ADJOINT SPACES AND FLAG VARIETIES OF p-COMPACT GROUPS

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ABSTRACT. For a compact Lie group G with maximal torus T, Pittie and Smith showed that the flag variety G/T is always a stably framed boundary. We generalize this to the category of p-compact groups, where the geometric argument is replaced by a homotopy theoretic argument showing that the class in the stable homotopy groups of spheres represented by G/T is trivial, even G-equivariantly. As an application, we consider an unstable construction of a G-space mimicking the adjoint representation sphere of G inspired by work of the second author and Kitchloo. This construction stably and G-equivariantly splits off its top cell, which is then shown to be a dualizing spectrum for G.

1. INTRODUCTION

Let G be a compact, connected Lie group of dimension d and rank r with maximal torus T. Left translation by elements of G on the tangent space $\mathfrak{g} = T_e G$ induces a framing of G. By the Pontryagin-Thom construction, G with this framing represents an element $[G]$ in the stable homotopy groups of spheres; this has been extensively studied for example in [Smi74, Woo76, Kna78, Oss82].

The following classical argument shows that the flag variety G/T, while not necessarily framed, is still a stably framed manifold: since every element in a compact Lie group is conjugate to an element in the maximal torus, the conjugation map $G \times T \to G$, $(g,t) \mapsto gtg^{-1}$ is surjective, and furthermore, it factors through $c\colon G/T \times T \to G$. An element $s \in T$ is called *regular* if the centralizer $C_G(s) \supseteq T$ equals T, or, equivalently, if $c|_{G/T\times\{s\}}$ is an embedding; it is a fact from Lie theory that the set of irregular elements has positive codimension in T. Thus there is a regular element s such that the derivative of c has full rank along $G/T \times \{s\}$, and by the tubular neighborhood theorem, it induces an embedding of $G/T \times U$, where U is a contractible neighborhood of s in T. Thus the framing of G can be pulled back to a stable framing of G/T.

Pittie and Smith showed in [Pit75, PS75] that G/T is always the boundary of another framed manifold M, and moreover, that M has a G-action which agrees with the standard G-action on G/T on the boundary. In terms of homotopy theory, this is saying that the class $[G/T] \in \pi^s_{d-r}$ induced by the Pontryagin-Thom construction is trivial.

The first main result of this paper generalizes this fact to p-compact groups.

Theorem 1.1. *Let G be a $\mathbf{Z}/p$-local, p-finite group with maximal torus T such that $dim(G) > dim(T)$. Then the Pontryagin-Thom construction $[G/T]\colon S_G \to S_T$ is G-equivariantly null-homotopic, with G acting trivially on S_T.*

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The statement of this theorem requires some explanation. A *p-compact group* [DW94] is a triple (G, BG, e) such that

- G is *p-finite*, i. e., $H_*(G; \mathbf{F}_p)$ is finite;
- BG is $\mathbf{Z}/p$*-local*, i. e. whenever $f\colon X \to Y$ is a mod-p homology equivalence of CW-complexes, then $f^*\colon [Y, BG] \to [X, BG]$ is an isomorphism;
- $e\colon G \to \Omega BG$ is a homotopy equivalence.

Clearly, G and e are determined by BG up to homotopy, making this definition somewhat redundant. Although a priori G is only a loop space, we will henceforth assume we have chosen a rigidification such that G is actually a topological group. This is always possible, for example by using the geometric realization of Kan's group model of the loops on a simplicial set [Kan56].

A $\mathbf{Z}/p$-local, p-finite loop space is only slightly more general than a p-compact group in that the latter also requires $\pi_0(G)$ to be a p-group. We will have no need to assume this in Theorem 1.1.

By [DW94], every p-compact group has a maximal torus T; that is, there is a monomorphism $T \to G$ with $T \simeq L_p(\mathbf{S}^1)^r$ and r is maximal with this property. By definition, a *monomorphism* of p-compact groups is a group monomorphism $H \to G$ such that G/H is p-finite (see [Bau04] for this slightly nonstandard point of view). Dwyer and Wilkerson show that T is essentially unique. Since a maximal torus is always contained in the identity component of a p-compact group, the same works for $\mathbf{Z}/p$-local, p-finite groups.

Denote by $\mathbf{S}^0[X]$ the suspension spectrum of a space X with a disjoint base point added.

Definition ([Kle01])**.** Let G be a topological group. Define S_G, the *dualizing spectrum of* G, to be the spectrum of homotopy fixed points of the right action of G on its own suspension spectrum. That is, $S_G = (\mathbf{S}^0[G])^{hG^{\mathrm{op}}}$ as left G-spectra.

In [Bau04], the first author showed that for a connected, d-dimensional p-compact group G, S_G is always homotopy equivalent to a $\mathbf{Z}/p$-local sphere of dimension d. Furthermore, there is a G-equivariant logarithm map $\mathbf{S}^0[G] \to S_G$, where G acts on the left by conjugation. If G is the $\mathbf{Z}/p$-localization of a connected Lie group, then S_G is canonically identified with the suspension spectrum of the one-point compactification of the Lie algebra of G. Thus we may call S_G the *adjoint (stable) sphere* of G.

