Deconstructing Hopf spaces

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Abstract. We characterize Hopf spaces with finitely generated cohomology as an algebra over the Steenrod algebra. We "deconstruct" the original space into an H-space Y with finite mod p cohomology and a finite number of p-torsion Eilenberg-Mac Lane spaces. We give a precise description of homotopy commutative H-spaces in this setting.

Introduction

Since their introduction in the 50's by Serre, H-spaces have produced some of the most beautiful results in Algebraic Topology. Some examples are Adams' solution of the Hopf invariant one conjecture [1], the criminal of Hilton-Roitberg [21], the construction of $DI(4)$ by Dwyer and Wilkerson [17], the recent proof that a finite loop space is of the homotopy type of a manifold [4], and the *new* example of a finite loop space in [3].

The structure of finite H-spaces is rather well understood. In one of the most important articles on finite H-spaces, [22], Hubbuck shows that there are no other finite connected homotopy commutative H-spaces than products of circles, which was proved for compact Lie groups by James. It was not until the early 90's that this result was extended by Slack to H-spaces with finitely generated mod 2 cohomology. With the aid of secondary operations, he shows in [36] that such homotopy commutative H-spaces are products of

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circles and other Eilenberg-Mac Lane spaces. In fact, using the modern techniques of Lannes' T functor, Broto et al. obtain a structure theorem for all H-spaces with noetherian mod p cohomology. They "deconstruct" such an H-space into mod p finite ones and copies of $K(\mathbb{Z}/p^r, 1)$ and $\mathbb{C}P^\infty$ in a functorial way. Recall that an H-space is said to be mod p finite if it is p-complete with finite mod p cohomology, which we denote simply by $H^*(-)$.

Our goal is to extend such results to an even larger class of infinite dimensional spaces and understand which are the basic pieces permitting to reconstruct the original H-space.

Natural examples of H-spaces, arising in connection with those which are finite, are their Postnikov sections and connected covers. The mod p cohomology of the n-connected cover of a finite H-space is not finite in general, but is finitely generated as an algebra over the Steenrod algebra $\mathcal{A}_p$ (we refer to the article [14] for more details). Up to p-completion, the basic examples of H-spaces satisfying this cohomological finiteness condition are all finite H-spaces, and Eilenberg-Mac Lane spaces of type $K(\mathbb{Z}/p^r, n)$ and $K(\mathbb{Z}_{p^\infty}, n)$.

We show that one can deconstruct any such H-space in terms of these basic H-spaces. We call an H-space which has only finitely many non-trivial homotopy groups an *H-Postnikov piece*.

Theorem 6.5. *Let X be a connected H-space such that $H^*(X)$ is a finitely generated algebra over the Steenrod algebra. Then X is the total space of an H-fibration*

$$F \longrightarrow X \longrightarrow Y,$$

where Y is an H-space with finite mod p cohomology and F is a p-torsion H-Postnikov piece whose homotopy groups are finite direct sums of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$.

The above fibration behaves well with respect to loop structures and a similar result holds for loop spaces. Our deconstruction theorem enables us to reduce questions on infinite dimensional H-spaces to finite ones. For instance, we use this technique to give a generalization of Hubbuck's Torus Theorem.

Corollary 6.6. *Let X be a connected homotopy commutative H-space such that the mod 2 cohomology $H^*(X)$ is finitely generated as an algebra over the Steenrod algebra $\mathcal{A}_2$. Then, up to 2-completion, X is homotopy equivalent to $(S^1)^n \times F$, where F is a connected 2-torsion H-Postnikov piece.*

When $H^*(X)$ is finitely generated as an algebra, we get back Slack's result [36], as well as their generalization by Lin and Williams in [27].

The arguments to prove our main theorem are the following. When $H^*(X)$ is finitely generated over $\mathcal{A}_p$, we show in Lemma 6.1 that

the unstable module of indecomposable elements $QH^*(X)$ belongs to some stage $\mathcal{U}_n$ of the Krull filtration in the category of unstable modules. This filtration has been studied in [34] by Schwartz in order to prove Kuhn's non-realizability conjecture [24].

The stage $\mathcal{U}_0$ of the Krull filtration is particularly interesting since it consists exactly of all locally finite modules (direct limits of finite modules). In fact, the condition that $QH^*(X)$ is locally finite is equivalent to requiring that the loop space ΩX be $B\mathbb{Z}/p$-local, i.e. the space of pointed maps $\mathrm{map}_*(B\mathbb{Z}/p, \Omega X)$ is contractible, see [18, Prop 3.2] and [32, Proposition 6.4.5].

We extend this topological characterization to H-spaces X with $QH^*(X) \in \mathcal{U}_n$. We use the standard notation T_V for Lannes' T functor.

Theorem 5.3. *Let X be a connected H-space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Then $QH^*(X)$ is in $\mathcal{U}_n$ if and only if $\Omega^{n+1}X$ is $B\mathbb{Z}/p$-local.*

We apply now Bousfield's results on the Postnikov-like tower associated to the $B\mathbb{Z}/p$-nullification functor $P_{B\mathbb{Z}/p}$ (relying on his "Key Lemma", [7, Chapter 7]). They enable us to reconstruct those H-spaces such that $\Omega^{n+1}X$ is $B\mathbb{Z}/p$-local from $P_{B\mathbb{Z}/p}X$ in a finite number of principal H-fibrations over p-torsion Eilenberg-Mac Lane spaces. When $n = 0$, we recover the results of Broto et al. from [10, 15, 11].

Theorem 5.5. *Let X be an H-space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Then $QH^*(X)$ is in $\mathcal{U}_n$ if and only if X is the total space of an H-fibration*

$$F \longrightarrow X \longrightarrow P_{B\mathbb{Z}/p}X$$

where F is a p-torsion H-Postnikov piece whose homotopy groups are finite direct sums of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$ concentrated in degrees 1 to $n+1$.

It is worthwhile to mention that working with H-spaces is crucial as illustrated by the example of BS^3. Its loop space S^3 is $B\mathbb{Z}/p$-local, but the fiber of the nullification map has infinitely many non-trivial homotopy groups (see Dwyer's computations in [16, Theorem 1.7, Lemma 6.2]).

