ON THE COHOMOLOGY OF HIGHLY CONNECTED COVERS OF FINITE HOPF SPACES

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Abstract. Relying on the computation of the André-Quillen homology groups for unstable Hopf algebras, we prove that if the mod p cohomology of both the fiber and the base in an H-fibration is finitely generated as algebra over the Steenrod algebra, then so is the mod p cohomology of the total space. In particular, the mod p cohomology of the n-connected cover of a finite H-space is always finitely generated as algebra over the Steenrod algebra.

Introduction

Consider the n-connected cover of a finite complex. Does its (mod p) cohomology satisfy some finiteness property? Such a question has already been raised by McGibbon and Møller in [MM97], but no satisfactory answer has been proposed. We do not ask here for an algorithm which would allow to make explicit computations. We rather look for a general structural statement which would tell us to what kind of class such cohomologies belong. The prototypical theorems we have in mind are the Evens-Venkov result, [Eve61], [Ven59], that the cohomology of a finite group is Noetherian, the analog for p-compact groups obtained by Dwyer and Wilkerson [DW94], and the fact that the mod p cohomology of an Eilenberg-Mac Lane space $K(A, n)$, with A abelian of finite type, is finitely generated as an algebra over the Steenrod algebra, which can easily been inferred from the work of Serre [Ser53] and Cartan [Car55].

This last observation leads us to ask first whether or not the mod p cohomology of a finite Postnikov piece is also finitely generated as an algebra over the Steenrod algebra and second, since a finite complex X and its n-connected cover $X\langle n\rangle$ only differ in a finite number of homotopy groups, if $H^*(X\langle n\rangle; \mathbb{F}_p)$ satisfies the same property. A positive solution to the question about Postnikov pieces was given in [CCSb, Proposition 2.1] when they are H-spaces.

In this paper we offer an affirmative answer to the second question when X is an H-space, based on the analysis of the fibration $P \to X\langle n\rangle \to X$, where P is a finite Postnikov piece (but we note that both questions are open in general). In fact we prove a strong closure property for H-fibrations, i.e. fibrations of H-spaces in which the maps preserve the H-space structure.

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Theorem 4.1. *Let $F \to E \to B$ be an H-fibration in which both $H^*(F;\mathbb{F}_p)$ and $H^*(B;\mathbb{F}_p)$ are finitely generated as algebras over the Steenrod algebra. Then so is $H^*(E;\mathbb{F}_p)$.*

In particular the mod p cohomology of highly connected covers of finite H-spaces is finitely generated as algebra over the Steenrod algebra, see Theorem 4.5. This harmless looking statement was the starting point of this work. In our previous work [CCS07], we announced this result, but referred wrongly to [CCSb] for a proof. The proof of Theorem 4.1 relies on our main result in [CCS07], which allows to "deconstruct" the base space B into Eilenberg-Mac Lane spaces and a mod p finite H-space. We have thus basically to prove the theorem in these two cases. We had previously obtained the result in the case of fibrations over an Eilenberg-Mac Lane space, but we need a stronger result, see Theorem 3.2, based on Smith's work [Smi70] on the Eilenberg-Moore spectral sequence. The key point is that we are able to keep control of the size of certain Hopf subalgebras thank to our main algebraic contribution:

Theorem 2.1. *Let B be an unstable Hopf algebra which is finitely generated as algebra over the Steenrod algebra. Then so is any unstable Hopf subalgebra.*

This reflects a property of André-Quillen homology of unstable Hopf algebras. Therefore we start this paper with the computation of the André-Quillen homology of unstable Hopf algebras which are finitely generated as algebras over the Steenrod algebra and establish Theorem 2.1. We then analyze H-fibrations over Eilenberg-Mac Lane spaces in Section 3 and prove finally Theorem 4.1 in the last section.

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1. André-Quillen homology of Hopf algebras

The functor $Q(-)$ of indecomposable elements takes a graded algebra to a graded vector space and an unstable algebra to an unstable module. To what extent this functor is not left exact is precisely measured by André-Quillen homology $H^Q_*(-)$. In this section, we compute André-Quillen homology for Hopf algebras, and introduce the action of the Steenrod algebra in the next one. A particular case of these calculations has been done in [CCS07] with similar methods, which explains why certain proofs resemble those in that article.