In a spectacular case of shortsightedness, [Bau04] restricts its scope to connected p-compact groups where everything would have worked for $\mathbf{Z}/p$-local, p-finite groups G as well. In this case, S_G has the mod-p homology of a d-dimensional sphere. Similarly, the proof of the following was given in [Bau04, Cor. 24] for connected groups, but immediately generalizes.

Let DM be the Spanier-Whitehead dual of a finite CW-spectrum M.

Lemma 1.2. *Let $H < G$ be a monomorphism of $\mathbf{Z}/p$-local, p-finite groups. Then there is a relative G-equivariant duality weak equivalence*

$$G_+ \wedge_H S_H \simeq D\left(\mathbf{S}^0[G/H]\right) \wedge S_G.$$

For any space X, there is a canonical map $\epsilon\colon \mathbf{S}^0[X] \to \mathbf{S}^0$ given by applying the functor $\mathbf{S}^0[-]$ to $X \to *$. If $T < G$ is a sub-torus in a $\mathbf{Z}/p$-local, p-finite group then there is a stable G-equivariant map

$$
(1.3)\quad [G/T]\colon S_G \xrightarrow{\mathrm{id}\wedge D\epsilon} S_G \wedge D\left(\mathbf{S}^0[G/T]\right) \underset{\text{Lemma 1.2}}{\simeq} G_+ \wedge_T S_T \simeq \mathbf{S}^0[G/T] \wedge S_T \xrightarrow{\epsilon} S_T
$$

where the homotopy equivalence on the right hand side holds because S_T has a homotopy trivial T-action as T is homotopy abelian. The first map is studied in [Bau04]. This is the map referred to in Theorem 1.1; it generalizes the Pontryagin-Thom construction.

In the second part of this paper, as an application of Theorem 1.1, we study the relationship between two notions of adjoint objects of p-compact groups. It is an interesting question to ask whether the action of G on S_G actually comes from an unstable action of G on $\mathbf{S}^d$. We will not be able to answer this question in this paper. However, there is an alternative, unstable construction of an adjoint object for a connected p-compact group G inspired by the following:

Theorem 1.4 ([CK02, Mit88])**.** *Let G be a semisimple, connected Lie group of rank r. There exist subgroups $G_I < G$ for every $I \subsetneq \{1, \ldots, r\}$ and a homeomorphism of G-spaces*

$$
A_G := \Sigma \operatorname*{hocolim}_{I \subsetneq \{1,\ldots,r\}} G/G_I \to \mathfrak{g} \cup \{\infty\}
$$

to the one-point compactification of the Lie algebra $\mathfrak{g}$ of G.

In the second part of this paper, we define a G-space A_G for every connected p-compact group G and show:

Theorem 1.5. *For any connected p-compact group G, there is a G-equivariant splitting $\mathbf{S}^0[A_G] \simeq S_G \vee R$ for some finite G-spectrum R.*

This result links the two notions of adjoint objects together. Thus stably, the adjoint sphere is a wedge summand of the adjoint space.

Unfortunately, A_G is in general not a sphere.

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2. The stable p-complete splitting of complex projective space

2.1. **Stable splittings from homotopy idempotents.** Let p be a prime. We denote by L_p the localization functor on topological spaces with respect to mod-p homology, which coincides with p-completion on nilpotent spaces [BK72]. Let $S = L_p\mathbf{S}^1$ be the p-complete 1-sphere, and set $P = \mathbf{S}^0[BS]$. It is a classical result that

$$
(2.1)\qquad P \simeq \bigvee_{s=0}^{p-2} P_s
$$

for certain $(2i-1)$-connected spectra P_i. In this section, we will investigate this splitting and its compatibility with certain transfer maps.

Let X be a spectrum, $e \in [X, X]$ and define

$$
eX = \operatorname{hocolim}\{X \xrightarrow{e} X \xrightarrow{e} \cdots\}.
$$

If e is idempotent, this is a homotopy theoretic analog of the image of e. Any such idempotent e yields a stable splitting $X \simeq eX \vee (1-e)X$. If $\{e_1, \dots, e_n\}$ are a complete set of orthogonal idempotents (this means that each e_i is idempotent, $e_i e_j \simeq *$, and $\mathrm{id}_X \simeq e_1 + \dots + e_n$), then they induce a splitting $X \simeq e_1 X \vee \dots \vee e_n X$.

Example 2.2. Let p be an odd prime. Denote by $\psi \colon P \to P$ the map induced by multiplication with a $(p-1)$st root of unity ζ. Define $e_s \colon P \to P$ by

$$e_s = \frac{1}{p-1} \left(\sum_{i=0}^{l-1} \zeta^{-is} \psi^i \right).$$

It is straightforward to check that $\{e_0, \dots, e_{p-2}\}$ are a complete set of orthogonal idempotents in $[P, P]$. They induce the splitting (2.1) by defining $P_s = e_s P$.