Notation. We say that $H^*(X)$ is of finite type if $H^n(X)$ is a finite $\mathbb{F}_p$-vector space for any integer $n \geq 0$.

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1. Lannes' T functor

Lannes' T functor was designed as a tool to compute the cohomology of mapping spaces with source BV, the classifying space of an elementary abelian p-group V. It was used also by Lannes to give an alternative proof of Miller's Theorem on the Sullivan's conjecture.

Let $\mathcal{U}$ (resp. $\mathcal{K}$) be the category of unstable modules (resp. algebras) over the Steenrod algebra. The functor T_V is the left adjoint of $-\otimes H^*(BV)$ in $\mathcal{K}$, where V is an elementary abelian p-group. The left adjoint of $-\otimes \widetilde{H}^*(BV)$ is called the reduced T functor and denoted by $\overline{T}_V$. For each unstable module $M\in\mathcal{U}$, we have a splitting of modules over the Steenrod algebra $T_VM = M\oplus \overline{T}_VM$. We will use T to denote $T_{\mathbb{Z}/p}$ and $\overline{T}$ to denote $\overline{T}_{\mathbb{Z}/p}$.

If $M = H^*(X)$, the evaluation map $BV\times \mathrm{map}(BV,X)\to X$ induces by adjunction a map $\lambda_V : T_VH^*(X)\longrightarrow H^*(\mathrm{map}(BV,X))$ of unstable algebras over $\mathcal{A}_p$, which is often an isomorphism.

Proposition 1.1. *Let X be an H-space such that $H^*(X)$ is of finite type. Assume that $T_VH^*(X)$ (or equivalently $H^*(\mathrm{map}(BV,X))$) is of finite type. Then*

$$T_VH^*(X)\cong H^*(\mathrm{map}(BV,X)),$$

as algebras over $\mathcal{A}_p$. Moreover, $\mathrm{map}(BV,\hat{X}_p)$ is p-complete.

Proof. If X is an H-space, then $\mathrm{map}(BV,X)$ is again an H-space, and so is the connected component $\mathrm{map}(BV,X)_c$ of the constant map (see [38]). Moreover, when X is connected, all connected components of the mapping space have the same homotopy type. Since the evaluation map is an H-map and has a section, there is a splitting

$$\mathrm{map}(BV,X)\simeq X\times \mathrm{map}_*(BV,X)\,.$$

By [29, Theorem 1.5] there is a weak equivalence $\mathrm{map}_*(BV,X)\simeq \mathrm{map}_*(BV,\hat{X}_p)$ for any elementary abelian p-group V. Since X is p-good, we can work with $\hat{X}_p$ and [25, Proposition 3.4.4] applies: $T_VH^*(\hat{X}_p)\cong H^*(\mathrm{map}(BV,\hat{X}_p))$ and we can conclude. □

When working with H-spaces, it is often handy to deal with the pointed mapping space instead of the full mapping space. This is possible since the above finiteness condition on $T_VH^*(X)$ can be given in terms of the pointed mapping space.

Lemma 1.2. *Let X be an H-space such that $H^*(X)$ is of finite type. Then, $T_VH^*(X)$ is of finite type if and only if $H^*(\mathrm{map}_*(BV,X))$ is of finite type. Moreover, if X is such that $T_VH^*(X)$ is of finite type for any elementary abelian p-group V then the same holds for $\mathrm{map}_*(BW,X)$ for any elementary abelian p-group W.*

Proof. By Proposition 1.1, if $T_V H^*(X)$ is of finite type then

$$H^*(\mathrm{map}(BV, X)) \cong H^*(X) \otimes H^*(\mathrm{map}_*(BV, X))$$

is of finite type. In particular, $H^*(\mathrm{map}_*(BV, X))$ is of finite type. On the other hand, if $H^*(X)$ and $H^*(\mathrm{map}_*(BV, X))$ are of finite type then $H^*(\mathrm{map}(BV, X))$ is of finite type, which is equivalent to $T_V H^*(X)$ being of finite type by Proposition 1.1.

The last statement follows from the fact that $T_{V\times W} = T_V T_W$ for any elementary abelian p-groups V and W. □

When X is connected, the evaluation $\mathrm{map}(BV, X) \to X$ is a homotopy equivalence if $T_V H^*(X) \cong H^*(X)$ (for finite spaces, this is the Sullivan conjecture proved by Miller [29, Theorems A,C]). Actually, spaces for which this happens have been cohomologically characterized by Lannes and Schwartz in [26]: their mod p cohomology is locally finite.

When one restricts the evaluation map to the connected component of the constant map, the module of indecomposable elements $QH^*(X)$ comes into play as observed by Dwyer and Wilkerson in [18, Proposition 3.2] (see also [32, 3.9.7 and 6.4.5]).

Proposition 1.3. *Let X be a connected H-space of finite type. Then $QH^*(X)$ is a locally finite $\mathcal{A}_p$-module if and only if $\mathrm{map}_*(BV, \Omega X)$ is contractible for some elementary abelian p-group V.*

Proof. Since $\hat{X}_p$ is a connected p-complete H-space, $QH^*(X)$ is a locally finite $\mathcal{A}_p$-module if and only if $\mathrm{map}_*(BV, \hat{X}_p)$ is homotopically discrete for any elementary abelian p-group V by [18, Proposition 3.2] and [32, Proposition 6.4.5].