A clear and short introduction to André-Quillen homology can be found in Bousfield's [Bou75, Appendix], see also Goerss' work [Goe90], and of course [Mil84]. We recall briefly from Schwartz's book [Sch94] how one computes André-Quillen homology in our setting. The symmetric algebra comonad $S(-)$ yields a simplicial resolution $S^\bullet(A)$ for any commutative algebra A. The André-Quillen homology group $H^Q_i(A)$ is the i-th homology group of the complex obtained from $S^\bullet(A)$ by taking the module of indecomposable elements (and the differential is the usual alternating sum).

This is a graded $\mathbb{F}_p$-vector space. Long exact sequences arise from certain extensions, just like in the dual situation for the primitive functor, [Bou75, Theorem 3.6].

Lemma 1.1. *Let A be a Hopf subalgebra of a Hopf algebra B of finite type. Then there is a long exact sequence*

$$\cdots \to H_2^Q(B//A) \to H_1^Q(A) \to H_1^Q(B) \to H_1^Q(B//A) \to QA \to QB \to Q(B//A)$$

in André-Quillen homology.

Proof. Long exact sequences in André-Quillen homology are induced by cofibrations of simplicial algebras. However, the inclusion $A \subset B$ of a sub-Hopf algebra is not a cofibration in general (seen as a constant simplicial object). To get around this difficulty we use Goerss' argument from [Goe90, Section 10]: For any morphism $f : A \to B$ of simplicial algebras, there is a spectral sequence, [Goe90, Proposition 4.7], $\mathrm{Tor}_p^{\pi_* A}(\mathbb{F}_p, \pi_* B)_q$ converging to the homotopy groups $\pi_{p+q} Cof(f)$ of the homotopy cofiber. Now, because B is of finite type, it is always a free A-module by the Milnor-Moore result [MM65, Theorem 4.4]. Thus the E_2-term is isomorphic to $\mathrm{Tor}_0^{\mathbb{F}_p}(\mathbb{F}_p, B//A)_* \cong B//A$. The spectral sequence collapses and hence $Cof(f)$ is weakly equivalent to $B//A$. ∎

Following the terminology used in [Smi70, Section 6], we introduce the following definition.

Definition 1.2. A sequence of (Hopf) algebras

$$\mathbb{F}_p \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \mathbb{F}_p$$

is *coexact* if the morphism $A \to B$ is a monomorphism and its cokernel $B//A$ is isomorphic to C as a (Hopf) algebra.

We can thus restate the previous lemma by saying that coexact sequences of Hopf algebras induce long exact sequences in André-Quillen homology.

By the Borel-Hopf decomposition theorem [MM65, Theorem 7.11], any Hopf algebra of finite type is isomorphic, as an algebra, to a tensor product of monogenic Hopf algebras, i.e. either a truncated polynomial algebra of the form $\mathbb{F}_p[x_i]/(x_i^{p^{k_i}})$, where p^{k_i} is the *height* of the generator x_i, or a polynomial algebra of the form $\mathbb{F}_p[y_j]$, or, when p is odd, an exterior algebra $\Lambda(z_i)$. Let us compute $H_1^Q(A)$ as a graded vector space and extract an explicit basis in the symmetric algebra resolution in order to be able in the next section to identify the action of the Steenrod algebra.

Proposition 1.3. *Let A be a Hopf algebra of finite type. Then $H_0^Q(A) = QA$ and $H_1^Q(A)$ is isomorphic to the $\mathbb{F}_p$-vector space generated by the elements $x_i^{\otimes p^{k_i}} \in S(A)$ of degree $p^{k_i} \cdot |x_i|$ where $x_i \in A$ is a generator of height p^{k_i}, $0 < k_i < \infty$. Moreover $H_n^Q(A) = 0$ if $n \geq 2$.*

Proof. Consider the symmetric algebra $S(QA)$ and construct an algebra map $S(QA) \twoheadrightarrow A$ by choosing representatives in A of the indecomposable elements. Let us denote by ξ the Frobenius map, sending an element x to its p-th power x^p. We have then a coexact sequence of algebras $\mathbb{F}_p[\xi^{k_i}x_i] \hookrightarrow S(QA) \twoheadrightarrow A$ and A, as Hopf algebra, can be seen as the quotient $S(QA)//\mathbb{F}_p[\xi^{k_i}x_i]$. Since $S(QA)$ is a free commutative algebra, $H_n^Q\big(S(QA)\big) = 0$ for all $n \geq 1$. Likewise $H_n^Q\big(\mathbb{F}_p[\xi^{k_i}x_i]\big) = 0$ for all $n \geq 1$. Lemma 1.1 yields isomorphisms $H_1^Q(A) \cong H_0^Q(\mathbb{F}_p[\xi^{k_i}x_i]) \cong \oplus_k \mathbb{F}_p\langle \xi^{k_i}x_i\rangle$, as graded vector spaces. It also shows that $H_n^Q(A) = 0$ for $n \geq 2$.

