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The homotopic adjoint representation for exotic p-compact groups

Abstract. The interpretation of properties of compact Lie groups in purely homotopic terms is the lei-motiv for studying properties of p-compact groups. The Lie algebra is an analytic object that has not been interpreted in terms of homotopy theory and it provides the complex and real adjoint representation of a Lie group. We construct complex homotopy representations for a family of exotic p-compact groups, that is maps into $BU(n)^\wedge_p$. These representations are the homotopic analogues to the adjoint representation. In particular they are monomorphism hence they solve the problem of existence of monomorphisms from a simply connected p-compact group into $U(n)^\wedge_p$ (p odd).

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1. Introduction

Since homotopy Lie groups (also called p-compact groups) were defined [DW94], its homotopy behaviour has been investigated in connection with homotopy properties of compact Lie groups. The basic lei-motiv in their study is the interpretation of properties of compact Lie groups (for example, the existence of maximal tori and the Weyl group) in purely homotopic terms [DWb] and their generalization to p-compact groups.

A p-compact group is a triple (X, BX, e) where X is a loop space such that X is $\mathbb{F}_p$-finite and BX is p-complete. The first examples came from the Lie group theory. If G is a compact Lie group then $(G^\wedge_p, BG^\wedge_p, e)$ is a p-compact group if $\pi_0(G)$ is a finite p-group.

But compact Lie groups possess an analytic object which has not been interpreted in terms of homotopy theory: the Lie algebra. The Lie algebra provides combinatorial information related to the root system associated to the Lie group. This information appears in the adjoint representation of the group.

Another approximation to the concept of the adjoint representation can be done in terms of purely algebraic data. In [MS], Mitchell and Stong define a particular Thom module associated to a polynomial algebra of invariants

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R: the adjoint Thom module Ad_R. In case of working with compact Lie groups, their method provides a way of reconstructing the Thom module associated to the adjoint representation in terms of its cohomology and Weyl group.

The attempt of this paper is to use of the combinatorial data provided by reflections in the Weyl group of a p-compact group in order to obtain a homotopic adjoint representation for some families of exotic p-compact groups. It is also shown that this concept fits nicely into the previous cited approximation. That means that we are going to construct mod p spherical fibrations over the p-compact groups X such that the cohomology of its Thom space is isomorphic to $Ad_{H^*(BX;\mathbb{F}_p)}$ as a Thom module.

Another motivation for this work is the results in K-theory obtained by Jeanneret and Osse. Recently Jeanneret and Osse [JO] have proved the Atiyah-Segal theorem for p-compact groups. The Atiyah-Segal theorem [AS] states that the complex K-theory ring $K^*(BG;\mathbb{Z})$ is isomorphic to the I-adic completion of the complex representation ring $R(G)$. Assuming G connected, this fact is equivalent to the following isomorphism:

$$K^*(BG;\mathbb{Z}) \cong K^*(BT;\mathbb{Z})^W$$

where T is a maximal torus of G and W the corresponding Weyl group.

The structure of p-compact groups is concentrated in a single prime p, hence it is natural to consider K-theory with p-adic coefficients. Jeanneret and Osse generalize the Atiyah-Segal theorem,

$$K^*(BX;\hat{\mathbb{Z}}_p) \cong K^*(BT;\hat{\mathbb{Z}}_p)$$

where X is a p-compact group, $i : T \subset X$ a maximal torus and W the corresponding Weyl group.

An important step in the proof is the fact that the $K^*(BT;\hat{\mathbb{Z}}_p)$ is a $K^*(BX;\hat{\mathbb{Z}}_p)$-module finitely generated. If there exists a monomorphism from the p-compact group into a unitary group $U(N)^\wedge_p$, the previous fact can be easily deduced. This observation is explicitly stated in [JO]. In the Lie group theory, it is known that every Lie group admits a monomorphism into a unitary group. Hence, it remains to show that every exotic p-compact group admits a monomorphism into some unitary group. We can generalize this question and we can ask for the existence of monomorphisms into infinite generalized grassmannians.

Theorem 1.1. *Every simply connected simple exotic p-compact group admits a monomorphism into $U(n)^\wedge_p$ for some n.*

Every simply connected p-compact group is homotopy equivalent to a product of almost simple p-compact groups ([DWa]). We know the result is known for compact Lie groups, hence we divide the proof in the study of the three families of exotic simple p-compact groups: the Clark-Ewing spaces, the Aguadé spaces and the infinite generalized grassmannians.

We also solve the question on the existence of a monomorphism into infinite Quillen grassmannians.

Theorem 1.2. *Every simply connected simple exotic p-compact group admits a monomorphism into an infinite generalized Quillen grassmannian.*

The close relation between some of these infinite Quillen grassmannians and the classification of mod p spherical fibrations [C2] is the key step to prove the following realization fact.

Theorem 1.3. *Let X be an exotic p-compact group of Clark-Ewing, Aguadé or an infinite generalized grassmannian of rank p. Then there exist mod p spherical fibrations $\eta \downarrow BX$ such that*

$$\tilde{H}^*(T(\eta); \mathbb{F}_p) \cong Ad_{H^*(BX;\mathbb{F}_p)}$$

where $Ad_R = Tor^{n,*}_{R\otimes R}(R, R)$ and n is the rank of X.

In Section 1 we review some standard facts on Thom modules and the construction of the adjoint Thom module. Section 2 presents some preliminaries on complex root systems and the description of the admissible morphism associated to the adjoint representation. The complex homotopy representations for Clark-Ewing spaces are described in Section 4. We discuss the complex representations of X_i in Section 5. In Section 6 we discuss the case of extending admissible morphisms of $SU(p)$ into unitary groups. Section 7 contains the proof of the main theorem in Section 5 with some consequences and Section 8 contains the analogous result for infinite generalized grassmannians of rank p. Section 9 is devoted to the study of factorizations of unitary representations through generalized grassmannians.

2. p-compact groups

A p-compact group is a p-local version of a finite loop spaces. Namely, a p-compact group is a triple $X = (X, BX, e)$ where BX is a p-complete pointed space, such that $H^*(X; \mathbb{F}_p)$ is finite and where $e : \Omega BX \to X$ is a homotopy equivalence.

The first examples of p-compact groups are the p completions (in the sense of Bousfield-Kan [BK]) of compact connected Lie groups and their classifying spaces. Many properties of compact Lie group theory can be reinterpreted as homotopy theoretic properties of the classifying spaces in such a way that the concept extends to the category of p-compact groups (see [DWb]). For example, they admit a concept of maximal torus and Weyl group.

A p-compact torus is a triple $(T^\wedge_p, BT^\wedge_p, \simeq)$ where $T = (S^1)^n$ is a torus. The dimension of the p-compact torus $(T^\wedge_p, BT^\wedge_p, \simeq)$ is the dimension of T. A p-compact toral group P is a compact Lie group such that its identity component P_0 is a p-compact torus and $\pi_0(P)$ is a finite p-group.

A homomorphism $f\colon X \to Y$ between two p-compact groups is a pointed map $Bf\colon BX \to BY$. Given a homomorphism $f\colon X \to Y$, we define the homogeneous space Y/X associated to f as the homotopy fiber of Bf.

A homomorphism f is said to be a monomorphism if the homotopy fiber Y/X is mod p finite (that is, $H^*(Y/X;\mathbb{F}_p)$ is a finite $\mathbb{F}_p$-vector space). Two morphisms $f,g : X \to Y$ are conjugate if Bf and Bg are freely homotopic.

A maximal torus of a p-compact group X is a monomorphism $f\colon T_X \to X$ from a p-compact torus of maximal dimension into X. Dwyer and Wilkerson [**?**] show that every p-compact group X has a maximal torus and that any two maximal tori are conjugate. The rank of X is defined as the dimension of T_X. Given a maximal torus, there is a definition for its normalizer and the Weyl group of a p-compact group ([**?**]). If X is connected, the Weyl group is the group, under composition, of all homotopy classes of self maps $\omega : BT_X \to BT_X$ such that $Bf \circ \omega \simeq Bf$.

If X is connected, the inclusion of the maximal torus induces an isomorphism

$$H^*(BX;\hat{\mathbb{Z}}_p)\otimes\mathbb{Q} \cong (H^*(BT_X;\hat{\mathbb{Z}}_p)\otimes\mathbb{Q})^{W_X}.$$

The induced representation $W_X \to GL(H^2(BT_X;\hat{\mathbb{Z}}_p)\otimes\mathbb{Q})$ is faithful and represents W_X as a pseudoreflection group over $\hat{\mathbb{Z}}_p\otimes\mathbb{Q}$ (as a finite group generated by elements of finite order in $GL(n,\hat{\mathbb{Z}}_p\otimes Q)$ fixing a hyperplane of codimension one. In fact, this representation is integral in the sense that it is induced by the action of the Weyl group on the torus, $W_X \to GL(H^2(BT_p^\wedge;\hat{\mathbb{Z}}_p))$.

A classification of all irreducible pseudoreflection groups over $\hat{\mathbb{Q}}_p$ was achieved by Clark and Ewing [**?**], using the classification of complex pseudoreflection groups by Shephard and Todd [**?**]. Each p-adic reflection group admits a complex representation as a complex reflection group. They are listed in a table that contains three infinite families and 34 exceptional groups.

Introduce centers and kernels.

3. Maps between classifying spaces

This section is devoted to the problem of lifting admissible maps $\phi_T : T_{SU(p)} \to T_{U(n(i))}$ previously defined in Section **??**. Recall that in order to construct maps out of $BSU(p)$ we can use its description as a homotopy colimit of p-compact toral subgroups over the corresponding p-stubborn category.

Let G be a compact Lie group. A p-toral subgroup $P \subset G$ is called p-stubborn if $N_G(P)/P$ is finite and contains no nontrivial normal p-subgroups. $\mathcal{R}_p(G)$ denotes the category whose objects are orbits G/P for each p-stubborn $P \subset G$ and $Mor(G/P, G/P')$ is the set of all G-maps between the orbits.

Theorem 3.1. *[JMO1] The induced map*

$$\operatorname*{hocolim}_{P\in\mathcal{R}_p(G)} EG/P \to BG$$

is an $\mathbb{F}_p$-homology equivalence.

This is the main tool used to study maps between classifying spaces of compact Lie groups.

Definition 3.1. *[JMO2] An $\mathcal{R}_p(G)$-invariant quasi representation of G into G', a compact Lie group, is $\rho : N_p(T)_\infty \to G'$ which extends to*

$$\varprojlim^0{}_{\mathcal{R}_p(G)} Rep(P_\infty, G').$$

Moreover, any $f : BG \to (BG')^\wedge_p$ gives, by restriction $f|_{BN_p(T)_\infty} \simeq B\rho$, a unique $\mathcal{R}_p(G)$-invariant quasi representation ρ (see [JMO2]).

The natural question is: when does an $\mathcal{R}_p(G)$-invariant quasi representation extend to a map from $B\hat{G_p}$? The setting for the answer is obstruction theory.

Theorem 3.2. *[JMO2] Let G' be any compact, connected Lie group, ρ an $\mathcal{R}_p(G)$-invariant representation. Then*

$$\mathrm{Map}(BG, (BG')^\wedge_p)_{[\rho]} \neq \emptyset \;\; if \;\; H^{i+1}(-; \Pi_i) = 0 \; for \; i \geq 1,$$

$$\mathrm{Map}(BG, (BG')^\wedge_p)_{[\rho]} \; is\ connected \;\; if \;\; H^i(-; \Pi_i) = 0 \; for \; i \geq 1,$$

where $\mathrm{Map}(BG, (BG')^\wedge_p)_{[\rho]}$ *denotes the set of components* $\mathrm{Map}(BG, (BG')^\wedge_p)_f$ *such that $f \simeq \rho$.*

From now on, we restrict our attention to the case $G = SU(p)$. For $p = 3$ we can describe explicitly the orbit category of the 3-stubborns in $SU(3)$. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

where $\xi = exp(\frac{2\pi i}{3})$. The only 3-stubborns of $SU(3)$ are (up to conjugacy)

$$N_3(T) = \langle T, B \rangle \;, \quad \Gamma = \langle A, B \rangle$$

with $N_{SU(3)}(N_3(T))/N_3(T) \cong \mathbb{Z}/2\mathbb{Z}$ and $N_{SU(3)}(\Gamma)/\Gamma \cong Sp_2(\mathbb{F}_3)$.