The weak equivalence $\mathrm{map}_*(BV, X) \simeq \mathrm{map}_*(BV, \hat{X}_p)$ given by [29, Theorem 1.5] shows that this is equivalent to $\mathrm{map}_*(BV, \Omega X)$ being contractible, i.e. the loop space ΩX is BV-local. □

2. The Krull filtration of $\mathcal{U}$

In [34], Schwartz proves the "strong realization conjecture" extending his previous results from [33]. This conjecture, given by Kuhn in [24], states that if the cohomology of a space lies in some stage of the Krull filtration of the category $\mathcal{U}$ of unstable modules, then it must be locally finite. The Krull filtration $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots$ is defined inductively, see [20]. It starts with the full subcategory $\mathcal{U}_0$ of $\mathcal{U}$ of locally finite unstable modules and the modules in $\mathcal{U}_n$ can be characterized as follows by means of the functor $\overline{T}$:

Theorem 2.1. *[32, Theorem 6.2.4] Let M be an unstable module. Then $M \in \mathcal{U}_n$ if and only if $\overline{T}^{n+1} M = 0$.* □

The proof of Kuhn's conjecture by Schwartz shows that under the usual finiteness conditions the cohomology of a space either lies in $\mathcal{U}_0$ or it is not in any $\mathcal{U}_n$. Instead of looking at when the full cohomology of a space is in $\mathcal{U}_n$, we will study the module of the indecomposable elements $QH^*(X)$. The Krull filtration induces a filtration of the category of H-spaces by looking at those H-spaces X for which $QH^*(X) \in \mathcal{U}_n$. There exist many spaces lying in each layer of this filtration, the most obvious ones being Eilenberg-Mac Lane spaces.

Example 2.2. The module $QH^*(K(\mathbb{Z}/p, n+1))$ is isomorphic to the suspension of the free unstable module $F(n)$ on one generator in degree n. In particular, the formula $\overline{T}F(n) = \oplus_{0\leq i\leq n-1}F(i)$ (see [32, Lemma 3.3.1]) yields that $QH^*(K(\mathbb{Z}/p, n+1)) \in \mathcal{U}_n$.

More generally, let G be any abelian discrete group such that $H^*(K(G, n+1))$ is of finite type. Then $QH^*(K(G, n+1)) \in \mathcal{U}_n$ (see [32, Section 9.8] for the explicit computations of the T functor).

From the above example, it is easy to see that the filtration is not exhaustive, since the infinite product $\prod_{n\geq 1} K(\mathbb{Z}/p, n)$ does not belong to any stage of the filtration.

Next lemma shows, by means of the reduced T functor, how $QH^*(X)$ is related to $QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X))$.

Lemma 2.3. *Let X be an H-space such that $TH^*(X)$ is of finite type. Then,*

$$\overline{T}QH^*(X) \cong QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X)) \ .$$

Proof. Under such assumptions, Proposition 1.1 applies and we know that the T functor computes the cohomology of the mapping space. Thus $QTH^*(X)$ is isomorphic to

$$QH^*(\mathrm{map}(B\mathbb{Z}/p, X)) \cong Q(H^*(\mathrm{map}_*(B\mathbb{Z}/p, X)) \otimes H^*(X))$$

Since T commutes with taking indecomposable elements [32, Lemma 6.4.2], it follows that $TQH^*(X) \cong QH^*(X) \oplus QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X))$. This is equivalent to $\overline{T}QH^*(X) \cong QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X))$. □

We end the section with a proposition which will allow us to perform an induction in the Krull filtration. Observe that Kuhn's strategy to move in the Krull filtration is to consider the cofiber of the inclusion $X \to \mathrm{map}(B\mathbb{Z}/p, X)$ in the component of the constant map, see [24]. In our context, Lemma 2.3 suggests to use the fiber of the evaluation $\mathrm{map}(B\mathbb{Z}/p, X) \to X$.

Proposition 2.4. *Let X be an H-space with $TH^*(X)$ of finite type. Then, for $n \geq 1$, $QH^*(X) \in \mathcal{U}_n$ if and only if $QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X))$ is in $\mathcal{U}_{n-1}$.*

Proof. By Theorem 2.1, the unstable module $QH^*(X)$ belongs to $\mathcal{U}_n$ if and only if $\overline{T}^{n+1}QH^*(X) = 0$. By Lemma 2.3, $\overline{T}^n\overline{T}QH^*(X) \cong \overline{T}^n(QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X))$, and we obtain that $QH^*(X) \in \mathcal{U}_n$ if and only if $QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X)) \in \mathcal{U}_{n-1}$. □

By repeatedly applying the previous proposition, one can give a more geometrical formulation to the condition $QH^*(X) \in \mathcal{U}_n$. This happens if and only if the pointed mapping space out of an $(n+1)$-fold smash product $\mathrm{map}_*(B\mathbb{Z}/p \wedge \ldots \wedge B\mathbb{Z}/p, X)$ is homotopically discrete.

3. Bousfield's $B\mathbb{Z}/p$-nullification filtration.

The plan of this section follows the preceding one step-by-step, replacing the algebraic filtration defined with the module of indecomposables by a topological one.

Dror-Farjoun and Bousfield have constructed a localization functor P_A from spaces to spaces together with a natural transformation $l : X \to P_AX$ which is an initial map among those having a local space as target (see [19] and [5]). This functor is known as the A-nullification. It preserves H-space structures since it commutes with finite products. Moreover, when X is an H-space, the map l is an H-map and its fiber is an H-space.

Bousfield has determined the structure of the fiber of the nullification map $l : X \to P_AX$ under certain assumptions on A. We are interested in the situation in which $A = \Sigma^n B\mathbb{Z}/p$.

Theorem 3.1. *[6, Theorem 7.2] Let $n \geq 1$ and X be a connected H-space such that $\Omega^n X$ is $B\mathbb{Z}/p$-local. The homotopy fiber of the localization map $X \to P_{\Sigma^{n-1}B\mathbb{Z}/p}X$ is then an Eilenberg-Mac Lane space $K(P, n)$ where P is an abelian p-torsion group (possibly infinite).* □

As mentioned by Bousfield in [6, p. 848], an inductive argument allows to obtain a precise description of the fiber of the $B\mathbb{Z}/p$-nullification map for H-spaces for which some iterated loop space is local.

Theorem 3.2. *Let $n \geq 0$ and X be a connected H-space such that $\Omega^n X$ is $B\mathbb{Z}/p$-local. Then there is an H-fibration*

$$F \longrightarrow X \longrightarrow P_{B\mathbb{Z}/p}X,$$

where F is a p-torsion H-Postnikov piece whose homotopy groups are concentrated in degrees from 1 to n. □

We introduce a "nullification filtration" by looking at those H-spaces X such that the iterated loop space $\Omega^n X$ is $B\mathbb{Z}/p$-local. The example of the Eilenberg-Mac Lane spaces shows that there are many spaces living in each stage of this filtration as well, compare with Example 2.2.