We identify now the generators $\xi^{k_i}x_i$ with explicit elements in the symmetric algebra resolution and compute therefore the first homology group of the complex $Q(S^\bullet(A))$:

$$\cdots \to S^2(A) \xrightarrow{d} S(A) \xrightarrow{m} A\,.$$

The morphisms are given by the alternating sums of the face maps. Let us use the symbols $\otimes$ for the tensor product in $S(A)$ and $\boxtimes$ for the next level in $S(S(A))$. If $\eta_A : S(A) \to A$ is the counit defined by $\eta_A(a \otimes b) = ab$, the two face maps $S^2(A) \to S^1(A)$ are then $S(\eta_A)$ and $\eta_{S(A)}$.

Thus $m(a) = S(\eta_A)(a) - \eta_{S(A)} = a - a = 0$ and $m(a \otimes b) = ab$, since $\eta_{S(A)}(a \otimes b) = a \otimes b$ is decomposable. Likewise $d(w) = \eta_A(w)$ on elements $w \in S(A)$ and $d(v \boxtimes w) = v \otimes w - \eta_A(v) \otimes \eta_A(w)$ for $v, w \in S(A)$. The elements $x_i^{\otimes p^{k_i}}$ clearly belong to the kernel of m. To compare them to the generators $\{\xi^{k_i}x_i\}$ of $H_0^Q(\mathbb{F}_p[\xi^{k_i}x_i]) \cong H_1^Q(A)$, we apply $S^\bullet$ to the coexact sequence of algebras $\mathbb{F}_p[\xi^{k_i}x_i] \hookrightarrow S(QA) \twoheadrightarrow A$. The snake Lemma yields a connecting morphism $\mathrm{Ker}(m) \to H_0^Q(\mathbb{F}_p[\xi^{k_i}x_i])$, which sends precisely $x_i^{\otimes p^{k_i}}$ to $\xi^{k_i}x_i$. ∎

The vanishing of the higher André-Quillen homology groups, or in other words the fact that the functor $Q(-)$ has homological dimension ≤ 1 for Hopf algebras, has been analyzed by Bousfield in the dual situation [Bou75, Theorem 4.1].

Remark 1.4. Alternatively, one could use the identification of the first André-Quillen homology group $H_1^Q(A)$ with the indecomposable elements of degree 2 in $\mathrm{Tor}_A(\mathbb{F}_p, \mathbb{F}_p)$, [Goe90, Section 10]. It is an $\mathbb{F}_p$-vector space generated by the elements $[x_i^{p^{k_i}-1}|x_i]$ defined via the bar construction. Explicit computations can be found for example in [Kan88, Section 29-2].

2. Bringing in the action of the Steenrod algebra

The results of the previous section apply to Hopf algebras which are finitely generated as algebras over the Steenrod algebra: They are of finite type. Our aim in this section is to identify the action of the Steenrod algebra on the unstable module $H_1^Q(A)$. This enables us then to prove our main algebraic result:

Theorem 2.1. *Let B be an unstable Hopf algebra which is finitely generated as algebra over the Steenrod algebra. Then so is any unstable Hopf subalgebra.*

Remark 2.2. For plain unstable algebras, Theorem 2.1 is false, as pointed out to us by Hans-Werner Henn. Consider indeed the unstable algebra

$$H^*(\mathbb{C}P^\infty \times S^2; \mathbb{F}_p) \cong \mathbb{F}_p[x] \otimes E(y)$$

where both x and y have degree 2. Take the ideal generated by y, and add 1 to turn it into an unstable subalgebra. Since $y^2 = 0$, this is isomorphic, as an unstable algebra, to $\mathbb{F}_p \oplus \Sigma^2\mathbb{F}_p \oplus \Sigma^2\widetilde{H}^*(\mathbb{C}P^\infty; \mathbb{F}_p)$, which is not finitely generated.

The $\mathbb{F}_p$-vector space $H_1^Q(A)$ is equipped with an action of $\mathcal{A}_p$ because the Steenrod algebra acts on the symmetric algebra via the Cartan formula. This yields the same unstable module $H_1^Q(A)$ as the derived functor computed with a resolution in the category of unstable algebras, [LS86] and [Sch94, Proposition 7.2.2].