For $p > 3$, the p-stubborn category of $SU(p)$ contains three objects corresponding to p-stubborns T, $N_p(T)$ and Γ where T is a maximal torus, $N_p(T)$ is the p-normalizer of the maximal torus and Γ is generated by the following matrices

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \xi & \cdots & 0 \\ \vdots & 0 & \ldots & 0 \\ 0 & 0 & \cdots & \xi^p \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ldots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $\xi = exp(\frac{2\pi i}{p})$. The automorphism groups are the following ones: $N_{SU(p)}(T)/T = \Sigma_p$, $N(N_p(T))/N_p(T) = \mathbb{Z}/(p-1)\mathbb{Z}$ and $N(\Gamma)/\Gamma = Sp_2(\mathbb{F}_p)$.

Recall that $(T_{SU(p)})_\infty \cong (\mathbb{Z}/p^\infty\mathbb{Z})^{p-1}$. Since

$$\hat{\mathbb{Z}}_p^{p-1} \cong Hom((\mathbb{Z}/p^\infty\mathbb{Z})^{p-1}, \mathbb{Z}/p^\infty\mathbb{Z})$$

, any $r \in \hat{\mathbb{Z}}_p^{p-1}$ defines a morphism $r : (\mathbb{Z}/p^\infty\mathbb{Z})^{p-1} \to S^1$. If we consider the inclusion of the symmetric group $\Sigma_p \cong \Sigma \leq G_i \leq GL_{p-1}(\hat{\mathbb{Z}}_p)$, we can use $r \in R$ to define an admissible morphism (see Section 2) ϕ_T by

$$\phi_T := \prod_{r \in R_W} : B(\mathbb{Z}/p^\infty\mathbb{Z})^{p-1} \to (BS^1)^{m(i)}$$

with respect to $\phi_G|_\sigma : \Sigma \to \Sigma_{n_i}$.

There are induced maps:

$$\phi_N : N_p T_\infty \to U(m(i)),$$

$$\phi_\Gamma : \Gamma_\infty \to U(m(i))$$

where $\phi_\Gamma = \phi_N|_{\Gamma_\infty}$ and ϕ_N is equivariant with respect the action of the automorphisms $\mathbb{Z}/(p-1)\mathbb{Z}$ in N_pT (it follows from the fact that ϕ_T is ϕ_W-equivariant). In order to obtain an $\mathcal{R}_p(G)$-invariant quasi representation, it only remains to show that ϕ_Γ is invariant with respect the action of $Sp_2(\mathbb{F}_p)$.

The following proposition shows how to obtain a $\mathcal{R}_p(G)$-invariant quasi representation under certain hypothesis.

Proposition 3.1. *Let $Bf_T : BT_\infty^{p-1} \to BT^n$ be an admissible morphism with respect to a group morphism $f_W : \Sigma_{p-1} \to \Sigma_n$ where*

$$Bf_T = \prod_{r \in R} Br$$

is defined by a set $R \subset Hom((\mathbb{Z}/p^\infty\mathbb{Z})^{p-1}, S^1)$. Assume that the p-Sylow subgroup S_p of Σ_{p-1} acts without fixed points on R ($Tr(f_W(B)) = 0$) and $Tr(f_T(A)) = -ap$ where $a \in \mathbb{Z}^+$. Then, the morphisms

$$f'_T := f_T \oplus Id_{(p-1)a},$$

$$f'_W := f_W \oplus a\rho,$$

where ρ is a faithful representation of Σ_p of dimension $p-1$, define a complex $\mathcal{R}_p(SU(p))$-invariant quasi representation.

Proof. It is easy to see from the definition of f' that

$$Tr(f'_W(B)) = Tr(f'_T(A)) = -a.$$

Let us see that this is enough to prove that f'_Γ is $Sp_2(\mathbb{F}_p)$-invariant. The other morphisms commute up to homotopy in the diagram by the definition of f'_T and f'_W. Using arguments of complex representation theory, it is enough to see that the trace of the morphisms f'_Γ does not change by the action of $Sp_2(\mathbb{F}_p)$.

The matrices of Γ are monomials and can be described as $\Sigma \cdot D$ where Σ is a permutation matrix and D is a diagonal matrix. Then

$$f'_\Gamma(\Sigma D) = f'_\Gamma(\Sigma) f'_\Gamma(D) = \left(\begin{array}{c|c} f_\Gamma(\Sigma) & 0 \\ 0 & a\rho(\Sigma) \end{array}\right) \left(\begin{array}{c|c} f_\Gamma(D) & 0 \\ 0 & Id \end{array}\right) =$$

$$\left(\begin{array}{c|c} f_\Gamma(\Sigma) f_\Gamma(D) & 0 \\ 0 & a\rho(\Sigma) \end{array}\right).$$

Notice that $Tr(f_\Gamma(\Sigma D)) = 0$ (the p-Sylow subgroup of W_i acts in the complex root system without fixed points) and $Tr(\rho(\Sigma)) = -1$.

It only remains to check what happens with the matrices in the centre of Γ, but they are fixed by the action of $Sp_2(\mathbb{F}_p) \cong N_W(\Gamma)/\Gamma$ by conjugation. □

Remark 3.1. The proof of Proposition 3.1 also shows that the morphism ϕ_N defines an $\mathcal{R}_p(SU(p))$-invariant quasi representation depending only on roots mod p because ξ is a p-th root of unity.

By Theorem 3.2, the next step is to look at higher limits for extension and unicity of maps. Recall the following result in [JMO2] which computes higher limits of functors over p-stubborn categories.

Proposition 3.2. *Let $F_1 : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}-Mod$ and $F_2 : \mathcal{R}_p(G) \to p-\mathcal{H}gr$ be functors from the orbit category of p-stubborns. If p^2 does not divide $|W_G|$, then $\varprojlim^2 F_2 = 0$ and $\varprojlim^n F_1 = 0$ for any $n \geq 2$.*

Corollary 3.1. $\mathrm{Map}(BSU(p), BU(N)\hat{}_p)_{[\phi_N]}$ *is nonempty and connected. That is, there exists $\phi : BSU(p) \to BU(N_i)\hat{}_p$ such that $\phi|_{BNT_\infty} \simeq B\phi_N$ and it is unique up to homotopy satisfying this condition.*

Proof. Recall that the obstruction to the existence lies in

$$\varprojlim{}^{i+1} \pi_i(\mathrm{Map}(BP_\infty, BU(N_i)\hat{}_p)_{\phi_N|_P}),$$

and to unicity in

$$\varprojlim{}^{i} \pi_i(\mathrm{Map}(BP_\infty, BU(N_i)\hat{}_p)_{\phi_N|_P}).$$

We are under the hypothesis of [JMO2] because p^2 does not divide $|\Sigma_p|$ and

$$\Pi_i(P) := \pi_i(\mathrm{Map}(BP_\infty, BU(N)\hat{}_p)_{\phi_N|_P}) \in \mathbb{Z}_{(p)} - mod \ \ for \ \ i \geq 2;$$

then $\varprojlim^j \Pi_i = 0$ for any $i \geq 2$ and $j \geq 2$. Moreover,

$$\Pi_1(P) = \pi_1(BC_{U(N_i)}(\phi_N(P)) = \pi_0(C_{U(N_i)}(\phi_N(P))) = 0$$

because this centralizer $C_{U(N_i)}(\phi_N(P))$ is connected.

By Theorems 3.2 and 3.1 we obtain an extension

$$\phi_{SU(p)} : BSU(p)\hat{}_p \longrightarrow BU(N_i)\hat{}_p$$

of ϕ_T.

Corollary 3.2. *$\phi_{SU(p)}$ is invariant under the action of $Z(W_i) \cong \mathbb{Z}/(p-1)\mathbb{Z}$ via Adams' operations.*

Proof. As a consequence of the uniqueness property we have that ϕ is homotopy equivariant with respect to the action of the Adams operations $\{\Psi^\xi\} \cong \mathbb{Z}/(p-1)\mathbb{Z}$ on $BSU(p)$. The reason is that $[\Psi^\xi \circ \phi_N] = [\phi_N] \in Rep(N_\infty T, U(N_i))$ and that the set Σ is invariant (that is the action of the Adams' operations on Σ corresponds to a permutation of the set of roots).

Remark 3.2. The method applied to construct a certain morphism from the p-compact group $SU(p)$ also applies for $U(p)$ because the p-stubborn categories of $SU(n)$ and $U(n)$ are isomorphic [Oli]. The bijection is given by the following:
$P \to \langle P, Z(U(n))\rangle$ for $P \subset SU(n)$;
$\tilde{P} \to \tilde{P} \cap SU(n)$ for $\tilde{P} \subset U(n)$.

We can obtain extra information about the homotopy groups of the mapping spaces $\pi_i(\mathrm{Map}(BSU(p), BG^\wedge_p)_\phi)$ using the Bousfield-Kan spectral sequence of the fourth quadrant

$$\varprojlim{}^i \Pi_{-j}(BP_\infty) \Longrightarrow \pi_{-i-j}(\mathrm{Map}(BSU(p), B\hat{G_p})_\phi)$$

and the connectedness of centralizers. The vanishing of higher limits $\varprojlim^j \Pi_i = 0$ for any $i \geq 2$, $j \geq 2$, and an analysis of the Bousfield-Kan spectral sequence:

	$\varprojlim^0$	$\varprojlim^1$	$\varprojlim^2$	$\cdots$
Π_0	$\star$	$\star$	$\star$	$\cdots$
Π_1	$\varprojlim^0 \Pi_1$	$\varprojlim^1 \Pi_1$	0	0
Π_2	$\varprojlim^0 \Pi_2$	$\varprojlim^1 \Pi_2$	0	0
$\vdots$	$\vdots$	$\vdots$	0	0
Π_i	$\varprojlim^0 \Pi_i$	$\varprojlim^1 \Pi_i$	0	0

shows that there exist short exact sequences for $i \geq 1$

$$\varprojlim{}^1 \Pi_{i+1}(BP_\infty) \rightarrowtail \pi_i(\mathrm{Map}(BSU(p), BG^\wedge_p)_\phi) \twoheadrightarrow \varprojlim{}^0 \Pi_i(BP_\infty).$$

Corollary 3.3. *$\pi_1(\mathrm{Map}(BSU(p), BG^\wedge_p)_\phi)$ is abelian if $C_G(\phi((N_pT)_\infty))$ and $C_G(\phi((\Gamma)_\infty))$ (and $C_G(\phi((T)_\infty))$ for $p \geq 3$) are connected.*

Proof. Using the exact sequence obtained from the Bousfield-Kan spectral sequence [BK] and the hypothesis of the corollary (i.e. $\Pi_1(BN_3T) = \Pi_1(B\Gamma) = 0$), we see that

$$\pi_1(\mathrm{Map}(BSU(p), B\hat{G_3})_\phi) \cong \varprojlim{}^1 \Pi_2(BP_\infty)$$

is abelian.

Remark 3.3. When G is a unitary group, it is well known that centralizers of subgroups are connected (in fact, a product of unitary groups). The hypothesis of Corollary 3.3 are satisfied.

4. Complex root systems

In this section we introduce the notion of complex root systems. We are concerned with the description of Weyl groups of p-compact groups as Weyl groups of complex root systems and we will use this information to define admissible morphisms into unitary groups. Our motivation is the following known result in Lie group theory (see [BrD]).

Theorem 4.1. *Let G be a compact connected Lie group with maximal torus T and let $V \subset LT^*$ be the subspace generated by the set R of real roots of G. Then R is a root system in V and the Weyl group W of G, viewed as a subgroup of $Aut(V)$, is the same as the Weyl group of this root system.*

In order to obtain a similar result using pseudoreflection groups, we consider complex root systems. In fact, the Weyl groups of p-compact groups we are considering are Weyl groups of complex root systems.

The notion of a complex root system was introduced by Cohen [Coh]. Let $V = \mathbb{C}^n$ with an hermitian, positive definite form.

Definition 4.1. *A complex reflection in $\mathbb{C}^n$ is a linear transformation of $\mathbb{C}^n$ of finite order with exactly $n-1$ eigenvalues equal to 1.*

A unitary root of a reflection is a unitary eigenvector corresponding to the unique nontrivial eigenvalue of the reflection.