Example 3.3. Let G be an abelian discrete group with non-trivial mod p cohomology. Then, the Eilenberg-Mac Lane space $K(G,n)$ enjoys the property that its n fold iterated loop space is $B\mathbb{Z}/p$-local (it is even discrete). The infinite product $\prod_{n\geq 1} K(\mathbb{Z}/p,n)$ does not live in any stage of this topological filtration.

Another source of examples of spaces in this filtration is provided by connected covers of finite H-spaces.

Example 3.4. Let X be a finite connected H-space and consider its n-connected cover $X\langle n\rangle$. Then $\Omega^{n-1}(X\langle n\rangle)$ is $B\mathbb{Z}/p$-local.

For a connected H-space X such that $\Omega^n X$ is $B\mathbb{Z}/p$-local, the study of the homotopy type of $\mathrm{map}_*(B\mathbb{Z}/p, X)$ is drastically simplified by Theorem 3.2, since this space is equivalent to $\mathrm{map}_*(B\mathbb{Z}/p, F)$ where F is a Postnikov piece, as we explain in the proof below. A complete study of the $B\mathbb{Z}/p$-homotopy theory of such H-spaces is undertaken in [13].

We prove now the topological analogue of Lemma 2.3.

Proposition 3.5. *Let X be a connected H-space such that $\Omega^n X$ is $B\mathbb{Z}/p$-local, then $\Omega^{n-1}\mathrm{map}_*(B\mathbb{Z}/p, X)$ is $B\mathbb{Z}/p$-local.*

Proof. Under the hypothesis that $\Omega^n X$ is $B\mathbb{Z}/p$-local, Theorem 3.2 tells us that we have a fibration

$$F \longrightarrow X \longrightarrow P_{B\mathbb{Z}/p}X,$$

where F is a p-torsion Postnikov system with homotopy concentrated in degrees from 1 to n. Thus, $\mathrm{map}_*(B\mathbb{Z}/p, X) \simeq \mathrm{map}_*(B\mathbb{Z}/p, F)$ because $P_{B\mathbb{Z}/p}X$ is a $B\mathbb{Z}/p$-local space. Now, $\Omega^{n-1}\mathrm{map}_*(B\mathbb{Z}/p, F)$ is $B\mathbb{Z}/p$-local (in fact, it is a homotopically discrete space) and thus so is $\Omega^{n-1}\mathrm{map}_*(B\mathbb{Z}/p, X)$. □

4. Infinite loop spaces

In order to compare the topological with the algebraic filtration, one of the key ingredients comes from the theory of infinite loop spaces. In this section we explain when a pointed mapping space $\mathrm{map}_*(A, X)$ is an infinite loop space, but we are of course specially interested in the case when A is $B\mathbb{Z}/p$. We make use of Segal's techniques of Γ-spaces and follow his notation from [35], which is better adapted

to our needs than that of Bousfield and Friedlander, see [8]. Recall that the category Γ is the category of finite sets, and a morphism $\theta : S \to T$ between two finite sets is a partition of a subset of T into $|S|$ disjoint subsets $\{\,\theta(\alpha)\,\}_{\alpha\in S}$. A Γ-space is a contravariant functor from Γ to the category of spaces with some extra conditions. We first construct a covariant functor $A_\bullet : \Gamma \to \mathit{Spaces}$ for any pointed space A by setting $A_n = A^n$ (so in particular $A_0 = *$) and a morphism $\theta : [n] \to [m]$ induces the map $\theta_* : A^n \to A^m$ sending $(a_1, \ldots, a_n)$ to the element $(b_1, \ldots, b_m)$ with $b_j = a_i$ if and only if $j \in \theta(i)$ and $b_j = *$ otherwise.

Hence, we get a contravariant functor for any pointed space X by taking the pointed mapping space $\mathrm{map}_*(-, X)$. For $\mathrm{map}_*(A_\bullet, X)$ to be a Γ-space one needs to check it is *special*, i.e. the n inclusions $i_k : [1] \to [n]$ sending 1 to k must induce a weak equivalence $\mathrm{map}_*(A^n, X) \to \mathrm{map}_*(A, X)^n$.

Lemma 4.1. *Let A and X be pointed spaces and assume that the inclusion $A^n \vee A \hookrightarrow A^n \times A$ induces for any $n \geq 1$ a weak equivalence $\mathrm{map}_*(A^n \times A, X) \to \mathrm{map}_*(A^n \vee A, X)$. Then, $\mathrm{map}_*(A_\bullet, X)$ is a Γ-space.* □

Proposition 4.2. *Let A be a pointed connected space and X an H-space. Assume that $\mathrm{map}_*(A, X)$ is A-local. Then $\mathrm{map}_*(A_\bullet, X)$ is a Γ-space.*

Proof. The cofiber sequence $A^n \vee A \to A^n \times A \to A^n \wedge A$ yields a fibration of pointed mapping spaces

$$\mathrm{map}_*(A^n \wedge A, X) \longrightarrow \mathrm{map}_*(A^n \times A, X) \longrightarrow \mathrm{map}_*(A^n \vee A, X).$$

By adjunction, the fiber $\mathrm{map}_*(A^n \wedge A, X) \simeq \mathrm{map}_*(A^n, \mathrm{map}_*(A, X))$ is contractible since any A-local space is also A^n-local (A^n is A-cellular or use Dwyer's version of Zabrodsky's Lemma in [16, Proposition 3.4]). Moreover, the inclusion $A^n \vee A \to A^n \times A$ induces a bijection on sets of homotopy classes $[A^n \times A, X] \to [A^n \vee A, X]$ by [38, Lemma 1.3.5]. Since all components of these pointed mapping spaces have the same homotopy type, we have a weak equivalence $\mathrm{map}_*(A^n \times A, X) \simeq \mathrm{map}_*(A^n \vee A, X)$ and conclude by the preceding proposition. □

Theorem 4.3. *Let A be a pointed connected space and let X be a loop space such that $\mathrm{map}_*(A, X)$ is A-local. Then, $\mathrm{map}_*(A, X)$ is an infinite loop space, and so is the corresponding connected component $\mathrm{map}_*(A, X)_c$ of the constant map.*

Proof. The Γ-space structure constructed above yields classifying spaces $B^n\mathrm{map}_*(A, X)$ and weak equivalences $\Omega B^{n+1}\mathrm{map}_*(A, X) \simeq B^n\mathrm{map}_*(A, X)$ for any $n \geq 1$. In our situation X is a loop space, and

so is the mapping space $\mathrm{map}_*(A,X)$. Therefore, Segal's result [35, Proposition 1.4] applies and shows that $\mathrm{map}_*(A,X)$ is equivalent to the loop space $\Omega B\mathrm{map}_*(A,X)$. □

We specialize now to the case $A = B\mathbb{Z}/p$, where we can even say more about the intriguing infinite loop space $\mathrm{map}_*(B\mathbb{Z}/p,X)_c$.