Setting $H_*(P) = \mathbf{Z}_p\{x_j\}$ with $|x_j| = 2j$, we have that $(e_i)_* \colon H_*(P) \to H_*(P)$ is given by

$$(e_i)_*(x_j) = \begin{cases} x_j; & j \equiv i \pmod{p-1} \\ 0; & \text{otherwise.} \end{cases} \tag{2.3}$$

2.2. **Transfers as splittings.** Let $1 \to H \xrightarrow{i} G \to W \to 1$ be an extension of compact Lie groups. Then associated to the fibration $W \to BH \to BG$ there are two versions of functorial stable transfer maps [BG75, BG76]:

(1) The Becker-Gottlieb transfer $\overline{\tau} \colon \mathbf{S}^0[BG] \to \mathbf{S}^0[BH]$
(2) The stable Umkehr map $\tau \colon BG^{\mathfrak{g}} \to BH^{\mathfrak{h}}$ of Thom spaces of the adjoint representation of the Lie groups.

Both versions can be generalized to a setting where the groups involved are not Lie groups but only $\mathbf{Z}/p$-local and p-finite [Dwy96, Bau04]. For such a group G, $BG^{\mathfrak{g}}$ is defined to be the homotopy orbit spectrum of G acting on the dualizing spectrum S_G; since $H_*(S_G) = H_*(\mathbf{S}^d; \mathbf{Z}_p)$, we have a (possibly twisted) Thom isomorphism between $H_*(BG)$ and $H_*(BG^{\mathfrak{g}})$.

Note that $\overline{\tau}$ factors through τ in the following way:

$$\begin{aligned} \mathbf{S}^0[BG] \xrightarrow{\tau'} BH^\nu \xrightarrow{\text{comult.}} BH^\nu \wedge_{BG} \mathbf{S}^0[BH] \\ \xrightarrow{\mathrm{id} \wedge \Delta} BH^\nu \wedge_{BG} \mathbf{S}^0[BH] \wedge_{BG} \mathbf{S}^0[BH] \xrightarrow{\mathrm{eval} \wedge \mathrm{id}} \mathbf{S}^0[BH] \end{aligned} \tag{2.4}$$

where $\nu = \mathfrak{h} - i^*\mathfrak{g}$ is the normal fibration along the fibers of $BH \to BG$, τ' is τ twisted by $-\mathfrak{g}$, and the right hand side evaluation map is defined by identifying BH^ν with the fiberwise Spanier-Whitehead dual of BH over BG.

Proposition 2.5. *Let $W = C_l$ be a finite cyclic group acting freely on S, with $l \mid p-1$. Denote by $N = S \rtimes W$ the semidirect product with respect to this action. Then the Becker-Gottlieb transfer map $\overline{\tau} \colon \mathbf{S}^0[BN] \to P$ factors through $fP \to P$ for some idempotent $f \colon P \to P$ which induces the same map in homology as $e_0 + e_l + \dots + e_{p-1-l}$, and the induced map $\mathbf{S}^0[BN] \to fP$ is a mod-p homology equivalence.*

Proof. Since $p \nmid |W|$, the Serre spectral sequence associated to the group extension $S \xrightarrow{i} N \to W$ is concentrated on the vertical axis and shows that

$$H^*(BN; \mathbf{Z}_p) \cong H^*(BS; \mathbf{Z}_p)^W \cong \mathbf{Z}_p[z^l] \hookrightarrow \mathbf{Z}_p[z] \cong H^*(BS; \mathbf{Z}_p).$$

In this case, the Becker-Gottlieb transfer is nothing but the usual transfer for finite coverings, therefore $i \circ \overline{\tau}$ is multiplication by $|W| = l \in \mathbf{Z}_p^\times$. Setting $I = l^{-1} i \colon P \to \mathbf{S}^0[L_p BN]$, we thus get orthogonal idempotents in $[P, P]$:

$$f = \overline{\tau} \circ I \quad \text{and} \quad e = \mathrm{id}_P - f.$$

Clearly, $e \circ \overline{\tau} \simeq *$, thus $\overline{\tau}$ factors through fP and induces an isomorphism $\mathbf{S}^0[L_p BN] \to fP$, in particular a mod-$p$ homology isomorphism between $\mathbf{S}^0[BN]$ and fP. The computation of the homology of BN together with (2.3) implies that $f_* = (e_0 + e_l + \cdots + e_{p-1-l})_*$. □

Corollary 2.6. *Let S, N, W be as above. Then the stable Umkehr map*

$$BN^{\mathfrak{n}} \to BS^{\mathfrak{s}} \simeq \Sigma P$$

factors through $\Sigma f P \to \Sigma P$ for some $f \colon P \to P$ which induces the same morphism in homology as $\sum_{i=0}^{\frac{p-1}{l}} e_{(i+1)l-1}$. The induced map $BN^{\mathfrak{n}} \to \Sigma f P$ is a mod-p homology equivalence.