Notice that the definition of a complex reflection group does not depend only on the group but also in a complex representation in some complex vector space $\mathbb{C}^n$.

Let r be a nonzero vector and ξ a root of the unity. With this data, we can define a reflection $S_{r,\xi} \in U(n)$ satisfying $S_{r,\xi}(r) = \xi r$ and $S_{r,\xi}(x) = 0$ if $\langle x, r\rangle = 0$. In fact, we have

$$S_{r,\xi}(x) = x + (\xi - 1)\frac{\langle x, r\rangle}{\langle r, r\rangle} r.$$

We will write $S_{r,m}$ when ξ is an m-th root of the unity and we say that r is a root of S.

A unitary root a_s associated to a reflection s defines a linear form which vanishes in the hyperplane of the reflection, $l_s = (\xi - 1)\langle x, a_s\rangle$ where ξ is a root of unity of order equal to the order of the reflection and $s(v) = v + l_s(v)a_s$.

The structure of finite complex reflection groups has been extensively studied (see e.g. [Spr]).

Let W be a finite complex reflection group. Define the set P by

$$P = \{\mathbb{C}^* a_s \mid s \in W \; reflection\}.$$

It is clear that there exists an action of W on P since the action of W permutes the hyperplanes associated to the reflections in W.

Proposition 4.1. *[Coh] The abelianization of W, W_{ab}, is isomorphic to the product of cyclic groups indexed by the W-orbits in P:*

$$W_{ab} \cong \prod_{O \in P/W} \mathbb{Z}/o(O)\mathbb{Z},$$

where $o(O)$ is the order of one of the reflections which defines the orbit O.

Next, we introduce the notion of a complex root system.

Definition 4.2. *[Coh] Let $\Sigma = (R, f)$ be a pair consisting of:*

1. *a finite set of nonzero elements in $\mathbb{C}^n$,*
2. *a map $f : R \to \mathbb{N} \setminus \{1\}$ such that for all $a, b \in R$*

$$S_{a,f(a)}(R) = R \text{ and } f(S_{a,f(a)}(b)) = f(b).$$

Σ is called a pre-root system and the group

$$W(\Sigma) = \langle S_{a,f(a)} | a \in R\rangle$$

is the Weyl group of this pre-root system.

A pre-root system is called a root system if, for all $a \in R$, there holds

$$ra \in R \iff ra \in W(\Sigma)a,$$

for any $r \in \mathbb{C}$.

If we consider $S_{r,f(r)}$ for each $r \in \Sigma$ as a unitary transformation of the complex vector space, they generate a finite group $W(\Sigma)$ called the Weyl group of the root system (notice that they have a natural representation into $U(n)$ by means of the matrices defining the reflections as linear transformations).

Example 4.1. If W is a finite complex reflection group acting on $V = \mathbb{C}^n$ as linear transformations, then we consider a unitary root a_s for each reflection $s \in W$. Let $R_0 = \{a_s \in V \mid s \; reflection\}$ and $f_0 : R_0 \to \mathbb{N} \setminus \{1\}$ defined by $f_0(a_s) = order(s)$. If we define $R = W \cdot R_0$ (i.e. the orbits of R_0 by the action of W) and a function $f : R \to \mathbb{N} \setminus \{1\}$ such that $f(ga_s) = f_0(a_s)$, then we obtain a pre-root system $\Sigma = (R, f)$.

The following lemma contains some of the properties of complex (pre)root systems we use later.

Lemma 4.1. *[Coh] Let $\Sigma = (R, f)$ be a pre-root system. Then,*

1. *$\{s^j_{a,f(a)} \mid a \in R,\ 0 < j < f(a)\}$ is the set of all reflections in $W(\Sigma)$.*
2. *There exists a complex root system $\Phi = (S, g)$ such that $W(\Sigma) = W(\Phi)$, $S \subset R$ and $g = f|_S$.*

It is interesting for the subsequent discussion to sketch the proof of the second statement in the lemma from [Coh]. Let $U = \{\mathbb{C}^* u \mid u \text{ root of } W(\Sigma)\}$ and $u_1, \ldots, u_l \in R$ such that $\{\mathbb{C}^* u_1, \ldots, \mathbb{C}^* u_l\}$ is a set of representatives of $W(\Sigma)$-orbits in U. Define $S = \cup_{i=1}^l W(\Sigma) u_i$ and $g = f|_S$. Then $\Phi = (S, g)$ is a root system and $W(\Phi) = W(\Sigma)$.

The following corollaries are direct consequences of Lemma 4.1.

Corollary 4.1. *If W is a finite complex reflection group, then W is the Weyl group of a complex root system Φ_W.*

Proof. If W is a finite complex reflection group, Example 4.1 shows how to define a pre-root system such that W is the corresponding Weyl group. The proof of Lemma 4.1 explains how to define a root system Φ_W from a pre-root system with the same Weyl group.

Corollary 4.2. *If W is a finite complex reflection group, there exists a homomorphism $\phi_W : W \to \Sigma_{|\Phi_W|}$ associated to the complex root system Φ_W.*

Proof. The homomorphism is defined by the permutation action of W on the roots of the system Φ_W which have been defined in the proof of Lemma 4.1.

Remark 4.1. [Coh] Let $\Sigma = (R, f)$ a complex root system and A a subring of $\mathbb{C}$ which contains $e^{\frac{2\pi i}{m}}$ for every $m \in f(\Sigma)$ and $(a|a)$, $\frac{(b|a)}{(a|a)}$ for any $a, b \in \Sigma$, then Σ and $W(\Sigma)$ are defined in the field of fractions of A.

Next we describe an example of a complex root system. Let us recall the definition of groups $G(q, r, n)$ of the family $2a$ in the list of Shephard and Todd [ST]. Let $q \geq 1$, and $\mu_q \subset \mathbb{C}$ the group of qth roots of unity (with a fixed isomorphism $\mu_q \cong \mathbb{Z}/q\mathbb{Z}$). For any $q \geq 1$ such that $r|q$, $n > 1$, define $A(q, r, n) = \{(z_1, \ldots, z_n) \in \mu_q^n \mid z_1 \cdots z_n \in \mu_r\}$. $G(q, r, n)$ is a split extension of $A(q, r, n)$ by Σ_n. $G(q, r, n)$ has a complex representation with $A(q, r, n)$ defined by diagonal matrices and Σ_n by permutation matrices:

$$G(q, r, n) \cong A(q, r, n) \rtimes \Sigma_n.$$

The groups in family $2b$ in the list correspond to $G(e, 1, 2)$. And the groups in family 3 in the list are the cyclic groups $\mathbb{Z}/n\mathbb{Z}$ generated by roots of unity.

Example 4.2. Let $G(m,r,n) = \Sigma_n \ltimes A(m,r,n)$. We describe this group as the Weyl group of a complex root system. Let $R(1,m,n)$ be the following set:

$$R(m,1,n) = \mu_2\mu_m\{e^{\frac{2\pi il}{m}}e_i - e_j \mid i,j,l \in \mathbb{N},\ i \neq j,\ 1 \leq i < j \leq n\}$$

and $f_{m,1,n}$ the function with constant value 2. Then $(R(m,1,n), f_{m,1,n})$ is a root system with Weyl group $W = G(m,1,n)$ and $|R(m,1,n)| = m^2n(n-1)/g.c.d.(2,m)$.

Now, define $R(m,r,n) = R(m,1,n) \cup \mu_m\{e_i | 1 \leq i \leq n\}$ and $f_{m,r,n}$ as the extension of $f_{m,1,n}$ determined by $f(e_i) = r$. Then $(R(m,r,n), f_{m,r,n})$ is a root system with Weyl group $G(m,r,n)$.

In this situation it is easy to prove that this root system can be defined in $\mathbb{Q}(\xi_m)$ where ξ_m is a primitive mth root of unity.

Corollary 4.3. *If W is a finite complex reflection group, W is the Weyl group of a root system defined in the character field of W, $\mathbb{Q}(\chi)$. Clearing denominators, if necessary, we can assume that roots lie in $\mathbb{Z}[\chi]$.*

Proof. It is known that a finite complex reflection group admits a representation in the character field $\mathbb{Q}(\chi)$ [CE]. If we consider the matrices of this representation in $\mathbb{Q}(\chi)$, we can build the root system associated considering the vectors in this subfield of $\mathbb{C}$.

Remark 4.2. Let a_s be a unitary root of W associated to a reflection s. Let $w \in W$ such that $w \cdot a_s = \xi a_r$. If we consider the root system defined in $\mathbb{Q}(\chi)$, then ξ has to be a unitary complex number in the subfield $\mathbb{Q}(\chi)$.

The classification of finite p-adic reflection groups is based on the classification of finite complex reflection groups by using the following theorem.

Theorem 4.2. *[CE] Let W be a finite group then W has a representation as a p-adic reflection group if and only if the following two conditions are satisfied:*

1. *W has a representation as a complex reflection group;*
2. *the character field $\mathbb{Q}(\chi) \subset \mathbb{Q}_p^\wedge$.*

For the primes satisfying the condition above we obtain that the complex root system can be represented by vectors in a $\mathbb{Q}_p^\wedge$-vector space and, cancelling denominators if necessary, in $\mathbb{Z}_p^\wedge$. So, roots lie in a $\hat{\mathbb{Z}}_p$-module.

The Weyl groups of complex root systems have been recently studied by means of generators and relations in [BMR]. These presentations "á la Coxeter" allow to compute W_{ab}. The following table contains these computations and the order of the center of W (recall that the center of W is a cyclic group by Schur lemma).

5. The Clark-Ewing spaces

The Clark-Ewing spaces are p-compact groups X such that the order of the Weyl group is prime to p. Let T be a fixed maximal torus of X and $W \subset GL(\pi_1(T) \otimes \hat{\mathbb{Z}}_p)$ the Weyl group endowed with a faithful p-adic representation. W acts on $(BT)^\wedge_p \simeq (BT_\infty)^\wedge_p \simeq K((\mathbb{Z}^\wedge_p)^r, 2)$ via the above representation. These p-compact groups are homotopy equivalent to the Borel construction:

$$X(W,p,T) := ((BT)^\wedge_p \times_W EW)^\wedge_p,$$

where EW is a contractible space with a free action of W. The cohomology of $BX(W,p,T)$ is $H^*(BX;R) \cong H^*(BT;R)^{W_X}$ where $R = \mathbb{F}_p, \hat{\mathbb{Z}}_p, \mathbb{Q}^\wedge_p$.

Maps between Clark-Ewing spaces have been described in [Woj2]. Wojtkowiak has also described maps between Clark-Ewing spaces and compact Lie groups.

Definition 5.1. *For maximal tori T and T' of X and X' respectively, a map $\phi : \pi_1 T \otimes \hat{\mathbb{Z}}_p \to \pi_1 T' \otimes \hat{\mathbb{Z}}_p$ is admissible if for all $w \in W_X$ there exists $w' \in W_{X'}$ such that $\phi \circ w = w' \circ \phi$.*

Two admissible maps ϕ and ψ are equivalent if there exists $w \in W_{X'}$ such that $w \circ \phi = \psi$.

$AHom_{\hat{\mathbb{Z}}_p}(T,T')$ is the set of classes of admissible maps from $\pi_1 T \otimes \hat{\mathbb{Z}}_p$ to $\pi_1 T' \otimes \hat{\mathbb{Z}}_p$.

Theorem 5.1. *Let G be a connected compact Lie group and suppose that p does not divide the order of W_X. Then the natural map*

$$\chi : [BX, BG^\wedge_p] \to AHom_{\hat{\mathbb{Z}}_p}(T_X, T_G)$$

is a bijection.

That means that we can describe complex homotopy representations of Clark-Ewing spaces. There is a bijection,

$$[BX, BU(n)^\wedge_p] \to AHom_{\hat{\mathbb{Z}}_p}(T_X, (S^1)^n).$$

Remark 5.1. If T and T' are tori of rank n, m respectively, we can identify

$$Hom(\pi_1 T \otimes \hat{\mathbb{Z}}_p, \pi_1 T' \otimes \hat{\mathbb{Z}}_p) = Hom(T_\infty, T'_\infty) = M_{m\times n}(\hat{\mathbb{Z}}_p).$$

The first consequence of this description is the following property.