Proposition 4.4. *Let X be a loop space such that $\mathrm{map}_*(B\mathbb{Z}/p,X)$ is $B\mathbb{Z}/p$-local. Then all homotopy groups of the infinite loop space $\mathrm{map}_*(B\mathbb{Z}/p,X)_c$ are $\mathbb{Z}/p$-vector spaces.*

Proof. Since $\pi_n\mathrm{map}_*(B\mathbb{Z}/p,X)_c \cong [B\mathbb{Z}/p,\Omega^n X]$, consider a map $B\mathbb{Z}/p \to \Omega^n X$. We claim that it is homotopic to an H-map. Indeed, by [38, Proposition 1.5.1], the obstruction lives in the set $[B\mathbb{Z}/p \wedge B\mathbb{Z}/p, \Omega^n X]$, which is trivial since $\mathrm{map}_*(B\mathbb{Z}/p,X)$ is $B\mathbb{Z}/p$-local. But any non-trivial H-map out of $B\mathbb{Z}/p$ has order p. □

5. Structure theorems for H-spaces

The purpose of this section is to give an inductive description of the H-spaces whose module of indecomposable elements lives in some stage of the Krull filtration. This is achieved by comparing this algebraic filtration with the topological one and by making use of Bousfield's result 3.2.

Proposition 5.1. *Let X be an H-space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Assume that $\Omega^n X$ is $B\mathbb{Z}/p$-local. Then $QH^*(X) \in \mathcal{U}_{n-1}$.*

Proof. We proceed by induction. For $n = 1$, assume that ΩX is $B\mathbb{Z}/p$-local, that is, $\Omega\mathrm{map}_*(B\mathbb{Z}/p,X)_c \simeq *$. Then, $\mathrm{map}_*(B\mathbb{Z}/p,X)$ is homotopically discrete since $\mathrm{map}_*(B\mathbb{Z}/p,X)_c$ is so and all components of the mapping space have the same homotopy type. Hence, $QH^*(\mathrm{map}_*(B\mathbb{Z}/p,X)) = 0$ and, by Lemma 2.3, $QH^*(X) \in \mathcal{U}_0$.

If $n > 1$, let X be an H-space such that $\Omega^n X$ is $B\mathbb{Z}/p$-local. We see by Proposition 3.5 that $\Omega^{n-1}\mathrm{map}_*(B\mathbb{Z}/p,X)_c$ is $B\mathbb{Z}/p$-local as well. Now, $\mathrm{map}_*(B\mathbb{Z}/p,X)_c$ is an H-space such that $\Omega^{n-1}\mathrm{map}_*(B\mathbb{Z}/p,X)_c$ is $B\mathbb{Z}/p$-local. Moreover, by Lemma 1.2, $T_V H^*(\mathrm{map}_*(B\mathbb{Z}/p,X))$ is of finite type for any elementary abelian p-group V. By induction hypothesis, $QH^*(\mathrm{map}_*(B\mathbb{Z}/p,X)_c) \in \mathcal{U}_{n-2}$. Since all components have the same homotopy type, we obtain that $QH^*(\mathrm{map}_*(B\mathbb{Z}/p,X)) \in \mathcal{U}_{n-2}$, and we conclude that $QH^*(X) \in \mathcal{U}_{n-1}$ by Corollary 2.4. □

Proposition 5.2. *Let X be a connected H-space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Suppose that $QH^*(X) \in \mathcal{U}_n$. Then $\Omega^{n+1}X$ is $B\mathbb{Z}/p$-local.*

Proof. As in the proof of Proposition 1.1 we can assume without loss of generality that X is p-complete.

Let us proceed by induction. The case $n = 0$ is given by Corollary 1.3. Now assume that the result is true for $n-1$, and consider a space X such that $QH^*(X) \in \mathcal{U}_n$. Then, by Lemma 2.4, $QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X)) \in \mathcal{U}_{n-1}$ and the induction hypothesis ensures that $\Omega^n \mathrm{map}_*(B\mathbb{Z}/p, X)_c \simeq \mathrm{map}_*(B\mathbb{Z}/p, \Omega^n X)$ is $B\mathbb{Z}/p$-local. Apply now Theorem 4.3 to deduce that the space $\mathrm{map}_*(B\mathbb{Z}/p, \Omega^n X)_c$ is an infinite loop space, with a p-torsion fundamental group by Proposition 4.4.

These are precisely the conditions of McGibbon's main theorem in [28]: the $B\mathbb{Z}/p$-nullification of connected infinite loop spaces with p-torsion fundamental group is trivial, up to p-completion. Moreover, our infinite loop space is $B\mathbb{Z}/p$-local, so

$$(\mathrm{map}_*(B\mathbb{Z}/p, \Omega^n X)_c)^\wedge_p \simeq (P_{B\mathbb{Z}/p}(\mathrm{map}_*(B\mathbb{Z}/p, \Omega^n X)_c))^\wedge_p \simeq *$$

As we assume that X is p-complete, so are the loop space $\Omega^n X$ and the pointed mapping space $\mathrm{map}_*(B\mathbb{Z}/p, \Omega^n X)_c$. Thus, we see that $\mathrm{map}_*(B\mathbb{Z}/p, \Omega^n X)_c$ must be contractible. Since all components of the pointed mapping space have the same homotopy type as the component of the constant map, we infer that $\mathrm{map}_*(B\mathbb{Z}/p, \Omega^n X)$ is homotopically discrete. Looping once again, one obtains the equivalence $\mathrm{map}_*(B\mathbb{Z}/p, \Omega^{n+1} X) \simeq *$, i.e. $\Omega^{n+1} X$ is $B\mathbb{Z}/p$-local, as we wanted to prove. □

Let us sum up these two results in one single statement, which extends widely Dwyer and Wilkerson's [18, Proposition 3.2] when X is assumed to be an H-space.

