each with the in-boundary drawn to the left):



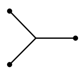
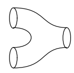
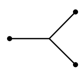
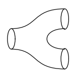
Two cobordisms are equivalent if they have the same genus and the same number of in- and out-boundaries. This gives a bunch of relations, and a complete description of the monoidal category ***2Cob*** in terms of generators and relations. Here are two examples of relations that hold in ***2Cob***:



(0.0.8)

These equations express that certain surfaces are topologically equivalent rel the boundary.

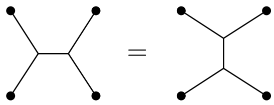
0.0.9 Topology of some basic algebraic operations. Some very basic principles are in play here: 'creation', 'coming together', 'splitting up', 'annihilation'. These principles have explicit mathematical manifestations as algebraic operations:

Principle	Feynman diagram	2D cobordism	Algebraic operation (in a $\Bbbk$-algebra A)	
merging			multiplication	$A \otimes A \to A$
creation			unit	$\Bbbk \to A$
splitting			comultiplication	$A \to A \otimes A$
annihilation			counit	$A \to \Bbbk$

Note that in the intuitive description there is a notion of time involved which accounts for the distinction between coming-together and splitting-up — or perhaps 'time' is too fancy a word, but at least there is a notion of *start* and *finish*. Correspondingly, in the algebraic or categorical description the notion of morphism involves a direction: morphisms are *arrows*, and they have well-defined *source* and *target*.

It is an important observation from category theory that many algebraic structures admit descriptions purely in terms of arrows (instead of referring to elements) and commutative diagrams (instead of equations among elements). In particular, this is true for the notion of an *algebra*: an algebra is a vector space A equipped with two maps $A \otimes A \to A$ and $\Bbbk \to A$, satisfying the associativity axiom and the unit axiom. Now according to the above dictionary, the left-hand relation of (0.0.8) is just the topological expression of associativity! Put in other words, the associativity equation has topological content: it expresses the topological equivalence of two surfaces (or two graphs).

It gives sense to other operations, like merging (or splitting) three particles: it makes no difference whether we first merge two of them and then merge the result with the third, or whether we merge the last two with the first. From the viewpoint of graphs, the basic axiom (equivalent to 0.0.8) is that two vertices can move past each other:



0.0.10 Frobenius algebras. In order to relate this to Frobenius algebras the definition given in the beginning of this Introduction is not the most convenient. It turns out one can characterise Frobenius algebras like this: It is an algebra (multiplication denoted ) which is simultaneously a *coalgebra* (comultiplication denoted ) with a certain compatibility condition between  and  . This compatibility condition is exactly the right-hand relation drawn in (0.0.8). (Note that by the dictionary, this is just a graphical expression of a precise algebraic requirement). In fact, the relations that hold in ***2Cob*** correspond precisely to the axioms of a commutative Frobenius algebra. This comparison leads to the main theorem:

0.0.11 Theorem. *There is an equivalence of categories*

$$\textbf{\textit{2TQFT}} \simeq \textbf{\textit{cFA}},$$

given by sending a TQFT to its value on the circle (the unique closed connected 1-*manifold).*

So in this sense, we can say, if we want, that Frobenius algebras are the same thing as linear representations of ***2Cob***.

The idea of the proof is this: let A be the image of the circle, under a TQFT $\mathscr{A}$. Now $\mathscr{A}$ sends each of the generators of ***2Cob*** to a linear map between tensor powers of A, just as tabulated above. The relations which hold in ***2Cob*** are preserved by $\mathscr{A}$ (since $\mathscr{A}$ by definition is a monoidal functor) and in its target category ***Vect*** they translate into the axioms for a commutative Frobenius algebra! (Conversely, every commutative Frobenius algebra can be used to define a 2-dimensional TQFT.)

0.0.12 Monoidal categories. As mentioned, just in order to define the category ***TQFT*** we need the notion of monoidal categories. In fact, monoidal categories is the good framework to understand all the concepts described above. The notion of associative multiplication with unit is precisely what the abstract concept of *monoid* encodes — and monoids live in monoidal categories.

The prime example of a monoidal category is the category $\boldsymbol{Vect}_{\Bbbk}$ of vector spaces and tensor products, with the ground field as neutral object. In general a monoidal category is a category equipped with some sort of 'product' like $\otimes$ or $\coprod$, satisfying certain properties. This 'product' serves as background for defining the multiplication maps, i.e., defining monoids: a monoid in $(\boldsymbol{Vect}_{\Bbbk}, \otimes, \Bbbk)$ is precisely a $\Bbbk$-algebra A, since the multiplication map is described as a $\Bbbk$-linear map $A \otimes A \to A$, etc. Another example of a monoid is the circle in ***2Cob***...