Corollary 5.1. *A non irreducible complex representation of a Clark-Ewing space splits in a sum of irreducible representations.*

Proof. If a complex representation $\rho : BX \to BU(n)^\wedge_p$ is not irreducible, the associated admissible map splits in the orbits by the action of the Weyl group. Let n_i be the cardinal of these orbits for $i = 1, \dots, s$. By Theorem 5.1, each of these orbits describes an admissible morphism which extends to a homotopy complex representation $\rho_i : BX \to BU(n_i)^\wedge_p$ unique up to homotopy. This uniqueness property assures that $\rho \simeq \rho_1 \oplus \cdots \oplus \rho_s$.

The root system defined in Section **??** defines an admissible morphism $\phi : T_X \to T_{U(n)}$ with respect to the group morphism $\phi_W : W \to \Sigma_n$ which represents W as permutations of the set of roots (n is the number of elements in Φ_W).

Each root r of the system defines a morphism $r : BT_\infty \to BS^1$ (a p-discrete approximation of the maximal torus is $BT_\infty = B(\mathbb{Z}/p^\infty\mathbb{Z})^{rank\ X}$ is a p-discrete approximation of the maximal torus) because it lies in a $\hat{\mathbb{Z}}_p$-module.

We define a complex representation from the following admissible morphism,

$$\prod_{r \in R_W} Br : BT_\infty \to BT_{U(n)}$$

where $n = |R_W|$. The following corollary is a direct consequence of Theorem 5.1.

Corollary 5.2. *The admissible map induced by the action of W_X in the complex root system extends to a complex homotopy representation which is unique up to homotopy.*

This representation is denoted by Ad.

Corollary 5.3. *The representation Ad is faithful.*

Proof. Notice that the p-discrete approximation of the kernel of Ad (see [MN]) is a p-discrete approximation of the center of X (see [DW95]). That is, Ad is defined by the linear forms which vanish on the hyperplanes of the reflections of W_X; then,

$$PreKer(Ad) = \cap_s \sigma(s),$$

because $N_pT = T$ and it follows that

$$BKerAd \simeq BC_p^\wedge \simeq map(BX, BX)_{id}.$$

The Clark-Ewing spaces are free of centre (Lemma 5.1), then Ad is a monomorphism:

$$BKer(Ad) \simeq *.$$

Lemma 5.1. *If X is a Clark-Ewing space, then,*

$$BZ(X) \simeq_p *.$$

Proof. If X is a Clark-Ewing space, then

$$\pi_*(BZ(X)) = \pi_*(map(BX, BX)_{id}).$$

But, $map(BX, BX)_{id} \simeq map(BT, BX)^{hW_X}_{Bi} \simeq BT^{hW_X}$. From the Bousfield-Kan spectral sequence [BK] which computes the homotopy groups of a homotopy invers limit we obtain $\pi_*(BZ(X)) \cong \pi_*(BT_p^\wedge)^{W_X}$. Then $\pi_i(BZ(X)) = 0$ for $i \neq 2$ and

$$\pi_2(BZ(X)) \cong H_2(BT; \hat{\mathbb{Z}}_p)^{W_X} = 0.$$

Corollary 5.4. *Ad is irreducible if and only if W_{ab} is cyclic.*

Proof. Recall that W_{ab} is a product of cyclic groups indexed by the orbits of hyperplanes of reflections (Proposition 4.1). The order of each of these cyclic groups is that of the reflection which defines the orbit.

If W_{ab} is cyclic, there is only one orbit of hyperplanes and, by the definition of root system ϕ_W (see 4.1), ϕ_W is an irreducible W-set (recall that $\xi a_s \in R_W \Leftrightarrow \xi a_s \in W a_s$).

In particular, if W contain reflections of different orders, it is clear that Ad is reducible.

Corollary 5.5. *Ad is invariant by the action of unstable Adams operations in $BU(n)^\wedge_p$ of order e which divides the order of the center of W_X.*

Proof. The centre of W_X consists of diagonal matrices ξId where ξ is a root of unity, (recall that $Z(W_X) = \mathbb{Z}/c\mathbb{Z}$) and it acts on the maximal torus as unstable Adams' operations.

Let ψ be an unstable Adams' operation of order e such that $e|c$. From the definition of admissible morphisms it is clear that $\psi Ad_T \simeq Ad_T \psi$. But, the action of ψ on the source corresponds to the action of $\langle \psi \rangle \leq W$ and there exists $w \in \Sigma_n$ such that $\psi Ad_T \simeq Ad_T w$. From the uniqueness property of extensions of equivalent admissible maps, we prove that $\psi Ad \simeq Ad$.

Remark 5.2. We can describe a standard method to construct complex homotopy representations. Consider a vector $e_1 \in T_\infty$ and its dual, which is a vector in $r_1 \in Hom(T_\infty, \mathbb{Z}/p^\infty\mathbb{Z})$. Let R_1 be the orbit of r_1 by the action of W. This orbit defines an admissible morphism with respect to the morphism between Weyl groups defined by the permutation of the elements in the orbit, $\phi_W : W \to \Sigma_{|R_1|}$.

6. The Aguadé spaces

This section provides an exposition of the techniques used in the proof of the main theorem. We start with a brief exposition without proofs of the mod p homology decomposition of the p-compact groups X_i.

The p-compact groups X_i were described for the first time by means of homotopy colimits by Aguadé [Ag]. The ones corresponding to groups of number 12 and 31 were first described by Zabrodsky using other techniques [Zab]. First of all, recall the following property that is satisfied by all of the groups.

Lemma 6.1. *[Ag] All the groups in the list contain a subgroup Σ isomorphic to a symmetric group. For groups of number* $12, 29, 31, 34$*, this subgroup is isomorphic to Σ_p.*

These p-compact groups will be described as homotopy colimits over an index category $\mathbb{I}$ with just two objects 0, 1 and morphisms $Hom(0,0) = Z$, $Hom(1,1) = W$, $Hom(1,0) = W/\Sigma$ and $Hom(0,1) = \emptyset$ where Z is the center of W. Let $F'_i : \mathbb{I} \to HoTop$ be the functor defined by $F'_i(0) = BSU(p)$, $F'_i(1) = BT^{p-1}$, and Z acting via unstable Adams' operations. This functor lifts to the topological category of spaces, $F_i : \mathbb{I} \to Top$, and we obtain the following decomposition:

$$(\operatorname*{hocolim}_{\mathbb{I}} F_i)^\wedge_p = BX_i.$$

This category is slightly different from the one used in [Ag] but the same methods apply to compute higher limits of functors $F : \mathbb{I} \to R-Mod$. More generally, the pair (W, Σ) satisfies the following lemma.

Lemma 6.2. *[Ag] Let $F : \mathbb{I} \to R - Mod$ and assume that $|Z|$ is invertible in R. Then we have,*

1. $\varprojlim^j F = 0$ *for* $j > 1$,
2. *there is an exact sequence*

$$0 \to \varprojlim{}^0 F \to F(0)^Z \to F(1)^{N_W(Z\times\Sigma)}/F(1)^W \to \varprojlim{}^1 F \to 0.$$

Proposition 6.1. *The p-compact groups X_i for $i = 12, 29, 31, 34$ are free of centre.*

Proof. We use the description of the centre as a mapping space $\operatorname{Map}(BX_i, BX_i)_{id}$. The following homotopy equivalences describe this mapping space as a homotopy limit:

$$\operatorname{Map}(BX_i, BX_i)_{id} \simeq \operatorname{Map}(\operatorname*{hocolim}_{\mathbb{I}} F_i, BX)_{id} \simeq_p \operatorname*{holim}_{\mathbb{I}} \operatorname{Map}(F_i, BX)_{inc}$$

where inc is the inclusion of $F_i(0) \simeq BSU(p)$ and $F_i(1) \simeq BT$ in $BX_i \simeq_p \operatorname{hocolim}_{\mathbb{I}} F_i$.

In order to compute the homotopy groups of the centre we use the fourth quadrant Bousfield-Kan spectral sequence [BK] with E_2-term

$$E_2^{i,j} = \varprojlim{}^i \pi_{-j}(\operatorname{Map}(F_i, BX_i)_{inc}),$$

which converges to $\pi_{-i-j}(\operatorname{holim} \operatorname{Map}(F_i, BX_i))$.

As a consequence of Proposition 4.6 in [Not2] and the homotopy uniqueness of special unitary groups (see [Not2]), we obtain the following homotopy equivalences:

$$\operatorname{Map}(BSU(p), BX_i)_{inc} \simeq \operatorname{Map}(BSU(p), BSU(p)^\wedge_p)_{id} \simeq BZ(SU(p)^\wedge_p);$$

if BT is a maximal torus in $BSU(p)$, then,

$$\operatorname{Map}(BT, BX_i)_{inc} \simeq map(BT, BSU(p)^\wedge_p)_{inc} \simeq BT^\wedge_p.$$

The homotopy groups of these mapping spaces define a functor Π_j which take values in the category of $\mathbb{Z}_{(p)}$-modules. In fact, $\pi_j(BZ(SU(p))) \cong \pi_j(B\mathbb{Z}/p\mathbb{Z})$ and $\pi_j(BT) = \pi_j(K((\hat{\mathbb{Z}}_p)^{p-1}, 2))$.

So, $\varprojlim^j \Pi_i = 0$ for all j and $i \geq 3$. By Lemma 6.2 we obtain $\varprojlim^i \Pi_j = 0$ for $i > 2$ and $j = 1, 2$.

Finally, using the exact sequence in Lemma 6.2 it is easy to see that,

$$\varprojlim{}^0 \Pi_2 = 0$$

$$\varprojlim{}^1 \Pi_1 = 0.$$

As Z acts as translations, we have

$$\varprojlim{}^1 \Pi_2 = ((\hat{\mathbb{Z}}_p)^{p-1})^{N_W(Z\times\Sigma)}/((\hat{\mathbb{Z}}_p)^{p-1})^W = 0,$$

$$\varprojlim{}^0 \Pi_1 = (\mathbb{Z}/p\mathbb{Z})^W = 0.$$

Then $\pi_j(BZ(X_i)) = 0$ for all j and $i = 12, 29, 31, 34$.

The main result of this section is the following description of complex representations of the p-compact groups X_i.

Proposition 6.2. *There is a bijection*

$$[BX_i, BU(N)\hat{}_p] \cong \lim_{\mathbb{I}} [F_i, BU(N)\hat{}_p]$$

for $i = 12\ (p = 3),\ 29\ (p = 5),\ 32\ (p = 5),\ 34\ (p = 7)$.

Remark 6.1. The inverse limit over $\mathbb{I}$ of complex homotopy representations can be described in the following way:

$$\lim_{\mathbb{I}} [F_i, BU(N)\hat{}_p] \cong \{[\phi] \in [BSU(p), BU(N)\hat{}_p] \,|\, [\phi_T] \in [BT, BU(N)\hat{}_p]^{W_i}\}.$$

Definition 6.1. *An $\mathbb{I}$-invariant representation of X_i into a connected p-compact group X is a morphism of p-compact groups $\rho : SU(p) \to X$ which extends to*

$$\varprojlim{}^0_{\mathbb{I}} Rep(\Omega F_i, X).$$

The following proposition gives conditions to assure when an $\mathbb{I}$-invariant representation defines a representation of X_i.

Proposition 6.3. *Let X be a connected p-compact group and ϕ an $\mathbb{I}$-invariant representation into X. Then the following hold:*

1. *If $\pi_1(map(BSU(p), BX)_\phi)$ is abelian then ϕ extends to a map $\phi : BX_i \to BX$.*
2. *Any morphism of p-compact groups $f : BX_i \to BX$ such that $\pi_1(map(BSU(p), BX)_{f|_{SU}})$ is abelian is unique up to homotopy (that is any other morphism of p-compact groups $g : BX_i \to BX$ such that $g|_{SU(p)} \simeq f|_{SU(p)}$ and $g|_T \simeq f|_T$ satisfies $g \simeq f$).*

Proof. The main tools used in the proof are obstruction theory and higher limits computation.