As expected with this type of questions, the case when $p = 2$ is slightly simpler than the case when p is odd. To write a unified proof, we use the well-known trick [LZ86] to consider, in the odd-primary case, the subalgebra of $\mathcal{A}_p$ concentrated in even degrees. If M is a module over $\mathcal{A}_p$, the module M' concentrated in even degrees is defined by $(M')^{2n} = M^{2n}$ and $(M')^{2n+1} = 0$. This is not an $\mathcal{A}_p$-submodule of M, but it is a module over $\mathcal{A}_p$ on which the Bockstein β acts trivially. Hence it can be seen as a module over the algebra $\mathcal{A}'_p$, the subalgebra of $\mathcal{A}_p$ generated by the operations $\mathcal{P}^i$. The category of such objects is denoted $\mathcal{U}'$. When $p = 2$ we adopt the convention that $\mathcal{A}'_2 = \mathcal{A}_2$, $\mathcal{U}' = \mathcal{U}$, and write $\mathcal{P}^i$ for Sq^i. Like in [Sch94, 1.2], for a sequence $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$ where the ε_k's are 0 or 1, we write $\mathcal{P}^I$ for the operation $\beta^{\varepsilon_0}\mathcal{P}^{i_1}\beta^{\varepsilon_1}\dots\mathcal{P}^{i_n}\beta^{\varepsilon_n}$.

In [LZ86, Appendice B], Lannes and Zarati prove that the category $\mathcal{U}$ of unstable modules over $\mathcal{A}_p$ is locally noetherian, which they do by reducing the proof to the case of $\mathcal{U}'$. We prove now a related statement.

Lemma 2.3. *Let M be an unstable module which is finitely generated over $\mathcal{A}_p$. Then so is the module M', over $\mathcal{A}'_p$.*

Proof. The statement is a tautology when $p = 2$. Let us assume that p is an odd prime. In the category $\mathcal{U}$, the object $F(n)$ is by definition the free unstable module on one generator in degree n. Likewise, in the category $\mathcal{U}'$, $F'(2n)$ is the free object on one generator ι_{2n} in degree $2n$, which should not be confused with $F(2n)'$. We must show that M' is a quotient of a finite direct sum of such modules. As we know that M is a quotient of a finite direct sum of $F(n)$'s, it is enough to prove the lemma for the free module $F(n)$.

A basis over $\mathbb{F}_p$ for the module $F(n)$ is given by the elements $\mathcal{P}^I\iota_n$ where I is admissible with excess $e(I) \leq n$. In particular there are at most n ϵ_i's in I which are non zero. As in the second part of the proof of [Sch94, Theorem 1.8.1], we filter $F(n)'$ by sub-$\mathcal{A}'_p$-modules by setting $F(n)'_k$ to be the span over $\mathbb{F}_p$ of the elements $\mathcal{P}^I\iota_n$ with $\sum \epsilon_i \geq k$, where $k = 0, \dots, n+1$. The $\mathcal{A}'_p$-module $F(n)'_k/F(n)'_{k+1}$ is zero when $k + n$ is odd, and, using Adem relations, it is generated over $\mathcal{A}'_p$ by the images of the elements $\mathcal{P}^I\iota_n$ where $\sum \epsilon_i = k$ and $\epsilon_i = 0$ for $i > k$. For each $0 \leq k \leq n$, there is a finite number of such elements and they generate $F(n)'$ as an $\mathcal{A}'_p$-module. ∎

The generators for $H_1^Q(A)$ will be related to certain elements in QA we describe next.

Lemma 2.4. *Let A be an unstable Hopf algebra which is finitely generated as algebra over the Steenrod algebra and let N be the $\mathcal{A}_p$-submodule of QA generated by the truncated polynomial generators x_i. Then N' is finitely generated in $\mathcal{U}'$ and one can choose an integer d and a finite set of generators $\{x_{k,i} \in N' \mid 1 \leq k \leq d, 1 \leq i \leq n_k\}$ such that any element in N' of height p^k can be written $\sum_i \theta_{k,i} x_{k,i}$ for some (admissible) operations $\theta_{k,i} \in \mathcal{A}'_p$.*

Proof. Observe that the property for an unstable algebra A to be a finitely generated algebra over $\mathcal{A}_p$ is equivalent to say that the module of the indecomposable elements QA is finitely generated as unstable module. Since $\mathcal{U}$ is locally noetherian, [Sch94, Theorem 1.8.1], the unstable module N is finitely generated, being a submodule of QA. Thus, by Lemma 2.3, N' is finitely generated over $\mathcal{A}'_p$. This implies in particular that the height of the truncated generators is bounded by some integer p^d (the action of the Steenrod algebra on x_i can only lower the height by the formulas [Sch94, 1.7.1]).