0.0.13 The simplex category Δ and what it means to monoids and algebras. There is a little monoidal category which bears some similarity with ***2Cob***: the simplex category Δ is roughly the category of finite ordered sets and order-preserving maps. It is a monoidal category under disjoint union. To be more precise, the objects of Δ are $\mathbf{n} = \{0, 1, 2, \ldots, n-1\}$, one for each $n \in \mathbb{N}$, and the arrows are the maps $f : \mathbf{m} \to \mathbf{n}$ such that $i \leq j \Rightarrow if \leq jf$. There are several other descriptions of this important category — one is in graphical terms, and reveals it as a subcategory of ***2Cob***. The object $\mathbf{1}$ is a monoid in Δ, and in a sense Δ is the smallest possible monoidal category which contains a nontrivial monoid. In fact the following universal property is shown to hold: *every monoid in any monoidal category* $\boldsymbol{V}$ *is the image of* $\mathbf{1}$ *under a unique monoidal functor* $\Delta \to \boldsymbol{V}$. This is to say that Δ is the free monoidal category containing a monoid. In particular, $\Bbbk$-algebras can be interpreted as 'linear representations' of Δ.

Observing that Δ can be described graphically, we see that this result is of exactly the same type as our Main Theorem.

0.0.14 Frobenius objects. Once we have taken the step of abstraction from $\Bbbk$-algebras to monoids in an arbitrary monoidal category, it is straightforward to define the notion of Frobenius object in a monoidal category: it is an object equipped with four maps as those listed in the table, and with the compatibility condition expressed in 0.0.8. In certain monoidal categories, called symmetric, it makes sense to ask whether a monoid or a Frobenius object is commutative, and of course these notions are defined in such a way that commutative Frobenius objects in $\boldsymbol{Vect}_{\Bbbk}$ are precisely commutative Frobenius algebras.

0.0.15 Universal Frobenius structure. With these general notions, it is immediate to generalise the Theorem: all the arguments of the proof do in fact carry over to the setting of an arbitrary (symmetric) monoidal category, and we find that ***2Cob*** is the free symmetric monoidal category containing a commutative Frobenius object. This means that *every commutative Frobenius object in any symmetric monoidal category **V** is the image of the circle under a unique symmetric monoidal functor from **2Cob***.

Since the proof of this result is the same as the proof of the original theorem, this is the natural generality of the statement. The interest in this generality is that it actually includes many natural examples of TQFTs which could not fit into the original definition. For example, in our treatment of Frobenius algebras in Chapter 2 we will see that cohomology rings are Frobenius algebras in a natural way, but typically they are not commutative but only graded-commutative. For this reason they cannot support a TQFT in the strict sense. But if instead of the usual symmetric monoidal category ***Vect*** we take for example the category of graded vector spaces with 'super-symmetry' structure, then all cohomology rings can support a TQFT (of this slightly generalised sort).

It is the good generalised version of the main theorem that makes this clear. In many sources on TQFTs, the questions of symmetry are swept under the carpet and the point about 'super-symmetric' TQFTs is missed.

In these notes, the whole question of symmetry is given a rather privileged rôle. The difficult thing about symmetry is to avoid mistaking it for identity! For example, for the cartesian product $\times$ (which is an important example of a monoidal structure), it is *not* true that $X \times Y = Y \times X$. What is true is that there is a natural isomorphism between the two sets (or spaces). Similar observations are due for disjoint union $\coprod$, and tensor product $\otimes$. . . While it requires some pedantry to treat symmetry properly, it is necessary in order to understand the super-symmetric examples just mentioned.

0.0.16 Organisation of these notes. The notes are divided into three chapters each of which should be read before the others! The first chapter is about topology: cobordisms and TQFTs; Chapter 2 is about algebra — Frobenius algebras; and Chapter 3 is mostly category theory. The reader is referred to the Table of Contents for more details on where to find what.

Although the logical order of the material is not completely linear, hopefully the order is justified pedagogically: We start with geometry! — the concrete and palpable — and then we gradually proceed to more abstract subjects, (or should we say: more abstract aspects of our subject), helped by drawings and intuition provided by the geometry. With the experience gained with these investigations we get ready to try to understand the abstract structures behind. The ending is about very abstract concepts and objects with universal properties, but we can cope with that because we know the underlying geometry — in fact we show that this very abstract thing with that universal property is precisely the cobordism category we described so carefully in Chapter 1.

0.0.17 Exercises. Each section ends with a collection of exercises of varying level and interest. Most of them are really easy, and the reader is encourages to do them all. A few of them are considered less straightforward and have been marked with a star.