1. The following diagram defined by the $\mathbb{I}$-invariant representation ϕ:

$$\begin{array}{ccc} BSU(p)\hat{}_p & \xrightarrow{\phi_{SU(p)}} & BX \\ \uparrow & \nearrow_{\phi_T} & \\ BT_\infty & & \end{array}$$

lifts to the topological category (and consequently defines a map from the homotopy colimit into BX) if the corresponding obstructions vanish. These obstructions to the lifting live in

$$\varprojlim{}^{i+1}{}_{\mathbb{I}}\pi_i(map(F_i(-), BX)_{\phi_i}),$$

for $i \geq 1$.
Notice that the centralizer of a p-discrete torus in a connected p-compact group is connected (see Dwyer-Wilkerson [DW95]), thus $\pi_1(map(BT_\infty, BX)_{\phi_T}) = 0$. Moreover, we are assuming that $\pi_1(map(BSU(p), BX)_\phi)$ is abelian, thus in this situation we obtain a functor $\Pi_i : \mathbb{I} \to \hat{\mathbb{Z}_p}$-Mod for $i \geq 1$ which takes the following values

$$\Pi_i(SU) = \pi_i(map(BSU(p), BX)_\Phi)$$

and

$$\Pi_i(T_\infty) = \pi_i(map(BT, BX)_{\phi_T}).$$

We can apply Lemma 6.2 to compute these higher limits:

$$\varprojlim{}^{i+1}{}_{\mathbb{I}}\pi_i(map(F_i(-), BX)_{\phi_i}) = 0$$

for $i \geq 1$.

2. Let $f : BX_i \to BX$. If we restrict f to the diagram, we obtain an $\mathbb{I}$-representation of X_i into X. Recall that the obstruction to uniqueness of extensions lies in

$$\varprojlim{}^{i}{}_{\mathbb{I}}\pi_i(map(F_i(-), BX)_{\phi_i},$$

for $i \geq 1$. Using the same arguments as above, we can infer that these limits vanish for $i \geq 2$ (Lemma 6.2).
Recall that $\pi_1(map(BT_\infty, BX)_{\phi_T}) = 0$. Thus, if $\pi_1(map(BSU(p), BX)_{\phi_{SU}})$ is abelian, we can apply Lemma 6.2 and we can deal with the following exact sequence:

$$\varprojlim{}^0\Pi_1 \rightarrowtail \Pi_1(SU(p))^{\mathbb{Z}/(p-1)\mathbb{Z}} \longrightarrow \Pi_1(T_\infty)^{N_{G_i}(\Sigma_p)}/\Pi_1(T_\infty)^{G_i} \twoheadrightarrow \varprojlim{}^1\Pi_1.$$

Clearly $\Pi_1(T_\infty) = 0$ implies the vanishing of lim 1:

$$\varprojlim{}^1{}_{\mathbb{I}}\pi_1(map(F_i(-), BX)_{\phi_i}) = 0.$$

We are mainly interested in applying the above considerations to the situation in which the target p-compact group X is the p-completion of a unitary group.

Proposition 6.4. *Let* $\phi : BSU(p) \to BU(N)^{\wedge}_p$. *Then* $\pi_1(\mathrm{Map}(BSU(p), BU(N))_\phi)$ *is abelian.*

The proof of the above proposition is given at the end of the next section where we describe some complex homotopy representations of the p-compact group $SU(p)$. The above discussion proves the following proposition.

Corollary 6.1. *Any* $\mathbb{I}$*-representation into* $U(N)$ *extends in an unique way to a complex representation on* X_i.

Proof (Proof of Proposition 6.2).
We use the description of the inverse limit in the Remark 6.1.

On the one hand, it is clear that any $f \in [BX_i, BU(N)\hat{}_p]$ gives rise, by restriction in the diagram, to maps f_{SU} and f_T satisfying the conditions described on the right hand side of the equality.

On the other hand, by Proposition 6.1, we are reduced to proving that for any map $f : BSU(p) \to BX$ satisfying conditions on the right hand side we can define an $\mathbb{I}$-representation into $U(N)$. But this is clear since f_T is equivariant with respect to the action of W_i and since the action of unstable Adams' operations on $BSU(p)$ corresponds to the action of the center $Z(W_i) \cong \mathbb{Z}/(p-1)\mathbb{Z}$.

Lemma 6.3. ϕ_Γ *is not equivariant with respect to the action of* $Sp_2(\mathbb{F}_p)$.

Proof. Notice that A and B are conjugate in $SU(p)$ but $Tr(\phi_\Gamma(A)) \neq Tr(\phi_\Gamma(B))$.

i	$Tr(\phi_\Gamma(A))$	$Tr(\phi_\Gamma(B))$
12	-3	0
19	-10	0
31	-10	0
34	-21	0

Thus, by representation theory arguments on characters, ϕ_Γ cannot be equivariant. Recall that B is a generator of the 3-Sylow subgroup of W_i, thus $Tr(\phi_\Gamma(B)) = 0$ because $\phi_\Gamma(B)$ acts on the set of roots as a permutation matrix without fixed points (see Lemma A.2). Using the computer system MAGMA, we have computed the following traces, $Tr(\phi_\Gamma(A)) = (-p|Z(W_i)|)/2$.

In order to solve this problem we modify the morphisms to obtain an $\mathcal{R}_p(G)$-invariant representation.

Recall the complex representations of W_i, $\rho_i : W_i \to U(p-1)$, described in Section 2. One can check that $Tr(\rho_i(B)) = -1$ (and $det(\rho_i(B)) = 1$).

Then, we can add this representation to our morphism in the following way:

$$\phi'_T := \phi_T \oplus Id_{(p-1)\frac{|Z|}{2}},$$

$$\phi'_W := \phi_W \oplus \frac{|Z|}{2}\rho_i.$$

Both maps describe an admissible map $\phi'_T : BT \to BU(N_i)$ with respect to ϕ'_W on $N_i = m(i) + (p-1)|Z|/2$. Our particular situation can be generalized.

We have constructed the maps ϕ_T defined by means of complex roots (see section **??**) and the extension ϕ_{SU}. They provide an $\mathbb{I}$-representation into $U(N_i)$ where $N_i = \frac{Z(W_i)}{2}((p-1)+2N)$ and N is the number of complex reflections in W_i.

$$\begin{array}{ccc} BSU(p)\hat{}_p & \xrightarrow{\phi_{SU(p)}} & BU(N_i)\hat{}_p \\ \uparrow & \nearrow_{\phi_T} & \\ BT_\infty & & \end{array}$$

By Proposition 6.1, these $\mathbb{I}$-representations extend to complex representations

$$Ad_i : BX_i \to BU(N_i)$$

where

i	12	29	31	34
N_i	26	168	248	774

Corollary 6.2. *The morphisms of p-compact groups Ad_i are monomorphisms (that is Ad_i is a faithful representation) and irreducible.*

Proof. Notice that a p-discrete approximation of the kernel of Ad_i (see [MN]) is a p-discrete approximation to the centre of X_i (see [DW95]). That is, the admissible morphism is defined by linear forms which vanish in the reflecting hyperplanes. The p-Sylow subgroup acts without fixed points on the complex root system, then we have

$$PreKer(Ad) = \cap_s \sigma(s),$$

$$BKerAd \simeq BC_p^\wedge \simeq map(BX, BX)_{id}.$$

These p-compact groups are free of centre (Lemma 6.1), hence Ad_i is a monomorphism (recall that $BKer(Ad) \simeq *$).

The representations Ad_i are irreducible by definition.

Corollary 6.3. *$Ad_i : BX_i \to BU(N_i)_p^\wedge$ is invariant with respect to the action of unstable Adams' operations of order q such that $q||Z| = p-1$ acting on $BU(N_i)_p^\wedge$.*

Proof. Let ψ be an unstable Adams' operation of order q such that $q||Z|$ acting on $BU(N_i)^\wedge_p$. From the definition of the admissible morphism it is clear that $\psi Ad_T \simeq Ad_T\psi$. The action of ψ corresponds to the action of $\langle\psi\rangle \leq W$ and $\psi Ad_T \simeq Ad_T w$ for some $w \in \Sigma_{N_i}$.

From the uniqueness property of extension of admissible morphisms out of $BSU(p)$, we obtain that ψAd_T extends to an $\mathbb{I}$-invariant representation homotopy equivalent to the $\mathbb{I}$-invariant representation Ad. Using the same argument of the uniqueness property of extensions we obtain that $\psi Ad \simeq Ad$.

7. Complex representations of infinite generalized grassmannians of rank p

The existence of faithful complex homotopy representations for generalized grassmannians is solved in [C2]. In this section we describe the existence of such a monomorphism for generalized grassmannians of rank p given by the combinatoric of the complex root system.

In Section **??** there is a description of the Weyl groups of this family of p-compact groups as the Weyl groups of a complex root systems as well as the action on the set of complex roots. Recall the description of the complex root system in Example 4.2.

Definition 7.1. *The set of roots $R(q,r,n)$ decomposes in the following two orbits by the action of $G(q,r,n)$:*

$$R_1 = \mu_2\mu_q\{e^{\frac{2\pi i l}{q}} e_i - e_j \mid i,j,l \in \mathbb{N},\ i \neq j,\ 1 \leq i < j \leq n\},$$

$$R_2 = \mu_q\{e_i | 1 \leq i \leq n\}.$$

Each one of these orbits gives rise to an admissible map

$$\phi^i_T = \prod_{r\in R_i} : BT_\infty \to BT^{|R_i|}$$

with respect to the action of $G(q,r,n)$ as permutations of the roots.

In order to extend these admissible morphisms we have to deal with extending it to unitary groups of type $U(p^k)$ and products of them. For this reason we concentrate in infinite generalized grassmannians of rank p.

These p-compact groups have a decomposition over an index category with two objects, $BU(p)$ and BT. More precisely, let $\mathbb{I}$ be the category with two objects $\{0,1\}$ and morphisms $\mathrm{Hom}(0,0) = N_{G(q,r,n)}(\Sigma_p)/\Sigma_p \cong Z(G(q,r,n))$, $\mathrm{Hom}(1,1) = G(q,r,n)$, $\mathrm{Hom}(0,1) = G(q,r,n)/\Sigma_p$, $\mathrm{Hom}(1,0) = \emptyset$. These p-compact groups are described as homotopy colimits of a functor $F : \mathbb{I}^{op} \to HoTop$ such that $F(1) = BT^\wedge_p$ and $F(0) = BU(p)^\wedge_p$ with natural definitions for the morphisms ($Z = Z(G(q,r,n))$ acting via Adams' operations in $U(p)$):

$$BX(q,r,p) \simeq (\operatorname{hocolim}_{\mathbb{I}} F)^{\wedge}_p.$$

With this decomposition and obstruction theory we prove the existence of the desired representation.

Proposition 7.1. *$BX(q,r,p)$ has complex homotopy representations extending the admissible morphisms ϕ_T^i for $i = 1,2$.*

The complexification map [C2] is an extension of ϕ_T^2. In fact, it exists for any infinite generalized grassmannnian. It remains to prove the existence of the extension for the admissible map ϕ_T^1.

First of all, we have to deal with the extension to $BU(p)^{\wedge}_p$. By the observation in Remark 3.2, it is clear that we can apply the same technique used for $SU(p)$, hence all the information we need is in $Tr(\phi(A))$ and the fixed points for the action of B in the root system.

Lemma 7.1. *1. The p-Sylow subgroup S_p of Σ_p acts without fixed points in the orbit R_1.*
2. $Tr(\phi(A)) = -p\frac{q}{g.c.d.(2,q)}$.

Proof. 1. It is easy to check that the action of S_p does not fix any $\xi^a e_i - \xi^b e_j$ for $1 \le i,j \le p, 0 \le a,b \le q-1$.
2. Recall that the matrix of A is

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \xi & \cdots & 0 \\ \vdots & 0 & \ldots & 0 \\ 0 & 0 & \cdots & \xi^p \end{pmatrix}$$

and R_1 is

$$R_1 = \{\xi^a e_i - \xi^b e_j \mid 1 \le i,j \le p, 0 \le a,b \le q-1\}.$$

The computations for $Tr(\phi(A))$ reduce to the following sum:

$$\sum_{l=0}^{q-1} \sum_{i,j=0, i\neq j}^{p-1} \xi_p^{\alpha^l i - j}$$

where $\xi_q = \alpha \bmod p$. Consider the morphism $\phi_{\alpha,l} : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ defined by $\phi_{\alpha,l}(i,j) = \alpha^l i - j$. We see that $\phi_{\alpha,l}$ is an epimorphism if $\alpha^l \in \mathbb{F}_p^*$. If $l = 0$ then the kernel of $\phi_{\alpha,l}$ is the diagonal in $\mathbb{F}_p \times \mathbb{F}_p$ and it is not onto since the image of elements is not in the diagonal. If $l \neq 0$ then it is an epimorphism out of the diagonal.
This sum is $-p$ if $l = 0$, and 0 if $l \neq 0$.
Finally, the required computation of the trace reduces to that of how many factors of each type appears in the trace. From the definition of the complex root system we see that there are as many factors as elements in $\mu_2\mu_q$, that is, $\frac{q}{g.c.d.(2,q)}$.