For $1 \leq k \leq d$, write $N'(k)$ for the (finitely generated) submodule of N' generated by the possibly infinite set of the x_i's of height p^k and choose generators $x_{k,i}$, with $1 \leq i \leq n_k$. The finite set $\{x_{k,i} \mid 1 \leq k \leq d, 1 \leq i \leq n_k\}$ generates N'. ∎

In Lemma 2.4, the relation $x = \sum_i \theta_{k,i} x_{k,i}$ holds in the module of indecomposable elements (in fact in N'). Beware that the same relation holds also in the algebra A, but only up to decomposable elements.

Proposition 2.5. *Let A be an unstable Hopf algebra which is finitely generated as algebra over $\mathcal{A}_p$. Then $H_0^Q(A) = QA$ and $H_1^Q(A)$ are both finitely generated unstable modules.*

Proof. The module $H_0^Q(A) = QA$ is finitely generated over $\mathcal{A}_p$ since A is finitely generated as algebra over $\mathcal{A}_p$. Lemma 1.3 allows us to identify $H_1^Q(A) \cong \oplus_k \mathbb{F}_p \langle x_i^{\otimes p^{k_i}} \rangle$, as a graded vector space. We must now identify the action of the Steenrod algebra.

We claim that the finite set of elements $x_{k,i}^{\otimes p^{k_i}}$ generates $H_1^Q(A)$ as unstable module. More precisely we show that the relation $x = \sum_i \theta_{k,i} x_{k,i}$ in QA given by Lemma 2.4 yields a relation for $x^{\otimes p^{k_i}}$ in $H_1^Q(A)$. To simplify the notation, let us assume that the height of x is p^k and that the relation is of the form $x = \sum_j \theta_j x_j$ for generators x_j of the same height. The relation for x holds in A up to decomposable elements which must have lower height. But if $a^{p^k} = 0 = b^{p^k}$, then, with the notation of the proof of Proposition 1.3,

$$d[a^{\otimes p^k} \boxtimes b^{\otimes p^k} - (a \otimes b)^{\boxtimes p^k}] = a^{\otimes p^k} \otimes b^{\otimes p^k} - a^{p^k} \otimes b^{p^k} - (a \otimes b)^{\otimes p^k} + (ab)^{\otimes p^k} = (ab)^{\otimes p^k}$$

and hence the decomposable elements are boundaries and disappear in $H_1^Q(A)$. Therefore $x^{\otimes p^k} = \left(\sum_i \theta_j x_j\right)^{\otimes p^k}$ in $H_1^Q(A)$. Since $(\mathcal{P}^n x)^{\otimes p^k} = \mathcal{P}^{p^k n}(x^{\otimes p^k})$ and because the operations θ_j live in $\mathcal{A}'_p$, there exist operations $\Theta_j \in \mathcal{A}'_p$ such that

$$x^{\otimes p^k} = \sum_j (\theta_j x_j)^{\otimes p^k} = \sum \Theta_j (x_j^{\otimes p^k})$$

and the claim is proven. ∎

Since the higher groups are all trivial (see Proposition 1.3), this gives a quite accurate description of André-Quillen homology in our situation. This can be compared to the result of Lannes and Schwartz, [LS86], that the module of indecomposable elements of an unstable algebra K is locally finite if and only if so are *all* André-Quillen homology groups of K. We are now ready to prove our main algebraic result.

Proof of Theorem 2.1. Consider an unstable Hopf subalgebra $A \subset B$ and the quotient $B//A$. By Lemma 1.1, we have an associated exact sequence in André-Quillen homology

$$H_1^Q(B//A) \to QA \to QB,$$

in which the unstable modules QB and $H_1^Q(B//A)$ are finitely generated by Proposition 2.5. Thus so is QA. ∎

We conclude the section with a computation of André-Quillen homology, which illustrates how the generators related to the truncated polynomial part arise, as explained in Lemma 2.4.