Theorem 5.3. *Let X be a connected H-space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Then, $QH^*(X)$ is in $\mathcal{U}_n$ if and only if $\Omega^{n+1} X$ is $B\mathbb{Z}/p$-local.* □

Combining these results with Theorem 3.2 (about the nullification functor $P_{B\mathbb{Z}/p}$) enables us to give a topological description of the H-spaces X for which the indecomposables $QH^*(X)$ live in some stage of the Krull filtration. Recall that the *Prüfer group* $\mathbb{Z}_{p^\infty}$ is defined as the union $\bigcup_{n\geq 1} \mathbb{Z}/p^n$. It is a p-torsion divisible abelian group.

Theorem 5.4. *Let X be a connected H-space of finite type such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Then $QH^*(X) \in \mathcal{U}_n$ if and only if X fits into a fibration*

$$K(P, n+1) \longrightarrow X \longrightarrow Y,$$

where Y is a connected H-space such that $QH^*(Y) \in \mathcal{U}_{n-1}$, and P is a p-torsion abelian group which is a finite direct sum of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$.

Proof. Let F be the homotopy fiber of the nullification map $X \to P_{\Sigma^n B\mathbb{Z}/p}(X)$. By Theorem 3.1, $F \simeq K(P, n+1)$ where P is an abelian p-group. Moreover, the equivalence $\mathrm{map}_*(\Sigma^n B\mathbb{Z}/p, K(P, n+1)) \simeq \mathrm{map}_*(\Sigma^n B\mathbb{Z}/p, X)$ shows that the set

$$\pi_n \mathrm{map}_*(B\mathbb{Z}/p, X) \cong \pi_1 \mathrm{map}_*(\Sigma^n B\mathbb{Z}/p, X) \cong \mathrm{Hom}(\mathbb{Z}/p, P)$$

is finite since all homotopy groups of $\mathrm{map}_*(B\mathbb{Z}/p, X)$ are p-torsion and its cohomology is of finite type. Thus, P is isomorphic to a finite direct sum of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$ by Lemma 5.8, which we prove at the end of the section.

We conclude by taking $Y = P_{\Sigma^n B\mathbb{Z}/p}(X)$. The cohomology $H^*(Y)$ is of finite type since $H^*(K(P, n+1))$ and $H^*(X)$ are of finite type, and so is $H^*(\mathrm{map}_*(BV, Y))$. Moreover, since $\Omega^n Y \simeq P_{B\mathbb{Z}/p}(\Omega^n X)$ is $B\mathbb{Z}/p$-local, Theorem 5.3 implies that $QH^*(Y) \in \mathcal{U}_{n-1}$. □

Equivalently, one can reformulate this result by describing the fiber of the $B\mathbb{Z}/p$-nullification map.

Theorem 5.5. *Let X be an H-space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Then, $QH^*(X)$ is in $\mathcal{U}_n$ if and only if X is the total space of an H-fibration*

$$F \longrightarrow X \longrightarrow P_{B\mathbb{Z}/p}X,$$

where F is a p-torsion H-Postnikov piece whose homotopy groups are finite direct sums of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$ concentrated in degrees 1 to $n+1$. □

In other words, the only H-spaces such that $QH^*(X)$ lies in $\mathcal{U}_n$ for some n are the $B\mathbb{Z}/p$-local H-spaces, the p-torsion Eilenberg-MacLane spaces introduced in Example 2.2, and extensions of the previous type.

Recall that the $B\mathbb{Z}/p$-nullification of a loop space is again a loop space. Moreover, by [19, Lemma 3.A.3], the nullification map is a loop map, and hence its homotopy fiber is also a loop space. Thus we obtain automatically the following result about loop spaces.

Corollary 5.6. *Let X be a loop space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Then $QH^*(X)$ is in $\mathcal{U}_n$ if and only if X is the total space of loop fibration*

$$F \longrightarrow X \longrightarrow P_{B\mathbb{Z}/p}X,$$

where the loop space F is a p-torsion Postnikov piece whose homotopy groups are finite direct sums of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$ concentrated in degrees 1 to $n+1$. □

If we restrict our attention to the case $n = 0$ in Theorem 5.5, our result reproves in a more conceptual way the theorems about H-spaces with locally finite module of indecomposable elements given by Broto, Saumell and the second named author in [10,15,11].

What do we learn from our study about H-spaces which do not belong to any stage of the filtration we have introduced in this paper? From a cohomological point of view, such H-spaces have a very large module of indecomposables. Let us discuss the example of BU.

Example 5.7. The module $QH^*(BU)$ is isomorphic to $\Sigma^2 H^*(BS^1)$. In particular, it is not a finitely generated $\mathcal{A}_p$-module. A computation of the value of the T functor on this module can be done using [32, Section 9.8] and shows that $QH^*(BU)$ does not belong to any $\mathcal{U}_n$.

Therefore the Krull filtration for the indecomposables detects in BU the fact that the $B\mathbb{Z}/p$-nullification Postnikov-like tower does not permit to deconstruct it into elementary pieces. In fact BU is $K(\mathbb{Z}/p, 2)$-local by a result of Mislin (see [30, Theorem 2.2]).

Finally, we prove the lemma about abelian p-torsion groups which was used in the proof of Theorem 5.4.