In this situation we can use a modification of the arguments used in Proposition 3.1 considering the complex representation of the group $G(q,r,p)$ restricted to Σ, $\rho : \Sigma_p \to U(p)$ in order to define an $\mathcal{R}_p(U(p))$-invariant quasi representation:

$$\phi'_T = \phi^1_T \oplus Id_{p\frac{q}{g.c.d.(2,q)}},$$

$$\phi'_\Gamma = \phi^1_\Gamma \oplus \frac{q}{g.c.d.(2,q)}\rho.$$

Corollary 7.1. *The morphisms ϕ'_T and ϕ'_Γ define an $\mathcal{R}_p(U(p))$-invariant quasi representation into $U(N)$ where $N = \frac{q^2 p(p-1)}{mcd(2,q)} + \frac{pq}{mcd(2,q)}$.*

Proposition 7.2. *The set* $\mathrm{Map}(BU(p), BU(N)^\wedge_p)_{[\phi_N]}$ *is non empty and connected, that is, there exists $\phi_U : BU(p) \to BU(N)^\wedge_p$ such that $\phi|_{BNT_\infty} \simeq B\phi_N$ and it is unique up to homotopy satisfying this condition.*

Corollary 7.2. *ϕ_U is invariant with respect to the action of $Z(G(q,r,n))$ on $BU(p)^\wedge_p$ via unstable Adams' operations.*

Hence we have defined an $\mathcal{O}_{\mathcal{H}}$-representation into $BU(N)^\wedge_p$.

Proof (Proof of Proposition 7.1). The obstructions to the existence of the morphism of p-compact groups extending the $\mathcal{O}_{\mathcal{H}}$-representation into $BU(N)^\wedge_p$ lie in

$$\varprojlim{}^{i+1}\pi_i(\mathrm{Map}(BG, BU(N))_\phi).$$

Using the same arguments as in the last section, we see that these higher limits vanish for $i \geq 2$ (Lemma 6.2) and $\pi_1(\mathrm{Map}(BU(p), BU(N))_{\phi_U})$ is abelian (Proposition 6.4). Moreover, notice that $\pi_1(\mathrm{Map}(BT, BU(N))_{\phi_U}) = 0$, because the centralizer of a torus in a unitary group is connected. Then these higher limits vanish for $i = 1$. Notice that we also have the vanishing of the following higher limits:

$$\varprojlim{}^{i}\pi_i(\mathrm{Map}(BG, BU(N))_\phi)$$

and then the extension is unique up to homotopy.

Let Ad^1 be the complex representation just constructed which extends the admissible map ϕ^1_T uniquely up to homotopy. The proof of the following corollary is the same as the one for Corollary 6.3.

Corollary 7.3. *The complex homotopy representation Ad^1 is invariant by the action of unstable Adams' operations on $BU(N)^\wedge_p$ of order e such that $e||Z|$.*

Corollary 7.4. *Ad^1 is faithful and irreducible.*

Proof. Notice that the p-Sylow subgroup of W acts without fixed points in the root system and that the representations ρ are faithful; then Ad^1 is a monomorphism if and only if Ad^1_T is a monomorphism.

$(Ad^1_T)^*$ is an epimorphism in cohomology mod p (the set R_1 contains a basis of $\mathbb{F}_p^n$). Hence, Ad^1_T is a monomorphism.

Ad^1 is irreducible by definition.

These complex homotopy representations defined by using the complex root system are closely related to the centre of the p-compact group. Let $Ad := Ad^1 \oplus Ad^2$.

Proposition 7.3. *Ad is a monomorphism if $q > 1$.*

Proof. $BKer(Ad) \simeq (BPreKer(Ad))^\wedge_p \simeq BC^\wedge_p \simeq *$. By the definition and the proof of the above corollary we know that

$$BPreKer(Ad) \simeq \cap_s \sigma(s) \simeq Z(X),$$

and then $BKer(Ad) \simeq BZ(X)$. The proposition follows from the fact that these p-compact groups are free of cente for $q > 1$.

8. Thom modules and mod p spherical fibrations

In this section we summarize without proofs the relevant material on the notion of the adjoint module introduced in [MS]. Previously, we have compiled some basic facts on Thom modules.

The notion of a Thom module appears for the first time in Handel [Han]. Let R be an unstable algebra over the Steenrod algebra $\mathcal{A}$.

Definition 8.1. *A Thom module M over R is a free $R \circ \mathcal{A}$-module of rank 1 as an R-module where $R \circ \mathcal{A}$ denotes the semi-tensor product. A Thom module is unstable if it is unstable as an $\mathcal{A}$-module.*

The natural example of a Thom module is the cohomology of the Thom space associated to an oriented vector bundle. Another example comes from the theory of invariants. Let $\chi : G \to \mathbb{F}_p^*$ be a character of a finite group G. Then the module of relative invariants $P(V)^G_\chi$ is a Thom module over $P(V)^G$. But the most relevant example for the subsequent discussion in the paper is provided by the theory of mod p spherical fibrations. A mod p spherical fibration is an orientable Hurewicz fibration $\pi : E \to B$ whose fibre has the homotopy type of a p-complete sphere. The associated Thom space is the homotopy cofiber of π and its reduced mod p cohomology is also an example of an $H^*(B; \mathbb{F}_p)$-Thom module.

Definition 8.2. *Let M be a Thom module over R. A Thom class is a homogeneous generator of M as an R-module. If U is the Thom class of M, we define the q-class $q = 1 + q_1 + q_2 + \cdots \in R$ by the formula $\mathcal{P}(U) = qU$ where $\mathcal{P} = \mathcal{P}^0 + \mathcal{P}^1 + \cdots$, $q_i \in R^{2i(p-1)}$.*

The q-class of M determines its structure as a module over the Steenrod algebra and then its isomorphism class.

The classification theorem for Thom modules involves the q-classes defined above. In order to state it, we have to introduce the following modules over the Steenrod algebra:

$$\mathcal{S}(n) = \mathbb{F}_p[q_1, \ldots, q_n] = \mathbb{F}_p[t_1, \ldots, t_n]^{G(p-1,p-1,n)},$$

where $q_i = \sigma_i(t_1^{p-1}, \ldots, t_n^{p-1})$, $\mid q_i \mid = 2i(p-1)$ and $G(p-1,p-1,n) = \Sigma_n \ltimes \mu_{p-1}^n$, ($\mu_q$ is the group of qth roots of unity). Let $\mathcal{S}$ be the module defined by

$$\mathcal{S} = \lim_{n\to\infty} \mathcal{S}(n) = \mathbb{F}_p[q_1, q_2, \ldots].$$

The q-class associated to a Thom module M over R determines an algebra homomorphism over the Steenrod algebra

$$q_M : \mathcal{S} \longrightarrow R.$$

Consider the Thom modules $T(n)$ over $\mathcal{S}(n)$ with Thom class $E = t_1 \cdots t_n$, $T(n) = E \cdot \mathcal{S}(n) \subset \mathbb{F}_p[t_1, \ldots, t_n]$. The q-class of $T(n)$ is $1 + q_1 + \cdots + q_n \in \mathcal{S}(n)$.

Recently, Notbohm [Not2] has described p-compact groups, namely infinite generalized grassmannians, realizing $\mathcal{S}(n)$ as cohomology algebras (the realizability of these algebras as the cohomology of spaces was known since [Qui] by Quillen). Moreover, there exist mod p spherical fibrations over the corresponding generalized grassmannians whose Thom space realizes $T(n)$ as the cohomology of a Thom space [C2] (the non-modular case was previously obtained by Broto in [Bro]). That means that there exist universal mod p spherical fibrations over generalized grassmannians whose characteristic classes are $1 + q_1 + \cdots + q_n$.

If M and N are Thom modules over R, then $M \otimes_R N$ is also a Thom module over R. Moreover, if M is a Thom module over R and $\phi : R \to S$ is a morphism of algebras over the Steenrod algebra, then $S \otimes_R M$ is a Thom module over S.

The main classification result can be stated in the following form. Let $Th(R)$ be the group of classes of isomorphic Thom modules over R.

Theorem 8.1. *[Bro] The natural transformation*

$$q : Th(R) \longrightarrow Hom_{\mathcal{K}}(\mathcal{S}, R)$$

is an equivalence of functors.

Mitchell and Stong [MS] describes a purely algebraic method of reconstructing the Thom module associated to the adjoint representation of a connected compact Lie group.

Definition 8.3. *Let R be an unstable algebra over the Steenrod algebra. The adjoint Thom module over R is $Ad_R = H_n R$ where $H_* R$ is the Hochschild homology $H_*(R) = Tor^{R\otimes R}(R, R)$.*

When R is polynomial, we can explicitly describe the Hochschild homology of R and therefore the adjoint Thom module.

Lemma 8.1. *[MS]* $H_*R \cong \Omega^*_R$ *as an* $R \circ \mathcal{A}$*-algebra where* Ω^*_R *is the* R*-module of differential forms.*

Notice that Ω^*_R is an exterior algebra on Ω^1_R and Ω^1_R is a free $R \circ \mathcal{A}$-module on generators $dz_1, \ldots, dz_n$ ($z_1, \ldots, z_n$ are the generators of the polynomial ring R).

Corollary 8.1. *[MS] If* R *is a polynomial ring with generators* $z_1, \ldots, z_n$*, then* $H_nR \cong \Omega^n_R$ *is a Thom module over* R *with Thom class* $dz_1 \cdots dz_n$*.*

The following theorem describes the main properties of this Thom module related to the Thom module of the adjoint representation of a compact connected Lie group.

Theorem 8.2. *[MS] Let* R *be a polynomial algebra of rank* n *and dimension* m *over* $\mathcal{A}$*. Then* Ad_R *satisfies the following properties:*

1. *If* $R = H^*(BG)$ *where* G *is a compact connected Lie group or* $O(n)$*, then* $Ad_R = H^*(T(ad_G \downarrow BG))$*.*
2. $dim Ad_R \leq m$ *(the dimension of a Thom module* M *is* i *if* q_i *is the biggest class such that* $q_i(M) \neq 0$*).*
3. *If* T *is abelian and* $i : R \to T$ *is a non-singular injection with Jacobian* J*, then* $Ad_R \cong R \cdot J \subset T$*.*

The first assertion in the theorem justifies the name for this particular Thom module and the last one allows us to make explicit computations in the case we are working with polynomial algebras.

Let $R = \mathbb{F}_p[V]^G = \mathbb{F}_p[z_1, \ldots, z_n]$ be a polynomial algebra of invariants where G is a finite group generated by pseudoreflections. Recall that an injection $R \to T$ is non-singular if and only if the induced morphism $H_nR \to H_nT$ is nonzero. In this situation, the injection defines an isomorphism $H_nR \cong R \cdot (J dy_1 \cdots dy_n)$ where $y_1, \ldots, y_n$ are the generators of the polynomial algebra T and J is the Jacobian of the injection, $J = det(\frac{\partial z_i}{\partial y_j})$.