Example 2.6. Let us consider the Hopf algebra $B = H^*(K(\mathbb{Z}/p, 2); \mathbb{F}_p)$. When p is odd, it is the tensor product of a polynomial algebra $\mathbb{F}_p[\iota_2, \beta\mathcal{P}^1\beta\iota_2, \beta\mathcal{P}^p\mathcal{P}^1\beta\iota_2, \cdots]$, concentrated in even degrees, with an exterior algebra $\Lambda(\beta\iota_2, \mathcal{P}^1\beta\iota_2, ...)$. Let A be the Hopf subalgebra given by the image of the Frobenius ξ. This is the polynomial subalgebra

$$\mathbb{F}_p[(\iota_2)^p, (\beta\mathcal{P}^1\beta\iota_2)^p, (\beta\mathcal{P}^p\mathcal{P}^1\beta\iota_2)^p, \dots]$$

The quotient $B//A$ has an exterior part and a truncated polynomial part where all generators have height p. The module of indecomposable elements $Q(B//A)$ is isomorphic to QB. It is a quotient of $F(2)$, and thus generated, as an unstable module, by a single generator ι_2 in degree 2. The submodule concentrated in even degree is a module over $\mathcal{A}'_p$. It is finitely generated as well, by Lemma 2.3, but one needs two generators ι_2 and $\beta\mathcal{P}^1\beta\iota_2$. Explicit computations of the action of the Steenrod algebra can be found in [Cre01].

Therefore $H_1^Q(B//A)$ is an unstable module, which is generated by the elements $\iota_2^{\otimes p}$ and $(\beta\mathcal{P}^1\beta\iota_2)^{\otimes p}$, as we saw in the proof of Proposition 2.5.

3. H-fibrations over Eilenberg-Mac Lane spaces

In the second part of this paper we turn our attention to cohomological finiteness and closure properties for H-fibrations. From now on we write simply $H^*(-)$ instead of $H^*(-; \mathbb{F}_p)$.

Definition 3.1. An H-space B satisfies the *weak cohomological closure property* when, for any H-fibration $F \to E \to B$, the cohomology $H^*(E)$ is finitely generated as algebra over $\mathcal{A}_p$ if so is $H^*(F)$. It satisfies the *strong cohomological closure property* when, for any H-fibration $F \to E \to B$, the cohomology $H^*(E)$ is finitely generated as algebra over $\mathcal{A}_p$ if and only if so is $H^*(F)$.

The aim of this section is to obtain the strong cohomological closure property for Eilenberg-Mac Lane spaces.

Theorem 3.2. *Let A be a finite direct sum of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$, and $n \geq 2$. Consider an H-fibration $F \xrightarrow{i} E \xrightarrow{\pi} K(A,n)$. Then $H^*(F)$ is a finitely generated algebra over $\mathcal{A}_p$ if and only if so is $H^*(E)$.*

We will recover in particular the weak cohomological closure property established in [CCS07, Theorem 6.1]. We first prove that both closure properties are themselves closed under extensions by fibrations.

Lemma 3.3. *Consider an H-space B and assume that there exists an H-fibration $B' \to B \to B''$ such that both B' and B'' satisfy the weak cohomological closure property. Then so does B. The same statement holds for the strong cohomological closure property.*

Proof. Consider an H-fibration $F \to E \to B$ where $H^*(F)$ is finitely generated as algebra over $\mathcal{A}_p$ and construct the following diagram of vertical and horizontal fibrations

$$\begin{array}{ccccc}
F & = & F & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
E' & \longrightarrow & E & \longrightarrow & B'' \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} & & \| \\
B' & \longrightarrow & B & \longrightarrow & B''
\end{array}$$

The weak cohomological closure property for B' implies that $H^*(E')$ is finitely generated as algebra over $\mathcal{A}_p$ and we conclude then by the closure property for B''. The proof for the strong cohomological closure property is analogous. ∎

Given $n \geq 2$, consider now a non-trivial H-fibration $F \xrightarrow{i} E \xrightarrow{\pi} K(A,n)$ where A is either $\mathbb{Z}/p$ or a Prüfer group $\mathbb{Z}_{p^\infty}$. This situation has been extensively and carefully studied by L. Smith in [Smi70]. The following proposition summarizes how the structure of the cohomology of the fiber relates to that of the base and total space.

Proposition 3.4. [Smi70, Proposition 7.3*] *Let $n \geq 2$ and consider an H-fibration $F \xrightarrow{i} E \xrightarrow{\pi} K(A,n)$ where A is either $\mathbb{Z}/p$ or a Prüfer group $\mathbb{Z}_{p^\infty}$. Then there is a coexact sequence of Hopf algebras*

$$\mathbb{F}_p \longrightarrow H^*(E)//\pi^* \xrightarrow{i^*} H^*(F) \longrightarrow R \longrightarrow \mathbb{F}_p,$$

and R is described in turn by a coexact sequence of Hopf algebras

$$\mathbb{F}_p \longrightarrow \Lambda \longrightarrow R \longrightarrow S \longrightarrow \mathbb{F}_p,$$

where Λ is an exterior algebra which is finitely generated as algebra over the Steenrod algebra, and $S \subseteq H^*(K(A,n-1))$ is an unstable Hopf subalgebra.