0.0.18 Further reading. My big sorrow about these notes is that I don't understand the physical background or interpretation of TQFTs. The physically inclined reader must resort to the existing literature, for example Atiyah's book [6] or the notes of Dijkgraaf [17]. I would also like to recommend John Baez's web site [8], where a lot of references can be found.

Within the categorical viewpoint, an important approach to Frobenius structures which has not been touched upon is the 2-categorical viewpoint, in terms of monads and adjunctions. This has recently been exploited to great depth by Müger [39]. Again, a pleasant introductory account is given by Baez [8], TWF 174 (and 173).

Last but not least, I warmly recommend the lecture notes of Quinn [43], which are detailed and go in depth with concrete topological quantum field theories.

Bibliography

[1] LOWELL ABRAMS. *Two-dimensional topological quantum field theories and Frobenius algebras*. J. Knot Theory Ramifications **5** (1996), 569–587. Available at http://home.gwu.edu/~labrams/docs/tqft.ps.

[2] LOWELL ABRAMS. *Modules, comodules, and cotensor products over Frobenius algebras*. J. Alg. **219** (1999), 201–213. Available at http://home.gwu.edu/~labrams/docs/cotensor.ps.

[3] LOWELL ABRAMS. *The quantum Euler class and the quantum cohomology of the Grassmannians*. Israel J. Math. **117** (2000), 335–352. Available at http://home.gwu.edu/~labrams/docs/qeuler.ps.

[4] HANS CHRISTIAN ANDERSEN. *Fyrtøjet*. København, 1835. (English translation, *The Tinder-Box*, available at http://www-math.unice.fr/~kock/privat/tinderbox.html.)

[5] MICHAEL ATIYAH. *Topological quantum field theories*. Inst. Hautes Études Sci. Publ. Math. **68** (1989), 175–186.

[6] MICHAEL ATIYAH. *The geometry and physics of knots*. Cambridge University Press, Cambridge, 1990.

[7] MICHAEL ATIYAH. *An introduction to topological quantum field theories*. Turkish J. Math. **21** (1997), 1–7.

[8] JOHN BAEZ's web site, http://math.ucr.edu/home/baez/.

[9] JOHN BAEZ and JAMES DOLAN. *Higher-dimensional algebra and topological quantum field theory*. J. Math. Phys. **36** (1995), 6073–6105 (q-alg/9503002).

[10] JOHN BAEZ and JAMES DOLAN. *From finite sets to Feynman diagrams*. In B. Engquist and W. Schmid, editors, *Mathematics unlimited—2001 and beyond*, pp. 29–50. Springer-Verlag, Berlin, 2001 (math.QA/0004133).

[11] JOHN W. BARRETT. *Quantum gravity as topological quantum field theory*. J. Math. Phys. **36** (1995), 6161–6179 (gr-qc/9506070).

[12] RAOUL BOTT and LORING W. TU. *Differential forms in algebraic topology*. No. 82 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1982.

[13] HAROLD S. M. COXETER and WILLIAM O. MOSER. *Generators and relations for discrete groups*. No. 14 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1957.

[14] LOUIS CRANE and DAVID YETTER. *On algebraic structures implicit in topological quantum field theories*. J. Knot Theory Ramifications **8** (1999), 125–163 (hep-th/9412025).

[15] CHARLES W. CURTIS and IRVING REINER. *Representation theory of finite groups and associative algebras*. Interscience Publishers, New York, 1962.

[16] ROBBERT DIJKGRAAF. *A geometric approach to two dimensional conformal field theory*. PhD thesis, University of Utrecht, 1989.

[17] ROBBERT DIJKGRAAF. *Fields, strings and duality*. In *Symétries quantiques (Les Houches, 1995)*, pp. 3–147. North-Holland, Amsterdam, 1998 (hep-th/9703136).

[18] ROBBERT DIJKGRAAF, ERIK VERLINDE, and HERMAN VERLINDE. *Notes on topological string theory and 2D quantum gravity*. In M. Green et. al., editor, *Proceedings of the Trieste Spring School, 1990: String theory and quantum gravity*, pp. 91–156. World Scientific, Singapore, 1991.

[19] BORIS DUBROVIN. *Geometry of 2D topological field theories*. In *Integrable systems and quantum groups*, no. 1620 in Lecture Notes in Mathematics, pp. 120–348. Springer-Verlag, New York, 1996 (hep-th/9407018).

[20] BERGFINNUR DURHUUS and THÓRDUR JÓNSSON. *Classification and construction of unitary topological field theories in two dimensions*. J. Math. Phys. **35** (1994), 5306–5313 (hep-th/9308043).

[21] DAVID EISENBUD. *Commutative algebra with a view toward algebraic geometry*. No. 150 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[22] GEORG FROBENIUS. *Theorie der hyperkomplexen Grössen*. Sitzungber. Königl. Preuss. Akad. Wiss. **24** (1903), 504–537; 634–645.