Corollary 8.2. $Ad_R \cong R \cdot J \subset \mathbb{F}_p[y_1, \ldots, y_n]$ *is a Thom module with Thom class* J*.*

The Jacobian is closely related to the structure of G as a reflection group,

Lemma 8.2. *[Ben] Let* G *be a finite group generated by pseudoreflections and let* k *a field such that* $(order(g) - 1, ch(k)) = 1$ *for any pseudoreflection* $g \in G$*. Then* J *is a scalar multiple of the product of all* $\alpha_g^{r_g - 1}$ *where* $\alpha_g \in k[V]$ *is of degree* 2 *such that it vanishes in the hyperplane* V^g *and* r *is the order of* g*:*

$$J = c \prod_{g \in G, pseudoreflection} \alpha_g^{r_g - 1}.$$

Thus, in the case in which the above lemma is satisfied, a reflection group is completely determined by its ring of invariants. The classification theorem asserts that the q-class of this Thom module over R determines it up to isomorphism (see Theorem 8.1). The q-class of the Thom module Ad_R can be computed as follows

$$\mathcal{P}(J) = \mathcal{P}(c \prod_{g\in G, pseudo} \alpha_g^{r_g-1}) = c \prod_{g\in G, pseudo} (\alpha_g + \alpha_g^p)^{r_g-1} = cJ \prod_{g\in G, pseudo} (1+\alpha_g^{p-1})^{r_g-1},$$

that is, elementary symmetric polynomials in the set $\cup_{g\in G}\{\alpha_g^{p-1}, \overset{r_g-1}{\ldots}, \alpha_g^{p-1}\}$.

An easy consequence of the Lemma 8.2 is the following formula for the dimension of a p-compact group which generalizes the known formula for compact Lie groups.

Recall that the dimension of a compact Lie group satisfies the following equality:

$$dim\, G = rank\,(G) + 2m,$$

where $2m$ is the number of roots of G (or, m is the number of reflections in the Weyl group). This formula follows directly from the decomposition of $L(G)$ as a T-module:

$$LG \cong LT \oplus (\oplus_{r\in R^+} \mathbb{R}^2),$$

where R^+ is the set of positive real roots of G.

In the situation of p-compact groups, we have also a concept of dimension. Let $(X, BX, *)$ be a p-compact group and W its Weyl group which is generated by pseudo reflections. We know that its rational p-adic cohomology is polynomial:

$$H^*(BX; \hat{\mathbb{Z}}_p) \otimes \mathbb{Q} \cong \mathbb{Q}_p^\wedge[f_1, \ldots, f_n],$$

where n is the rank of X. We define the dimension of a p-compact group as the cohomological dimension of the loop space X:

$$dim_p\, X = \sum_{i=1}^n deg(f_i) - 1.$$

Generalizing the situation for compact Lie groups, we can state a relation between this definition and the number and order of the pseudo reflections in the Weyl group.

Corollary 8.3.

$$dim\, X = rank\, X + 2(\sum_{g\in W} (o(g) - 1)).$$

where the sum is over $g \in G$ pseudo reflection and $o(g)$ is its order.

Proof. Notice that $dim\, X = |J| + rank\, X$ and use the decomposition of the Jacobian in Lemma 8.2.

In this section we will restrict our attention to the factorizations of the adjoint complex representations through generalized grassmannians. The crucial fact is the following description of certain generalized grassmanians as homotopy fixed points on unitary groups (see [C1]).

Proposition 8.1. *[C1]*

$$BU(nq)^{h\mathbb{Z}/q\mathbb{Z}} \simeq_p BX(q,q,n).$$

Corollary 8.4. *[C1] If X is a p-complete space, then*

$$[X, BU(N)]^{h\mathbb{Z}/q\mathbb{Z}} \cong [X, BX(q,q,\left[\tfrac{n}{q}\right])].$$

Proof. It is an easy consequence of Proposition 8.1 and the Bousfield-Kan spectral sequence [BK] for the homotopy groups of homotopy inverse limits.

Definition 8.4. *Let X be a p-compact group. The adjoint homotopy representation of X is a homotopy representation of X into an infinite generalized grassmannian of type $X(p-1,p-1,n)$ such that the Thom space T associated to the mod p spherical fibration induced over BX satisfies $H^*(T;\mathbb{F}_p) \simeq Ad_{H^*(BX;\mathbb{F}_p)}$.*

The non-modular case has been studied by C. Broto in [Bro]. The main result of this section is the following one.

Theorem 8.3. *[Bro] Let M be a Thom module over $P(V)^G$ where G is a non-modular subgroup of $GL_n(\mathbb{F}_p)$. There exist a space X such that $H^*(X;\mathbb{F}_p) = P(V)^G$ and a mod p spherical fibration η such that*

$$\tilde{H}^*(T(\eta);\mathbb{F}_p) \cong M$$

as a Thom module.

Corollary 8.5. *Let X be a Clark-Ewing space. There exists a mod p spherical fibration η over X such that*

$$\tilde{H}^*(T(\eta);\mathbb{F}_p) \cong Ad_{H^*(BX;\mathbb{F}_p)}.$$

In Section **??** we have constructed complex homotopy representations of Aguadé spaces invariants for the action of unstable Adams' operations on $BU(n)^\wedge_p$ of order e such that $e \mid |Z| = p-1$. Let $dim_{\mathbb{C}} X_i$ be the rank of the complex homotopy representation defined in Section **??**, where

i	12	29	31	34
$dim_{\mathbb{C}} X_i$	26	168	248	774

The following is a direct consequence of Corollary 6.3.

Corollary 8.6. *If $Z = Z(W_i) \cong \mathbb{Z}/n_i\mathbb{Z}$ then for each $e|n_i$ such that $e|p-1$, the representation Ad factors through an infinite generalized grassmannian,*

$$BX_i \to BX(e, e, \tfrac{N_i}{e}).$$

Corollary 8.7. *There exists a mod p spherical fibration η_{Ad_i} over BX_i such that the cohomology of the corresponding Thom space is isomorphic to $Ad_{H^*(BX_i;\mathbb{F}_p)}$ for each $i = 12, 29, 31, 34$.*

Proof. For $i = 12, 29, 31, 34$, recall that $Z(W_i) \cong \mathbb{Z}/(p-1)\mathbb{Z}$, then, using Corollary 8.6 we see that there exist representations

$$GAd_i : BX_i \to BX(p-1, p-1, \tfrac{N_i}{(p-1)}).$$

The pullback along GAd_i of the mod p spherical fibration over $BX(p-1, p-1, n)$ (see [C2]) induces a mod p spherical fibration over BX_i such that the q-classes $1 + q_1 + \cdots + q_{\frac{N_i}{p-1}}$ satisfies

$$\mathcal{P}(J) = J(1 + q_1 + \cdots + q_{\frac{N_i}{p-1}}).$$

It is easy to see by construction that the Thom class decomposes as a product of linear forms vanishing in the reflecting hyperplanes.

The following relation is satisfied:

$$dim_{\mathbb{C}} X = \frac{|Z|}{mcd(|Z|, 2)} dim_p X.$$

Next we will deal with infinite generalized grassmannians of rank p. In this situation the Jacobian decomposes as a product of orbits of vanishing forms on the reflecting hyperplanes.

First of all, notice that the homotopy representations defined from the root system in Section **??** factor through infinite generalized grassmannians of type $X(p-1, p-1, N)$.

Corollary 8.8. *If $Z(W_i) \cong \mathbb{Z}/n_i\mathbb{Z}$, then, for each $e|n_i$ such that $e|p-1$, the complex homotopy representation*

$$Ad^1 : BX(q, r, p) \to BU(N)$$

factors in the following way:

$$Ad^1_e : BX(q, r, p) \to BX(e, e, \tfrac{N}{e}).$$

Corollary 8.9. *The adjoint representation of $X(q, r, p)$ is the composition $Ad_p = \Phi(Ad^1_p \times (r-1)i)$ where Ad^1_p is the composition of $Ad^1_{|Z|}$ with the inclusion $BX(|Z|, |Z|, N/|Z|) \to BX(p-1, p-1, N/|Z|)$,*

$$BX(q, r, p) \overset{Ad^1_p \times (r-1)i}{\longrightarrow} BX(p-1, p-1, N_1) \times BX(p-1, p-1, p)^{r-1},$$

$$BX(p-1, p-1, N_1) \times BX(p-1, p-1, p)^{r-1} \overset{\Phi}{\to} BX(p-1, p-1, N_1 + (r-1)p).$$

It is an irreducible representation if $r \neq 1$.

Proof. We can reproduce the same arguments in the situation for Aguadé spaces to prove that the Thom class of the mod p spherical fibration induced by the pullback along the representations is the Jacobian. Notice that $|Z| \mid p-1)$

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A. APPENDIX: On complex reflection groups 12, 29, 31, 37.

These p-compact groups correspond to the groups of number 12 ($p = 3$), 29 ($p = 5$), 31 ($p = 5$), 34 ($p = 7$), 36 ($p = 5, 7$) and 37($p = 7$) in the Clark-Ewing list. They are described by means of homotopy colimits by Aguadé [Ag]. Among them, only those of number $12, 29, 31$ and 34 correspond to exotic p-compact groups (the groups of number 36 and 37 are realized by the exceptional Lie groups E_7 and E_8). Zabrodsky [Zab] described spaces realizing groups of number 12 and 31 by other means.

Notice that each representation of the complex reflection groups of number $12, 29, 31, 34$ lies, in fact, in the character ring $\mathbb{Q}(\xi_{n(i)})$; thus we can restrict ourselves to the subfield $\mathbb{Q}(\xi_{n(i)})$ in $\mathbb{C}$ (two proportional roots are proportional up to a unitary vectors in $\mathbb{Q}(\xi_{n(i)})$, i.e. up to $\langle \xi_{n(i)} \rangle$).

i	12	29	31	34
χ-field	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\xi_4)$	$\mathbb{Q}(\xi_4)$	$\mathbb{Q}(\xi_6)$
$\|Z\|$	2	4	4	6

Let $\Delta \subset (\hat{\mathbb{Z}}_p)^n = Hom((\mathbb{Z}/p^\infty\mathbb{Z})^n, S^1)$ be the subset containing the unitary normal vectors associated to the reflecting hyperplanes. W acting on Δ defines a complex root system $\Sigma(\Delta, W) := W \cdot \Delta$ with Weyl group W.

Lemma A.1. *Let W be one of the groups of number* $12, 29, 31, 34$.

1. *The complex root system lies in the subfield $\mathbb{Q}(\chi) \subset \mathbb{C}$.*
2. *$R_W = W \cdot e$, that is R_W consists of a single orbit. R_W is an irreducible W-set.*
3. *If Z is the center of W, then $R_W = Z \cdot \Delta$ thus $|\Phi_W| = |Z| \cdot |\Delta| = |Z| \cdot N$, where N is the number of reflections in W.*

	12	*29*	*31*	*34*
N	*12*	*40*	*60*	*126*

Proof. 1. It is a consequence of the fact that these groups admit complex representations in the character field (see corollary 4.3).

2. First of all recall that W_{ab} is a finitely generated abelian group which is a product of cyclic groups. The product is indexed by the W-orbits of hyperplanes associated to the reflections in W and each one has order equal to the order of a reflection defining the orbit (Proposition 4.1). These computations can be found in Section **??**.
We can check that $W_{ab} \cong \mathbb{Z}/2\mathbb{Z}$, then there is only one orbit of hyperplanes. Then R_W is an irreducible W-set, i.e., $R_W = W \cdot e$.
3. Clearing denominators, if necessary, we can assume that all roots in R_W lie in $\mathbb{Z}[\chi]$. These complex root systems are described in the following subsections. With these explicit computations it is easy to check that $2\langle e, a_s\rangle/\langle e, e\rangle \in \mathbb{Z}[\chi]$. If $ra_s \in R_W$ then $r \in \mathbb{Z}[\chi]$ and it is unitary. Then, in fact, $r = \xi_{n(i)}$ where $\xi_{n(i)}$ is an $n(i)$-th root of unity, $n(i) = 2, 4, 4, 6$ for $i = 12, 29, 31, 34$. Observe that these multiples are obtained by the action of the centre of W in each case.

We will study the groups in the list by Aguadé in [Ag] as Weyl groups of complex root systems (the groups of number 12 ($p = 3$), 29 ($p = 5$), 31 ($p = 5$) and 34 ($p = 7$)).

The description of the groups W_i for $i = 12, 29, 31, 34$ as Weyl groups of complex root systems allows us to describe admissible morphisms (see subsections A.1,A.2 and A.3) because they act on roots as permutation groups.

From the previous discussion, the following lemma is straight forward.