Proof. The only point which is not explicit in Smith's proposition is the fact that the exterior algebra Λ is finitely generated as algebra over $\mathcal{A}_p$. Let L be the Hopf algebra kernel of π^*, which is finitely generated as algebra over $\mathcal{A}_p$ by Theorem 2.1. It follows from Smith's analysis that Λ is taken over a desuspended subquotient of $(QL)'$, the even degree part of the module of indecomposable elements of L. As $(QL)'$ is finitely generated by Lemma 2.3, so is any subquotient since $\mathcal{U}'$ is locally noetherian. Thus Λ is finitely generated as algebra over $\mathcal{A}_p$. ∎

Proof of Theorem 3.2. If we consider the fibration of Eilenberg-Mac Lane spaces induced by a group extension $A' \to A \to A$, we see from Lemma 3.3 that we can assume that $A = \mathbb{Z}/p$ or $\mathbb{Z}_{p^\infty}$.

Since $H^*(K(A,n))$ is finitely generated as algebra over $\mathcal{A}_p$, so is its image $\mathrm{Im}(\pi^*) \subseteq H^*(E)$. Hence, to prove the theorem, it is enough to show that the module of indecomposable elements $Q(H^*(E)//\pi^*)$ is a finitely generated $\mathcal{A}_p$-module if and only if so is $QH^*(F)$.

Let us now apply Lemma 1.1 to the coexact sequences from Proposition 3.4. The unstable Hopf algebra S is an unstable Hopf subalgebra of $H^*(K(A,n))$. Thus Theorem 2.1 implies that S is finitely generated over $\mathcal{A}_p$, and so is the exterior algebra Λ. The exact sequence in André-Quillen homology for the coexact sequence involving R and Proposition 2.5 show that both QR and $H_1^Q(R)$ are finitely generated unstable modules. Finally, since $\mathcal{U}$ is a locally noetherian category, [Sch94, Theorem 1.8.1], the exactness of the sequence

$$H_1^Q(R) \to Q(H^*(E)//\pi^*) \to QH^*(F) \to QR$$

implies that QH^*F is a finitely generated $\mathcal{A}_p$-module if and only if so is $Q(H^*(E)//\pi^*)$. ∎

In fact, Theorem 3.2 can be easily generalized to p-torsion *H-Postnikov pieces*, i.e. H-spaces which have only finitely many non-trivial homotopy groups.

Corollary 3.5. *Consider an H-fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$, where B is a p-torsion H-Postnikov piece whose homotopy groups are finite direct sums of cyclic groups and Prüfer groups. Then $H^*(E)$ is a finitely generated algebra over $\mathcal{A}_p$, if and only if so is $H^*(F)$.*

Proof. An induction on the number of homotopy groups of B with Lemma 3.3 reduces the proof to the case when B is an Eilenberg-Mac Lane space $K(A,n)$. ∎

The next corollaries also deal with Postnikov pieces. The first one gives a certain control on the size of the cohomology of H-Postnikov pieces. The second one yields a characterization of the H-spaces which satisfy the strong cohomological closure property.

Corollary 3.6. [CCSb, Proposition 2.1] *Let F be an H-Postnikov piece of finite type. Then $H^*(F)$ is finitely generated as algebra over the Steenrod algebra.* ∎

Proposition 3.7. *Let X be an H-space X which satisfies the strong cohomological closure property. Then X is, up to p-completion, a p-torsion Postnikov piece.*

Proof. If X satisfies the strong cohomological closure property, observe that $H^*(\Omega X)$ is a finitely generated $\mathcal{A}_p$-algebra (look at the universal path fibration). But in this case, by [CCS07, Corollary 7.4], ΩX is, up to p-completion, a p-torsion Postnikov piece. ∎

Remark 3.8. Another approach to Theorem 3.2 is to dualize the work of Goerss, Lannes, and Morel in [GLM92, Section 2]. Consider an H-fibration $F \to E \to K(A, n)$. The complex

$$H^*(K(A, n)) \xrightarrow{\pi^*} H^*E \xrightarrow{i^*} H^*F \longrightarrow H^*K(A, n-1).$$

is then exact at H^*E, [Smi67, Proposition 5.5], and its homology at H^*F is isomorphic to $U\Omega_1 N$, where U is Steenrod-Epstein's functor, left adjoint to the forgetful functor $\mathcal{K} \to \mathcal{U}$, Ω_1 is the first left derived functor of Ω, left adjoint of the suspension, and N is a certain quotient of $PH^*K(A, n)$. Theorem 3.2 then follows from the fact that $\Omega_1 N$ is a finitely generated unstable module.

