1. Cohomological finiteness conditions: spaces versus H-spaces

We wish to ask a very naive and classically flavored question. Consider a finite complex X and an integer n. Does its n-connected cover $X\langle n\rangle$ satisfy any cohomological finiteness property? When X is an H-space we have:

THEOREM 1.1. *[CCS] Let X be a finite H-space and n an integer. Then $H^*(X\langle n\rangle; \mathbf{F}_p)$ is finitely generated as an algebra over the Steenrod algebra.*

This leads naturally to ask whether the same statement holds for arbitrary spaces. Of course, some restriction on the fundamental group will be needed, as the universal cover of $S^1 \vee S^2$ is an infinite wedge of copies of S^2.

QUESTION 1.2. *Let X be a finite space with $\pi_1 X$ finite and $n \geq 1$. Is $H^*(X\langle n\rangle; \mathbf{F}_p)$ finitely generated as an algebra over the Steenrod algebra?*

The "difference" between a space and its n-connected cover is a Postnikov piece. Thus, a first step towards a solution to Question 1.2 would be to understand the cohomology of finite type Postnikov pieces.

QUESTION 1.3. *Is the cohomology of a finite type Postnikov piece finitely generated as an algebra over the Steenrod algebra?*

Again, we know from [CCS, Corollary 3.8] that the answer is yes if the Postnikov piece is an H-space. The proof of Theorem 1.1 is based on Larry Smith's analysis of the Eilenberg-Moore spectral sequence, and the following algebraic result, whose proof relies deeply on the Borel-Hopf structure theorem.

THEOREM 1.4. *[CCS] Let A be an unstable, connected, associative, and commutative Hopf algebra which is finitely generated as an algebra over the Steenrod algebra. Then so is any unstable Hopf subalgebra $B \subset A$.*

For plain unstable algebras, this is false, as pointed out to us by Hans-Werner Henn. Consider the unstable algebra $H^*(\mathbf{C}P^\infty \times S^2; \mathbf{F}_p) \cong \mathbf{F}_p[x] \otimes E(y)$ where both x and y have degree 2. Turn the ideal generated by y into an unstable subalgebra by adding 1. This is isomorphic, as an unstable algebra, to $\mathbf{F}_p \oplus \Sigma^2 \mathbf{F}_p \oplus \Sigma^2 \widetilde{H}^*(\mathbf{C}P^\infty; \mathbf{F}_p)$, which is not finitely generated.

Where do these questions come from? The condition that $H^*(X; \mathbf{F}_p)$ is finitely generated as an algebra over the Steenrod algebra is equivalent to the fact that the indecomposables $QH^*(X;\mathbf{F}_p)$ are finitely generated as a module over the Steenrod algebra. This guarantees that $QH^*(X;\mathbf{F}_p)$ lives in the Krull filtration of the category $\mathcal{U}$ of unstable modules, introduced by Schwartz in [Sch94]: An unstable module M lives in $\mathcal{U}_n$ if and only if $\bar{T}^{n+1}M = 0$, where $\bar{T}$ denotes Lannes' reduced T functor. This algebraic filtration can be compared with Bousfield's $B\mathbf{Z}/p$-nullification filtration, [Bou94] (a connected space X is $B\mathbf{Z}/p$-null if the space of pointed maps $\mathrm{map}_*(B\mathbf{Z}/p, X)$ is contractible).

THEOREM 1.5. *[CCS06] Let X be a connected H-space satisfying that $T_V H^*(X;\mathbf{F}_p)$ is of finite type for any elementary abelian p-group V. Then $QH^*(X;\mathbf{F}_p)$ is in $\mathcal{U}_n$ if and only if $\Omega^{n+1}X$ is $B\mathbf{Z}/p$-null.*

Dwyer and Wilkerson have shown in [DW90] that the case $n=0$ holds for arbitrary spaces. However, our methods rely so deeply on the H-structure that we still don't know if one should look for a positive or negative answer to our last question.

QUESTION 1.6. *Let X be a connected space such that $T_V H^*(X;\mathbf{F}_p)$ is of finite type for any elementary abelian p-group V, and let $n \geq 1$. Is it true that $QH^*(X;\mathbf{F}_p)$ is in $\mathcal{U}_n$ if and only if $\Omega^{n+1}X$ is $B\mathbf{Z}/p$-null?*

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Natàlia Castellana and Jérôme Scherer

Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain. *E-mail:* `natalia@mat.uab.es`, `jscherer@mat.uab.es`

Juan A. Crespo

Departamento de Economía, Universidad Carlos III de Madrid, E-28903 Getafe, Spain. *E-mail:* `jacrespo@eco.uc3m.es`