[23] WILLIAM FULTON. *Algebraic topology: A first course*. No. 153 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[24] PAUL G. GOERSS and JOHN F. JARDINE. *Simplicial homotopy theory*, vol. 174 of Progress in Mathematics. Birkhäuser, Basel, 1999.

[25] PHILLIP GRIFFITHS and JOE HARRIS. *Principles of Algebraic Geometry*. John Wiley and Sons, 1978.

[26] M. Hazewinkel, editor. *Encyclopaedia of mathematics. Vol. 1–6. A–Zyg, index*. Kluwer Academic Publishers, Dordrecht, 1995. Translated from the Russian, Reprint of the 1988–1994 English translation.

[27] MORRIS HIRSCH. *Differential Topology*. No. 33 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1976.

[28] ANDRÉ JOYAL and ROSS STREET. *Braided tensor categories*. Adv. in Math. **102** (1993), 20–78.

[29] CHRISTIAN KASSEL. *Quantum groups*. No. 155 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[30] SERGE LANG. *Algebra*. Addison-Wesley, Reading, Mass., 1971.

[31] H. BLAINE LAWSON JR. *The theory of gauge fields in four dimensions*. CBMS Regional Conference Series in Mathematics, **58**, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1985.

[32] F. WILLIAM LAWVERE. *Ordinal sums and equational doctrines*. In B. Eckmann, editor, *Seminar on triples and categorical homology theory, ETH 1966/67*, no. 80 in Lecture Notes in Mathematics, pp. 141–155. Springer-Verlag, New York, 1967.

[33] F. WILLIAM LAWVERE. *Metric spaces, generalized logic, and closed categories*. Rend. Sem. Mat. Fis. Milano **43** (1973), 135–166. Reprinted in *Repr. Theory Appl. Categ.* **1** (2002), 36pp. (electronic), with an author commentary: Enriched categories in the logic of geometry and analysis. (Available at http://www.tac.mta.ca/tac/reprints/articles/1/tr1abs.html.)

[34] SAUNDERS MAC LANE. *Categories for the working mathematician, second edition*. No. 5 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[35] JOHN MILNOR. *On manifolds homeomorphic to the* 7*-sphere*. Ann. of Math. **64** (1956), 399–405.

[36] JOHN MILNOR. *Lectures on the h-cobordism theorem*. Princeton University Press, Princeton, N.J., 1965.

[37] EDWIN E. MOISE. *Geometric topology in dimensions* 2 *and* 3. No. 47 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1977.

[38] ELIAKIM HASTINGS MOORE. *Concerning the abstract groups of order* $k!$ *and* $\frac{1}{2}k!$ *holohedrically isomorphic with the symmetric and the alternating substitution-groups on k letters*. Lond. Soc. Math. Proc. **28** (1897), 357–366.

[39] MICHAEL MÜGER. *From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories*. To appear in J. Pure Appl. Alg. (math.CT/0111204).

[40] JAMES MUNKRES. *Elementary differential topology*. No. 54 in Annals of Mathematics Study. Princeton University Press, Princeton, N.J., 1963.

[41] TADASI NAKAYAMA. *On Frobeniusean algebras I*. Ann. of Math. **40** (1939), 611–633.

[42] CECIL NESBITT. *On the regular representations of algebras*. Ann. of Math. **39** (1938), 634–658.

[43] FRANK QUINN. *Lectures on axiomatic topological quantum field theory*. In *Geometry and quantum field theory (Park City, UT, 1991)*, pp. 323–453. Amer. Math. Soc., Providence, RI, 1995.

[44] STEPHEN SAWIN. *Direct sum decompositions and indecomposable TQFTs*. J. Math. Phys. **36** (1995), 6673–6680 (q-alg/9505026).

[45] GRAEME SEGAL. *Two-dimensional conformal field theories and modular functors*. In *IXth International Congress on Mathematical Physics (Swansea, 1988)*, pp. 22–37. Hilger, Bristol, 1989.

[46] MOSS E. SWEEDLER. *Hopf algebras*. W.A. Benjamin, Inc., New York, 1969.

[47] RENÉ THOM. *Quelques propriétés globales des variétés différentiables*. Comment. Math. Helv. **28** (1954), 17–86.

[48] ANDREW H. WALLACE. *Differential topology: First steps*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[49] JOSEPH H. M. WEDDERBURN. *On hypercomplex numbers*. Proc. London Math. Soc. **6** (1908), 77–118.

[50] EDWARD WITTEN. *Topological quantum field theory*. Comm. Math. Phys. **117** (1988), 353–386.