Proof. This follows from a similarly simple homological consideration. The S-fibration $\mathfrak{n}$ is not orientable, thus we have a twisted Thom isomorphism

$$\tilde{H}^{n+1}(BN^{\mathfrak{n}}) \cong H^n(BN; \mathcal{H}^1(S; \mathbf{Z}_p))$$

where $\pi_1(BN) = \mathbf{Z}/l$ acts on $H^1(S; \mathbf{Z}_p) \cong \mathbf{Z}_p$ by multiplication by an lth root of unity. Thus

$$H^i(BN^{\mathfrak{n}}; \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p; & i \equiv -1 \pmod{l} \\ 0; & \text{otherwise.} \end{cases}$$

The factorization (2.4) of $\overline{\tau}$ through τ

$$\mathbf{S}^0[BN] \xrightarrow{\tau'} BS^\nu \xrightarrow{\text{comult.}} BS^\nu \wedge_{BN} \mathbf{S}^0[BS]$$
$$\xrightarrow{\mathrm{id} \wedge \Delta} BS^\nu \wedge_{BN} \mathbf{S}^0[BS] \wedge_{BN} \mathbf{S}^0[BS] \xrightarrow{\mathrm{eval} \wedge \mathrm{id}} \mathbf{S}^0[BS]$$

simplifies considerably since $i^*\nu$ is the trivial 1-dimensional fibration over BS, and the composition of the three right hand side maps is an equivalence.

In Prop. 2.5 it was shown that $I \circ \overline{\tau} = \mathrm{id}_{\mathbf{S}^0[L_p BN]}$, thus the same holds after twisting with $\mathfrak{n}$:

$$\mathrm{id}_{BN^{\mathfrak{n}}} \colon L_p BN^{\mathfrak{n}} \xrightarrow{\tau} BS^{\mathfrak{s}} \to BS^{i^*\mathfrak{n}} \xrightarrow{I^{\mathfrak{n}}} L_p BN^{\mathfrak{n}}.$$

If we denote the composition $BS^{\mathfrak{s}} \to BS^{i^*\mathfrak{n}} \xrightarrow{I^{\mathfrak{n}}} L_p BN^{\mathfrak{n}}$ by I, overriding its previous meaning, the argument now proceeds as in Prop. 2.5. Using the computation of $H^*(BN^{\mathfrak{n}}; \mathbf{Z}_p)$, we find that $L_p BN^{\mathfrak{n}} \simeq (\tau \circ I)P$, and

$$(\tau \circ I)_* = \sum_{i=0}^{\frac{p-1}{l}} (e_{(i+1)l-1})_*$$

□

3. Framing p-compact flag varieties

Before proving Theorem 1.1, we need an alternative description of the Pontryagin-Thom construction (1.3) on G/T.

Lemma 3.1. *The map $[G/T]$ is G-equivariantly homotopic to the map*

$$S_G \xrightarrow{\text{incl}} BG^{\mathfrak{g}} \xrightarrow{\tau} BT^{\mathfrak{t}} \simeq \Sigma^r \mathbf{S}^0[BT] \xrightarrow{\Sigma^r \epsilon} \mathbf{S}^r,$$

where $BG^{\mathfrak{g}}$, $BT^{\mathfrak{t}}$, and τ are as in Section 2.2, and all spectra except S_G have a trivial G-action.

Proof. Applying homotopy G-orbits to (1.3), we get a G-equivariant diagram

$$\begin{tikzcd} S_G \arrow[r] \arrow[d, "\text{incl}"] & S_G \wedge D(\mathbf{S}^0[G/T]) \arrow[r, "\sim"] & \mathbf{S}^0[G/T] \wedge S_T \arrow[r] \arrow[d] & S_T \arrow[d, equal] \\ BG^{\mathfrak{g}} \arrow[r, "\tau"] & BT^{\mathfrak{t}} \arrow[r, "\sim"] & \mathbf{S}^0[BT] \wedge S_T \arrow[r] & S_T \end{tikzcd}$$

which is commutative by the definition of τ [Bau04, Def. 25]. $\square$

In the proof of Theorem 1.1, certain special subgroups will play an important role. In order to define them we need to recall certain facts about the Weyl group of a p-compact group.

Dwyer and Wilkerson showed in their ground-breaking paper [DW94] that given any connected p-compact group G with maximal torus T, there is an associated *Weyl group* $W(G)$, which is defined as the group of components of the homotopy discrete space of automorphisms of the fibration $BT \to BG$. This generalizes the notion of Weyl groups of compact Lie groups; they are canonically subgroups of $\mathrm{GL}(H_1(T)) = \mathrm{GL}_r(\mathbf{Z}_p)$, and they are so-called finite complex reflection groups. This means that they are generated by elements (called reflections or, more classically, pseudo-reflections) that fix hyperplanes in $\mathbf{Z}_p^r$. The complete classification of complex reflection groups over $\mathbf{C}$ is classical and due to Shephard and Todd [ST54], the refinement to the p-adics is due to Clark and Ewing [CE74].

Call a reflection $s \in W$ *primitive* if there is no reflection $s' \in W$ of strictly larger order such that $s = (s')^k$ for some k.