Lemma 5.8. *Let P be a p-torsion discrete group. If $\mathrm{Hom}(\mathbb{Z}/p, P)$ is finite then P is a finite direct sum of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$.*

Proof. By Kulikov's theorem (see [31, Theorem 10.36]), P admits a basic subgroup, which is a direct sum of cyclic groups. It must be of bounded order since $\mathrm{Hom}(\mathbb{Z}/p, P)$ is finite, and a result of Prüfer (see [31, Corollary 10.41]) shows now that this subgroup is a direct summand. Since the quotient is divisible and $\mathrm{Hom}(\mathbb{Z}/p, P)$ is finite, P is a finite direct sum of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$. □

6. H-spaces with finitely generated algebra over $\mathcal{A}_p$

We will assume in this section that $H^*(X)$ is finitely generated as an algebra over the Steenrod algebra. Then, the $B\mathbb{Z}/p$-nullification of X is a mod p finite H-space up to p-completion, as we prove in Theorem 6.4.

The next lemma shows that, under this finiteness condition, the H-spaces considered in this section satisfy the hypothesis of Theorem 5.3 (they belong to some stage of the filtration we study in this paper).

Lemma 6.1. *Let K be a finitely generated unstable $\mathcal{A}_p$-algebra. Then there exists some integer n such that the module of indecomposables QK belongs to $\mathcal{U}_n$. Moreover $T_V K$ is a finitely generated unstable $\mathcal{A}_p$-algebra for any elementary abelian group V.*

Proof. First of all, QK is a finitely generated module over $\mathcal{A}_p$, i.e. it is a quotient of a finite direct sum of free modules. Hence, there exists an epimorphism $\oplus_{i=1}^{k} F(n_i) \to QK$. Since $\overline{T}$ is an exact functor, it follows that $\overline{T}^m(QK) = 0$, where m is the largest of the n_i's, and so $QK \in \mathcal{U}_{m-1}$.

Moreover, T_V commutes with taking indecomposables elements [34, Lemma 6.4.2]. Therefore, $Q(T_V K)$ is a finitely generated unstable module. Then, the above discussion shows that $T_V K$ is a finitely generated $\mathcal{A}_p$-algebra. □

Our first proposition is inspired by the situation studied by Smith in [37].

Proposition 6.2. *Let $p : X \to B$ be a principal H-fibration classified by an H-map $\varphi : B \to BF$. Then, there is an isomorphism of algebras $H^*(X) \cong (H^*(B)//\varphi^*) \otimes A$, where A is a subalgebra of $H^*(X)$ and $H^*(B)//\varphi^*$ is the quotient by the ideal generated by the positive degree elements in $\mathrm{Im}(\varphi^*)$. Moreover, if $H^*(BF)$ and $H^*(X)$ are finitely generated $\mathcal{A}_p$-algebras, then so is $H^*(B)$.*

Proof. Consider the Serre spectral sequence associated to the principal fibration $F \to X \to B$, with E_2-term $E_2^{*,*} = H^*(F) \otimes H^*(B)$ which converges to $H^*(X)$. Let $\{E'_r\}$ be the spectral sequence associated to the universal path fibration $F \to PBF \to BF$. The map φ induces a morphism of spectral sequences $E'_r \to E_r$ which, in the E_2-term, is the identity on the vertical axes and φ^* on the horizontal axes.

Since the spectral sequence for the universal path fibration converges to $\mathbb{F}_p$, it follows from naturality that all elements in the image of φ^* are hit by some differential. Moreover, an element on the horizontal axis is hit by a differential exactly if it lies in the ideal generated by the elements which are killed by an element on the vertical axis. Therefore, $E_\infty^{*,0} \cong H^*(B)//\varphi^*$.

The morphism p^* induced in cohomology factors through the edge homomorphism $H^*(B) \to H^*(B)//\varphi^* \subset H^*(X)$. In particular, we know that $H^*(B)//\varphi^* \cong \mathrm{Im}(p^*)$ is an $\mathcal{A}_p$-Hopf subalgebra. Therefore, by the proof of the Borel-Hopf decomposition theorem in [23, Section 2.2], there is a complement A such that $H^*(X) \cong (H^*(B)//\varphi^*) \otimes A$, as algebras.

If $H^*(X)$ is a finitely generated algebra over $\mathcal{A}_p$, then $H^*(B)//\varphi^*$ is also a finitely generated algebra over $\mathcal{A}_p$. On the other hand, $\mathrm{Im}(\varphi^*) \subset H^*(B)$ is a $\mathcal{A}_p$-Hopf subalgebra which is also finitely generated as algebra over $\mathcal{A}_p$, being the image of the finitely generated one $H^*(BF)$. Thus so is $H^*(B)$. □

The proof of the next theorem is done by induction, in which the reduction step relies again on mapping spaces. Thus, we need to control the finiteness conditions of such mapping spaces.

Lemma 6.3. *Let F be a p-torsion H-Postnikov piece. If the cohomology $H^*(\mathrm{map}_*(B\mathbb{Z}/p, F))$ is a finitely generated algebra over $\mathcal{A}_p$, then so is $H^*(F)$.*

Proof. Since $\mathrm{map}_*(B\mathbb{Z}/p, K(P,n))$ is homotopy equivalent to a product $K(P_{n-1}, n-1)\times\cdots\times K(P_0, 0)$ of lower dimensional Eilenberg-Mac Lane spaces, the result holds for Eilenberg-MacLane spaces. In particular, $P_{n-1} \cong \mathrm{Hom}(\mathbb{Z}/p, P)$ must be finite and P is a finite direct sum of cyclic and Prüfer groups by Lemma 5.8.

By induction, the same holds for Postnikov pieces. Let F be a Postnikov piece with homotopy concentrated in degrees from 1 to n and consider the principal fibration

$$K(P,n) \longrightarrow F \longrightarrow F'.$$

The highest non-trivial homotopy group of $\mathrm{map}_*(B\mathbb{Z}/p, F)$ is isomorphic to $\mathrm{Hom}(\mathbb{Z}/p, P)$. Since the mod p cohomology is of finite type, this must be a finite group. Hence, by Lemma 5.8, P is a finite direct sum of copies of $\mathbb{Z}/p^r$.