Lemma A.2. *For each i in the table, there exists a homomorphism $\phi_W : W_i \to \Sigma_{n(i)}$ describing the permutation action of the roots such that the p-Sylow subgroup S acts without fixed points on the set R_W. Let W_e be the stabilizer of a root $e \in R_W$.*

i	12	29	31	34
$n(i)$	24	160	240	756
$\|W_e\|$	2	$2^4 \cdot 3$	$2^6 \cdot 3$	$2^7 \cdot 3^4 \cdot 5$
p	3	5	5	7

Proof. First part is clear from the description of the groups as Weyl groups of complex root systems (see subsections A.1,A.2 and A.3), second part comes from the fact that $\Sigma(W, \Delta) = W \cdot e$ and that p does not divide $|W_e|$. If $S = \langle g \rangle$ and $gr = r$ then there exists $h \in G$ such that $ghe_1 = he_1$ and $h^{-1}gh \in W_e$ but $o(h^{-1}gh) = p$.

$|W_e|$ are computed with the following formula. We know that there is only one orbit of hyperplanes and that each hyperplane determines a reflection. Let e be a unitary root which determines a generator of the orbit, then $|W : W_e| = N \cdot |Z|$ where N is the number of reflections in W.

Each $r \in R_W$ lies in $\mathbb{Q}(\xi_{n(i)})^n$. If we consider the p-adic representation of each of these groups, we can assume $r \in \hat{\mathbb{Z}}_p$. It induces a map at the level of classifying spaces $Br : B(\mathbb{Z}/p^\infty\mathbb{Z})^n \to BS^1$. The whole set of roots define a map

$$\phi_T := \prod_{r \in R_W} Br : B(\mathbb{Z}/p^\infty\mathbb{Z})^n \to (BS^1)^{n(i)}.$$

Corollary A.1. *The map ϕ_T is admissible with respect to ϕ_W.*

A.1. Group of number 12

Looking at Clark-Ewing list [CE], we notice that this group can be realized as a 3-adic reflection group. As an abstract group it has 48 elements and all reflections are of order 2 (i.e. reflections). It contains exactly twelve reflections and it is generated by 3 of them (so it is not the Weyl group of a root graph).

	dim	$\|G\|$	degrees	χ-field	$\|Z\|$
12	2	48	$6, 8$	$\mathbb{Q}(\sqrt{-2})$	2

We are going to study this group using its faithful representation into $GL_2(\hat{\mathbb{Z}}_3)$. Let us start with some notation. Consider the following matrices in $GL_2(\hat{\mathbb{Z}}_3)$:

$$S = \begin{pmatrix} w & \frac{1}{2} \\ -\frac{1}{2} & \bar{w} \end{pmatrix}, T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} \frac{\sqrt{2}i}{2} & -\frac{\sqrt{2}i}{2} \\ -\frac{\sqrt{2}i}{2} & -\frac{\sqrt{2}i}{2} \end{pmatrix}$$

where $w = \frac{-1+\sqrt{-2}}{2} \in \hat{\mathbb{Z}}_3$.

Lemma A.3. *[Smi] G_{12} contains exactly the following twelve reflections:*

$$\pm RX1 = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm RX2 = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm RX3 = \pm \begin{pmatrix} -\frac{1}{2} & \bar{w} \\ w & \frac{1}{2} \end{pmatrix}$$

$$\pm RX4 = \pm \begin{pmatrix} \frac{1}{2} & w \\ \bar{w} & -\frac{1}{2} \end{pmatrix}, \pm RX5 = \pm \begin{pmatrix} -\frac{1}{2} & w \\ \bar{w} & \frac{1}{2} \end{pmatrix}, \pm RX6 = \pm \begin{pmatrix} \frac{1}{2} & \bar{w} \\ w & -\frac{1}{2} \end{pmatrix}.$$

Each of the above reflections has an associated complex unitary root. A complex unitary root defines a linear form vanishing in the fixed hyperplane, $-2\frac{\langle x,r\rangle}{\langle r,r\rangle}$. Here we have the list of these unitary root vectors associated to the above reflections:

$$r1 = (\frac{\sqrt{2}i}{2}, -\frac{\sqrt{2}i}{2}), r2 = (0,1), r3 = (\bar{w}, -\frac{1}{2}), r4 = (\frac{1}{2}, -\bar{w}),$$

$$r5 = (w, -\frac{1}{2}), r6 = (\frac{1}{2}, -w),$$

$$r1n = (\frac{\sqrt{2}i}{2}, \frac{\sqrt{2}i}{2}), r2n = (1,0), r3n = (\frac{1}{2}, w), r4n = (w, \frac{1}{2}),$$

$$r5n = (\frac{1}{2}, \bar{w}), r6n = (\bar{w}, \frac{1}{2}).$$

Remark A.1. Clearing denominators (multiplying by 2), notice that the roots lie in a $\mathbb{Z}[\sqrt{-2}]$-module generated by the roots of generating reflections.

The action of W_{12} on unitary roots up to sign is described in the following table:

	R1	R2	R3	R4	R5	R6
r1	-r1	r1n	r6	r5	-r4	-r3
r2	r2n	-r2	r4n	r3	r6n	r5
r3	-r4	r6n	r3n	r2	-r3	-r1
r4	-r3	r5n	-r2n	-r4n	-r1	-r4
r5	-r6	r4n	-r5	r1	r5n	r2
r6	-r5	r3n	r1	-r6	-r2n	-r6n
r1n	r1n	r1	r6n	r5n	-r4n	-r3n
r2n	r2	r2n	-r4	r3n	-r6	r5n
r3n	r4n	r6	r3	r2n	r3n	-r1n
r4n	r3n	r5	r2	-r4	-r1n	r4n
r5n	r6n	r4	r5n	r1n	r5	r2n
r6n	r5n	r3	r1n	r6n	r2	-r6

This action allows us to define a morphism $\phi_W : W_{12} \to \Sigma_{24}$ describing the permutation (up to sign) action of G_{12} on $R_{W_{12}} = \Delta_{12} \cup -\Delta_{12}$.

A.2. Groups of number 29,31

Groups of number 29 and 31 in the Shephard-Todd list can be realized as a 5-adic reflection groups ($G_i \to GL_4(\hat{\mathbb{Z}}_5)$).

	dim	$\|G\|$	degrees	χ-field	$\|Z\|$
29	4	7680	$4, 8, 12, 20$	$\mathbb{Q}(\xi_4)$	4
31	4	$64 \cdot 6!$	$8, 12, 20, 24$	$\mathbb{Q}(\xi_4)$	4

W_{29} is generated by four reflections and contains exactly 40 reflections. W_{31} is generated by five reflections and contains 60 reflections.

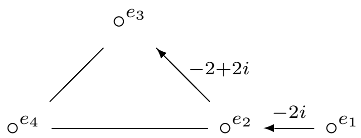
W_{29} is generated by

$$\begin{pmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

W_{31} is generated by the above reflections plus

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Group of number 29 is the Weyl group of a complex root system denoted by $N4$ [Coh] and it corresponds to the Weyl group of a root graph;



where

$$\begin{aligned} e_1 &= (1, 1, 1, 1), \\ e_2 &= (-1+i, 1+i, 0, 0), \\ e_3 &= (1+i, -1-i, 0, 0), \\ e_4 &= (0, 1+i, -1-i, 0). \end{aligned}$$

Notice that all the roots have norm 2, hence, in order to obtain unitary roots we have to divide all of them by 2.

Remark A.2. Recall that R_W consists of a single orbit $W \cdot e_1$. As $2\langle e_1, e\rangle / \langle e, e\rangle \in \mathbb{Z}[i]$ for all root e, it is clear that the $\mathbb{Z}[i]$-module generated by the roots is $\sum_{i=1}^{4} \mathbb{Z}[i] e_i$.

These root system $N4$ admits a neat extension $EN4$ ($R_{W_{31}} = R_{W_{29}} \cup W_{31} \cdot (0, 0, 1, 0)$) as it is described in [Coh] and its Weyl group is the group of number 31. In this case, $W(EN4)$ is not the Weyl group of a root graph.

Notice that the action of G_i on $R_{W_i} = Z(W_i) \cdot \Delta_i$ for $i = 29, 31$ defines morphisms $\phi_W : W_{29} \to \Sigma_{160}$ and $\phi_W : W_{31} \to \Sigma_{240}$ respectively.

A.3. Group of number 34

The group of number 34 in the Shephard-Todd list can be realized as a 7-adic reflection group ($W_{34} \to GL_6(\hat{\mathbb{Z}}_7)$).

	dim	$\|G\|$	degrees	χ-field	$\|Z\|$
34	6	$108 \cdot 9!$	$6, 12, 18, 24, 30, 42$	$\mathbb{Q}(\xi_3)$	6

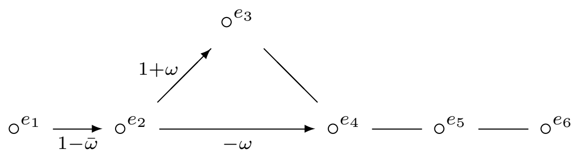
W_{34} is generated by six reflections and contains exactly 126 reflections. The generating reflections are the following ones:

$$\begin{pmatrix} 2/3 & -1/3 & -1/3 & -1/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 & -1/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & 2/3 & -1/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & -1/3 & 2/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & -1/3 & -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & 2/3 \end{pmatrix}, \begin{pmatrix} 0 & \omega^2 & 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \end{pmatrix},$$

where ω is a 3rd primitive root of unity. This group is the Weyl group of a complex root system denoted by $K6$ [Coh] and it corresponds to the Weyl group of a root graph.



where

$$e_1 = (1, 1, 1, 1, 1, 1),$$
$$e_2 = (1, -\omega, 0, 0, 0, 0),$$
$$e_3 = (1, -1, 0, 0, 0, 0,)$$
$$e_4 = (0, 1, -1, 0, 0, 0),$$

$$e_5 = (0, 0, 1, -1, 0, 0),$$
$$e_6 = (0, 0, 0, 1, -1, 0).$$

Remark A.3. Recall that R_W consists of a single orbit $W \cdot e_1$. As $2\langle e_1, e\rangle/\langle e, e\rangle \in \mathbb{Z}[\xi_6]$, where ξ_6 is a primitive 6th root of unity, for all roots e, it is clear that the $\mathbb{Z}[\xi_6]$-module generated by the roots is $\sum_{i=1}^{4} \mathbb{Z}[\xi_6]e_i$.

The action of W_{34} on $R_{W_{34}} = Z \cdot \Delta_{34}$ defines a morphism $\phi_W : W_{34} \to \Sigma_{756}$.

B. Table of W_{ab}

Number	W_{ab}	$\lvert Z(W)\rvert$
1	$\mathbb{Z}/2\mathbb{Z}$	$\begin{cases} 1 & n \geq 2 \\ 2 & n = 1 \end{cases}$
$2a$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}$	$r(n, \frac{m}{r})$
$2b$	$\begin{cases} \mathbb{Z}/2\mathbb{Z} & e\ odd \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & e\ even \end{cases}$	$(2, e)$
3	$\mathbb{Z}/n\mathbb{Z}$	n
4	$\mathbb{Z}/3\mathbb{Z}$	2
5	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	6
6	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	4
7	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	12
8	$\mathbb{Z}/4\mathbb{Z}$	4
9	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	8
10	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	12
11	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	24
12	$\mathbb{Z}/2\mathbb{Z}$	2
13	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	4
14	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	6
15	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	12
16	$\mathbb{Z}/5\mathbb{Z}$	10
17	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	20
18	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	30
19	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	60
20	$\mathbb{Z}/3\mathbb{Z}$	6
21	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	12
22	$\mathbb{Z}/2\mathbb{Z}$	4
23	$\mathbb{Z}/2\mathbb{Z}$	2
24	$\mathbb{Z}/2\mathbb{Z}$	2
25	$\mathbb{Z}/3\mathbb{Z}$	3
26	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	6
27	$\mathbb{Z}/2\mathbb{Z}$	6
28	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2
29	$\mathbb{Z}/2\mathbb{Z}$	4
30	$\mathbb{Z}/2\mathbb{Z}$	2
31	$\mathbb{Z}/2\mathbb{Z}$	4
32	$\mathbb{Z}/3\mathbb{Z}$	6
33	$\mathbb{Z}/2\mathbb{Z}$	2
34	$\mathbb{Z}/2\mathbb{Z}$	6
35	$\mathbb{Z}/2\mathbb{Z}$	1
36	$\mathbb{Z}/2\mathbb{Z}$	2
37	$\mathbb{Z}/2\mathbb{Z}$	2