Denote by $s \in W$ a primitive reflection of minimal order $l > 1$. Let $T^s < T$ be the fixed point subtorus under s. Since s is primitive,

$$\langle s \rangle = \{ w \in W \mid w|_{T^s} = \mathrm{id}_{T^s} \}.$$

Definition. Given connected p-compact group G and a primitive reflection $s \in W(G)$ of minimal order $l > 1$, define C_s to be the centralizer of T^s in G.

Since G is connected, so is the subgroup C_s [DW95, Lemma 7.8]. Furthermore, C_s has maximal rank because $T < C_s$ by definition, and the inclusion $C_s < G$ induces the inclusion of Weyl groups $\langle s \rangle < W$ [DW95, Thm. 7.6]. Since the Weyl group of C_s is $\mathbf{Z}/l$, the quotient of C_s by its p-compact center, $C_s/Z(C_s)$, can have rank at most 1. By the (almost trivial) classification of rank-1 p-compact groups, we find that its rank is equal to 1 and

$$C_s \cong \left(L_p(\mathbf{S}^1)^{r-1} \times L_p \mathbf{S}^{2l-1} \right) / \Gamma, \tag{3.2}$$

where $L_p\mathbf{S}^{2l-1}$ is simply $L_p\,\mathrm{SU}(2)$ for $l=2$, and the Sullivan group given by

$$L_p\mathbf{S}^{2l-1} = \Omega L_p\left(L_p(B\mathbf{S}^1)_{h\mathbf{Z}/l}\right)$$

for p odd, and Γ is a finite central subgroup.

Proof of Thm. 1.1. By Lemma 3.1, showing equivariant null-homotopy is equivalent to showing that the map

$$h(G/T)\colon BG^{\mathfrak{g}} \xrightarrow{\tau} BT^{\mathfrak{t}} \xrightarrow{\text{proj}} \mathbf{S}^r$$

is null. Note that for any given subgroup $H < G$ of maximal rank, there is a factorization of τ through $BH^{\mathfrak{h}}$. In particular, we may assume that G is connected. By the dimension hypothesis of the theorem, $W(G)$ is nontrivial. If $H = C_s$ is the subgroup associated to a primitive reflection $s \in W(G)$ of minimal order $l > 1$, then the map $h(C_s/T)$ is the $(r-1)$-fold suspension of $h(L_p\mathbf{S}^{2l-1}/S)$ by (3.2). Therefore, it is enough to prove the theorem for those p-compact groups C_s.

We distinguish two cases.

First suppose that $l=2$. By the classification of complex reflection groups [ST54], and with the terminology of that paper, this is always the case except when W is a product of any number of groups from the list

$$\{G_4, G_5, G_{16}, G_{18}, G_{20}, G_{25}, G_{32}\}.$$

This comment is only meant to intimidate the reader and is insubstantial for what follows.

In this case, the map $h(L_p\,\mathrm{SU}(2)/S)$ is null by Pittie [Pit75, PS75] since the spaces involved are Lie groups, thus $h(G/T) \simeq *$.

Now suppose that $l > 2$. This forces $p > 2$ as well, and since $\langle s \rangle$ acts faithfully on some line in $H^1(T;\mathbf{Z}_p)$ while fixing the complementary hyperplane, we must have that it acts by an lth root of unity, and thus $l \mid p-1$. The proof is finished if we can show that

$$h(L_p\mathbf{S}^{2l-1}/S) = 0$$

where S is the 1-dimensional maximal torus in the Sullivan group $G' := L_p\mathbf{S}^{2l-1}$. To see this, note that the inclusion $S \to G'$ factors through the maximal torus normalizer $N_{G'}(S) \cong S \rtimes \mathbf{Z}/l$, and thus

$$h(G'/S)\colon BG'^{\mathfrak{g}'} \xrightarrow{\tau_1} BN^{\mathfrak{n}} \xrightarrow{\tau_2} \Sigma\mathbf{S}^0[BS] \to \mathbf{S}^1.$$

If $P \simeq \bigvee_{i=0}^{p-2} e_iP$ is any stable splitting of the p-completed complex projective space $P = \mathbf{S}^0[BS]$ induced by idempotents e_i as in the previous section, then the rightmost projection map clearly factors through e_0P, which is the part containing the bottom cell. Since $p > 2$, Corollary 2.6 shows that there is an idempotent $f \in [P,P]$ such that $\tau \simeq f \circ \tau$ and $\tau \circ e_0 = e_0 \circ \tau = 0$, proving the theorem. □

4. THE ADJOINT REPRESENTATION

Let G be a d-dimensional connected p-compact group with maximal torus T of rank r. Choose a set $\{s_1, \ldots, s_{r'}\}$ of generating reflections of $W = W(G)$ with r' minimal. The classification of pseudo-reflection groups [ST54, CE74] implies that for G semisimple, most of the time $r = r'$, but there are cases where $r' = r+1$.