Applying $\mathrm{map}_*(B\mathbb{Z}/p, -)$ to the fibration $F \to F' \to K(P, n+1)$, we notice that $\mathrm{map}_*(B\mathbb{Z}/p, F')$ has finitely generated cohomology as an $\mathcal{A}_p$-algebra by the assumption on $\mathrm{map}_*(B\mathbb{Z}/p, F)$ and Proposition 6.2. By induction hypothesis, $H^*(F')$ enjoys the same property and so, the same holds for $H^*(F)$ by Proposition 6.2 again. □

We can now state our main finiteness result. It enables us to understand better the $B\mathbb{Z}/p$-nullification, which is the first building block in our deconstruction process (Theorem 5.4).

Theorem 6.4. *Let X be a connected H-space such that $H^*(X)$ is finitely generated as algebra over the Steenrod algebra. Then, $P_{B\mathbb{Z}/p}X$ is an H-space with finite mod p cohomology.*

Proof. By Lemma 6.1, there exists an integer n such that $QH^*(X) \in \mathcal{U}_{n-1}$, so Theorem 5.3 applies and we know that $\Omega^n X$ is $B\mathbb{Z}/p$-local.

We will show that if $H^*(X)$ is finitely generated as an algebra over $\mathcal{A}_p$ and $\Omega^n X$ is $B\mathbb{Z}/p$-local, then $H^*(P_{B\mathbb{Z}/p}X)$ is finitely generated as an algebra over $\mathcal{A}_p$. We proceed by induction on n. When $n = 0$ the statement is clear. Assume the statement holds for $n-1$.

Since $H^*(X)$ is a finitely generated $\mathcal{A}_p$-algebra, so is $TH^*(X)$ by Lemma 6.1. Lemma 2.3 shows that $H^*(\mathrm{map}_*(B\mathbb{Z}/p, X))$ is a finitely generated algebra over $\mathcal{A}_p$ as well. Let F be the homotopy fiber of the nullification map $X \to P_{B\mathbb{Z}/p}X$. Because of the weak equivalence $\mathrm{map}_*(B\mathbb{Z}/p, F) \simeq \mathrm{map}_*(B\mathbb{Z}/p, X)$, $H^*(\mathrm{map}_*(B\mathbb{Z}/p, F))$ is finitely generated as algebra over $\mathcal{A}_p$ and, by Lemma 6.3, the same holds for $H^*(F)$ since F is a p-torsion Postnikov piece. By Theorem 3.1, there is a principal H-fibration

$$K(P,n) \longrightarrow X \longrightarrow P_{\Sigma^{n-1}B\mathbb{Z}/p}X,$$

where P is a finite direct sum of copies of $\mathbb{Z}/p^r$, $1 \leq r \leq \infty$ and the nth homotopy group of F is precisely P. In particular, $H^*(K(P,n))$ is a finitely generated $\mathcal{A}_p$-algebra as well as $H^*(K(P,n+1))$.

It follows from Lemma 6.2 that $H^*(P_{\Sigma^{n-1}B\mathbb{Z}/p}X)$ is finitely generated as an algebra over $\mathcal{A}_p$. Moreover, $\Omega^{n-1}P_{\Sigma^{n-1}B\mathbb{Z}/p}X$ is weakly equivalent to $P_{B\mathbb{Z}/p}\Omega^{n-1}X$, which is $B\mathbb{Z}/p$-local, so the induction hypothesis applies. The cohomology of $P_{B\mathbb{Z}/p}X \simeq P_{B\mathbb{Z}/p}P_{\Sigma^{n-1}B\mathbb{Z}/p}X$ is finitely generated as an algebra over the Steenrod algebra.

Since $H^*(P_{B\mathbb{Z}/p}X)$ is locally finite, this implies that $P_{B\mathbb{Z}/p}X$ has finite mod p cohomology. □

Combining this last result with Theorem 5.5 we obtain:

Theorem 6.5. *Let X be a connected H-space such that $H^*(X)$ is a finitely generated algebra over the Steenrod algebra. Then, X is the total space of an H-fibration*

$$F \longrightarrow X \longrightarrow Y$$

where Y is an H-space with finite mod p cohomology and F is a p-torsion H-Postnikov piece whose homotopy groups are finite direct sums of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$. □

In the special case in which the H-space we are working with is a loop space, a result analogous to Corollary 5.6 follows.

We propose finally an extension of Hubbuck's Torus Theorem on homotopy commutative H-spaces. At the prime 2, we have:

Corollary 6.6. *Let X be a connected homotopy commutative H-space such that the mod 2 cohomology $H^*(X)$ is finitely generated as algebra over the Steenrod algebra $\mathcal{A}_2$. Then, up to 2-completion, X is homotopy equivalent to $(S^1)^n \times F$, where F is a connected 2-torsion H-Postnikov piece.*

Proof. Consider the fibration $F \to X \to P_{B\mathbb{Z}/p}X$. We know from the preceding theorem that the fiber is a p-torsion Postnikov piece and the basis is an H-space with finite mod p cohomology. Both are homotopy commutative. In particular, the mod 2 Torus Theorem of Hubbuck (see [22]) implies that $P_{B\mathbb{Z}/2}X$ is, up to 2-completion, a finite product of circles $(S^1)^n$. Since the fiber is 2-torsion, the above fibration is split and the result follows. □

When X is a mod 2 finite H-space, this corollary is the original Torus Theorem due to Hubbuck. When X is an H-space with noetherian cohomology, $QH^*(X) \in \mathcal{U}_0$, the Postnikov piece F is an Eilenberg-Mac Lane space $K(P,1)$ where P is a 2-torsion abelian group, and we get back Slack's results [36], as well as their generalization by Lin and Williams in [27]: up to 2-completion, X is the

direct product of a finite number of S^1's, $K(\mathbb{Z}/p^r, 1)$'s, and $K(\mathbb{Z}, 2)$'s. Of course, in our setting it is no longer true that the fiber F in Theorem 6.5 is a product of Eilenberg-Mac Lane spaces.

At odd primes, there are many more finite H-spaces which are homotopy commutative (all odd dimensional spheres for example). However, Hubbuck's result still holds for finite loop spaces of H-spaces, as was shown in [2] by Aguadé and Smith. Therefore, replacing the original Torus Theorem by the Aguadé-Smith version, the same conclusion as in Corollary 6.6 holds at odd primes for the loop space on an H-space.

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