Example 4.1 (The group no. 7)**.** Let $p \equiv 1 \pmod{12}$. Let G_7 be the finite group generated by the reflection s of order 2 and the two reflections $t,\ u$ of order 3, where $s,\ t,\ u \in GL_2(\mathbf{Z}_p)$ are given by

$$s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t = \frac{1}{\sqrt{2}} \begin{pmatrix} -\zeta & \zeta^7 \\ -\zeta & -\zeta^7 \end{pmatrix}, \quad u = \frac{1}{\sqrt{2}} \begin{pmatrix} -\zeta^7 & -\zeta^7 \\ \zeta & -\zeta \end{pmatrix}.$$

Here ζ is a 24th primitive root of unity. Note that although possibly $\zeta \notin \mathbf{Z}_p$, $\frac{1}{\sqrt{2}}\zeta \in \mathbf{Z}_p$. In Shephard and Todd's classification, this is the restriction to $\mathbf{Z}_p$ of the complex pseudo-reflection group no. 7. They show that even over $\mathbf{C}$, G_7 cannot be generated by two reflections. The associated p-compact group is given by

$$\Omega L_p((BT^2)_{hG_7}).$$

If G is not semisimple (i. e. it contains a nontrivial normal torus subgroup), then r' may be smaller than r. Set $\kappa = r + 1 - r' \geq 0$.

Let $\mathcal{I}_{r'}$ be the set of proper subsets of $\{1, \ldots, r'\}$, and for $I \subseteq \{1, \ldots, r'\}$, let T_I be the fixed point subtorus $T^{\langle s_i | i \in I\rangle}$ and $C_I = C_G(T_I)$ be the centralizer in G, which is connected by [DW95, Lemma 7.8].

Definition. Let G be a connected p-compact group. Define the *adjoint space* A_G by the homotopy colimit

$$A_G = \Sigma^\kappa \operatorname*{hocolim}_{I \in \mathcal{I}_r} G/C_I$$

with the induced left G-action, and the trivial G-action on the suspension coordinates.

Theorem 1.4 shows that if G is a the p-completion of a connected, semisimple Lie group (in this case $r = r'$ and $\kappa = 1$), then A_G is a d-dimensional sphere G-equivariantly homotopy equivalent to $\mathfrak{g} \cup \{\infty\}$. This holds more generally: if G is a connected, compact Lie group with maximal normal torus T^k then

$$A_G \cong \Sigma^k A_{G/T^k} = (\mathfrak{t} \cup \{\infty\}) \wedge (\mathfrak{g}/\mathfrak{t} \cup \{\infty\}) = \mathfrak{g} \cup \{\infty\}.$$

Lemma 4.2. *Let $\mathcal{I}_r$ be the poset category of proper subsets of $\{1, \ldots, r\}$ and*

$$F, G \colon \mathcal{I}_k \to \{\text{finite CW-complexes or finite CW-spectra}\}.$$

be two functors. Then

(1) *If F is has the property that* $\dim F(\emptyset) > \dim F(I)$ *for every $I \neq \emptyset$, then*
$$\dim \operatorname{hocolim} F = \dim F(\emptyset) + k - 1.$$
(2) *If $f \colon F \to G$ is a natural transformation of two such functors such that*
$$f_*(\emptyset) \colon H_{\dim F(\emptyset)}(F(\emptyset)) \xrightarrow{\cong} H_{\dim G(\emptyset)}(G(\emptyset)),$$
then f induces an isomorphism
$$\operatorname{hocolim} f_* \colon H_{\dim \operatorname{hocolim} F}(\operatorname{hocolim} F) \to H_{\dim \operatorname{hocolim} G}(\operatorname{hocolim} G).$$
(3) *Let $F \colon \mathcal{I}_r \to$ Top be the functor given by $F(\emptyset) = \mathbf{S}^n$, $F(I) = *$ for $I \neq \emptyset$. Then* $\operatorname{hocolim}_{\mathcal{I}_r} F \simeq \mathbf{S}^{n+r-1}$.

Proof. The first two assertions follow from the Mayer-Vietoris spectral sequence [BK72, Chapter XII.5],

$$E^1_{p,q} = \bigoplus_{I \in \mathcal{I}_k,\, |I|=k-1-p} H_q(F(I)) \Longrightarrow H_{p+q}(\operatorname{hocolim} F),$$

along with the observation that under the dimension assumptions of (1), $E^1_{p,q} = 0$ for $q \geq \dim F(\emptyset)$ except for $E^1_{k-1,\dim F(\emptyset)} = H_{\dim F(\emptyset)}(F(\emptyset))$. In particular, this group cannot support a nonzero differential and thus

$$H_i(F(\emptyset)) \cong H_{i+k-1}(\operatorname{hocolim} F) \quad \text{for } i \geq \dim F(\emptyset).$$

The third one is an immediate consequence of the Mayer-Vietoris spectral sequence. □

Corollary 4.3. *For any connected p-compact group G, A_G is a d-dimensional G-space.*

Proof. This follows from Lemma 4.2. Indeed, since any C_I ($I \neq \emptyset$) is connected and has the nontrivial Weyl group W_I, its dimension is greater than $\dim T$. So the condition

$$\dim F(\emptyset) = \dim G/T > \dim F(I)$$

is satisfied, and

$$\dim \operatorname{hocolim} F = d - r + r' - 1 = d - \kappa.$$

□

As mentioned at the end of the introduction, for p-compact groups G, A_G is not usually a sphere, as the following example illustrates.

Example 4.4. Let $p \geq 5$ be a prime, and let $G = \mathbf{S}^{2p-3}$ be the Sullivan sphere, whose group structure is given by $BG = L_p\left(BS_{hC_{p-1}}\right)$, where $C_{p-1} \subseteq \mathbf{Z}_p^\times$ acts on $BS = K(\mathbf{Z}_p, 2)$ by multiplication on $\mathbf{Z}_p$. Clearly, G has rank 1, and $\mathcal{I}_1$ consists only of a point, thus $A_G = \Sigma G/T \simeq L_p\Sigma\mathbf{C}P^{p-2}$. Since $p \geq 5$, this is not a sphere.

For the proof of Theorem 1.5 we need a preparatory result.

Proposition 4.5. *Let P be a p-compact subgroup of maximal rank in a p-compact group G. Denote by T a maximal torus of P (and thus also of G). Then the following composition is G-equivariantly null-homotopic:*

$$f_{G,P}\colon S_G \wedge DS_T \to \mathbf{S}^0[G/T] \to \mathbf{S}^0[G/P].$$

The second map is the canonical projection, whereas the first map is given by using the duality isomorphism

$$\begin{aligned} S_G \wedge DS_T \to S_G \wedge D(\mathbf{S}^0[G/T]) \wedge DS_T &\xrightarrow{\sim} G_+ \wedge_T S_T \wedge DS_T \\ &\simeq \mathbf{S}^0[G/T] \wedge S_T \wedge DS_T \xrightarrow{\mathrm{id}\wedge\mathrm{ev}} \mathbf{S}^0[G/T]. \end{aligned}$$

Proof. In [Bau04, Cor. 24] it was shown that the relative duality isomorphism from Lemma 1.2 is natural in the sense that the following diagram commutes:

$$\begin{tikzcd} (\mathbf{S}^0[G])^{hP^{\mathrm{op}}} \arrow[r, "\sim"] \arrow[d, "\mathrm{res}"] & G_+ \wedge_P S_P \arrow[r, "\sim"] & D(\mathbf{S}^0[G/P]) \wedge S_G \arrow[d, "D(\mathrm{proj})\wedge\mathrm{id}"] \\ (\mathbf{S}^0[G])^{hT^{\mathrm{op}}} \arrow[r, "\sim"] & G_+ \wedge_T S_T \arrow[r, "\sim"] & D(\mathbf{S}^0[G/T]) \wedge S_G \end{tikzcd}$$

Taking duals and smashing with DS_G, we find that the map of the proposition is the left hand column in the diagram

$$\begin{tikzcd}
S_G \wedge D(G_+ \wedge_P S_P) \arrow[r, "\sim"] & \mathbf{S}^0[G/P] \\
S_G \wedge D(G_+ \wedge_T S_T) \arrow[r, "\sim"] \arrow[u] & \mathbf{S}^0[G/T] \arrow[u] \\
S_G \wedge D(\mathbf{S}^0[G/T]) \wedge DS_T \arrow[u, "\sim"] & \\
S_G \wedge DS_T \arrow[u] &
\end{tikzcd}$$

Thus we need to show that the composition

$$G_+ \wedge_P S_P \to G_+ \wedge_T S_T \simeq \mathbf{S}^0[G/T] \wedge S_T \to S_T$$

is G-equivariantly trivial, or equivalently, that

$$S_P \to P_+ \wedge_T S_T \to S_T$$

is P-equivariantly trivial. But this map is exactly the homotopy class represented by $[P/T]$, thus the assertion follows from Theorem 1.1. $\square$

Proof of Thm. 1.5. Let G be a connected p-compact group whose Weyl group is generated by a minimal set of r' reflections. Let $F, A\colon \mathcal{I}_{r'} \to \mathrm{Top}$ be the functors given by $F(\emptyset) = S_G \wedge DS_T$, $F(I) = *$ for $I \neq \emptyset$, and $A(I) = G/C_I$. Note that, since G is connected, $C_G(T) = T$ [DW94, Proposition 9.1] and $A(\emptyset) = G/T$. There is a map $\Phi\colon F \to A$ of $\mathcal{I}_{r'}$-diagrams in the homotopy category of G-spectra which is fully described by defining

$$\Phi(\emptyset) = f_{G,T}\colon F(\emptyset) = S_G \wedge DS_T \to \mathbf{S}^0[G/T]$$

as the map given in Prop. 4.5. The strategy of the proof is to obtain a functor $F\colon \mathcal{I}_{r'} \to \mathrm{Top}$ such that $F(\emptyset) = S_G \wedge DS_T$, $F(I) \simeq *$ for $I \neq \emptyset$, and a map of $\mathcal{I}_{r'}$-diagrams $\Phi\colon F \to A$ in the category of G-spectra such that $\Phi(\emptyset) = f_{G,T}$. From this we get a G-equivariant map

$$S_G \simeq \mathbf{S}^\kappa \wedge \Sigma^{r'-1} S_G \wedge DS_T \simeq \mathbf{S}^\kappa \wedge \operatorname*{hocolim}_{\mathcal{I}_{r'}} F \to \Sigma^\kappa \operatorname*{hocolim}_{\mathcal{I}_{r'}} \mathbf{S}^0[G/C_I] \simeq \mathbf{S}^0[A_G],$$

which will give us the splitting.

We will proceed by induction on the number of generating reflections r'. If $r' = 1$ then A_G is $\mathbf{S}^\kappa \wedge G/T$ and $\Phi(\emptyset) = \mathbf{S}^\kappa \wedge f_{G,T}$. We can construct the functor F and the natural transformation Φ step by step. Fix a subset I of cardinality k, and assume that F and Φ have been defined for all vertices in the diagram corresponding to I' with $|I'| < k$.

Let $\mathcal{P}(I)$ be the poset category of all proper subsets of I. Since F and Φ are defined over $\mathcal{P}(I)$ by induction hypothesis, we can consider $\operatorname{hocolim}_{\mathcal{P}(I)} F \simeq \Sigma^{k-1} S_G \wedge DS_T \to \mathbf{S}^0[G/C_I]$. It is enough to show that this map is G-equivariantly nullhomotopic. Then, we can fix a null-homotopy and extend the map to the cone of $\operatorname{hocolim}_{\mathcal{P}(I)} F$. Finally, we define $F(I) = C(\operatorname{hocolim}_{\mathcal{P}(I)} F)$ and $\Phi(I)$ is the corresponding extension.

Note that $S_G \wedge DS_T \to \mathbf{S}^0[G/T]$ factors through $G_+ \wedge_{C_I} S_{C_I} \wedge DS_T$. By induction, we know there is a map

$$\Sigma^{k-1} S_{C_I} \wedge DS_T \to \mathbf{S}^0[\operatorname*{hocolim}_{J \in \mathcal{I}_k} C_I/C_J],$$

which splits the top cell. We get a factorization

$$\begin{aligned}(4.6)\quad \Sigma^{k-1} S_G \wedge DS_T &\to \Sigma^{k-1} G_+ \wedge_{C_I} S_{C_I} \wedge DS_T \\ &\to G_+ \wedge_{C_I} \mathbf{S}^0[\operatorname*{hocolim}_{J \in \mathcal{I}_k} C_I/C_J] \to G_+ \wedge_{C_I} \mathbf{S}^0.\end{aligned}$$

It thus suffices to show that in the $\mathcal{I}_k$-diagram

$$\begin{tikzcd}
S_{C_I} \wedge DS_T \arrow[r, Rrightarrow] \arrow[d] & \{*\}_{J \in \mathcal{I}_k - \{\emptyset\}} \arrow[r, "\text{hocolim}"] \arrow[d] & \Sigma^{k-1} S_{C_I} \wedge DS_T \arrow[d] \\
\mathbf{S}^0[C_I/T] \arrow[r, Rrightarrow] \arrow[d] & \{\mathbf{S}^0[C_I/C_J]\}_{J \in \mathcal{I}_k - \{\emptyset\}} \arrow[r, "\text{hocolim}"] \arrow[d] & \mathbf{S}^0[\operatorname{hocolim}_{J \in \mathcal{I}_k} C_I/C_J] \arrow[d] \\
\mathbf{S}^0 \arrow[r, Rrightarrow] & \{\mathbf{S}^0\}_{I \in \mathcal{I}_k - \{\emptyset\}} \arrow[r, "\text{hocolim}"] & \mathbf{S}^0
\end{tikzcd}$$

the right hand side composition $\Sigma^{k-1} S_{C_I} \wedge DS_T \to \mathbf{S}^0$ is C_I-equivariantly null-homotopic. In the latter diagram, it makes no difference whether the centralizers are taken in C_I or in G. But by Theorem 1.1, the left hand column is already null-homotopic, thus, as a colimit of null-homotopic maps over a contractible diagram, so is the right hand column. $\square$

Conclusion and questions. In this paper, we have compared two imperfect notions of adjoint representations of a p-compact group G. One (S_G) is a sphere, but has a G-action only stably; the other (A_G) is an unstable G-space, but fails to be a sphere. The question remains whether there is an unstable G-sphere whose suspension spectrum is S_G. It might even be true that A_G splits off its top cell after only one suspension, yielding a solution to this problem in the cases where the Weyl group of the rank-r group G is generated by r reflections.

There are also a number of interesting open questions about the flag variety G/T of a p-compact groups:

- By the classification of p-compact groups, $H^*(G/T; \mathbf{Z}_p)$ is torsion free and generated in degree 2. Can this be seen directly?
- Is there a manifold M such that $L_p M \simeq G/T$, analogous to smoothings of G [BKNP04, BP06]? Is it a boundary of a manifold?
- If such a manifold M exists, can it be given a complex structure?

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