4. Cohomological closure properties of H-fibrations

In this section we explain how our results from [CCS07] allow to reduce the proof of the main theorem to the study of fibrations whose base space is either an Eilenberg-Mac Lane spaces or mod p finite. Our main result is:

Theorem 4.1. *Consider an H-fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$. If $H^*(F)$ and $H^*(B)$ are finitely generated as algebras over the Steenrod algebra, then so is $H^*(E)$.*

Remark 4.2. Theorem 4.1 cannot be improved to an "if and only if" statement. Consider for example the path-fibration for the 3-dimensional sphere $\Omega S^3 \to PS^3 \to S^3$. It is well-known that $H^*(\Omega S^3)$ is a divided power algebra, which is not finitely generated over $\mathcal{A}_p$.

In order to prove Theorem 4.1, we need some input from the theory of localization. Recall (cf. [Far96]) that, given a pointed connected space A, a space X is *A-local* if the evaluation at the base point in A induces a weak equivalence of mapping spaces $\mathrm{map}(A, X) \simeq X$. When X is an H-space, it is sufficient to require that the pointed mapping space $\mathrm{map}_*(A, X)$ be contractible.

Dror-Farjoun and Bousfield have constructed a localization functor P_A from spaces to spaces together with a natural transformation $l : X \to P_A X$ which is an initial map among those having an A-local space as target (see [Far96] and [Bou77]). This functor is known as the A-nullification. Since it commutes with finite products, the map l is an H-map when X is an H-space and its fiber is an H-space. One of the key properties of P_A is that it preserves fibrations whose base space is A-local (see [Far96, Corollary 3.D.3]).

For any elementary abelian group V, tensoring with H^*V has a left adjoint, Lannes' T-functor T_V, [Lan92]. When $V = \mathbb{Z}/p$, the notation T is usually used instead of $T_{\mathbb{Z}/p}$ and $\bar{T}$ is the reduced T-functor, left adjoint to tensoring with the reduced cohomology of $\mathbb{Z}/p$. This allows to characterize the Krull filtration of the category $\mathcal{U}$ of unstable modules as follows: $M \in \mathcal{U}_n$ if and only if $\overline{T}^{n+1} M =$

0, [Sch94, Theorem 6.2.4]. When X is an H-space whose mod p cohomology is finitely generated as algebra over $\mathcal{A}_p$, $TH^*(X) \cong H^*(\mathrm{map}(B\mathbb{Z}/p, X))$ and $\overline{T}QH^*(X) \cong QH^*(\mathrm{map}_*(B\mathbb{Z}/p, X))$, [CCS07].

The interaction between algebraic cohomological properties and the homotopical localization is well illustrated by the following lemma about Bousfield's localization tower. This result is an improvement of [CCS07, Theorem 7.2], where the "if" part was proved.

Lemma 4.3. *Let X be an H-space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Then $H^*(P_{B\mathbb{Z}/p}X)$ is finite if and only if, for some n, $H^*(P_{\Sigma^n B\mathbb{Z}/p}X)$ is a finitely generated $\mathcal{A}_p$-algebra.*

Proof. By [Bou94, Theorem 7.2], there are fibrations $P_{\Sigma^n B\mathbb{Z}/p}X \to P_{\Sigma^{n-1} B\mathbb{Z}/p}X \to K(A_n, n+1)$ for any n, where A_n is a p-torsion abelian group. Since $T_V H^*(X)$ is of finite type, [CCS07, Theorem 5.4] applies and we see that A_n is a finite direct sum of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$. Hence, Theorem 3.2 implies that $H^*(P_{\Sigma^n B\mathbb{Z}/p}X)$ is finitely generated as algebra over $\mathcal{A}_p$ if and only if $H^*(P_{\Sigma^{n-1} B\mathbb{Z}/p}X)$ is so. The statement follows by induction since $H^*(P_{B\mathbb{Z}/p}X)$ is always locally finite, [Sch94, Corollary 8.6.2]. ∎

Our strategy in [CCS07] was to obtain an algebraic characterization of $\Sigma^n B\mathbb{Z}/p$-local H-spaces in terms of the Krull filtration. In a first step towards the proof of Theorem 4.1 we use one implication to understand the total space from a $B\mathbb{Z}/p$-homotopy theoretical point of view.

Lemma 4.4. *Consider an H-fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$. If $H^*(F)$ and $H^*(B)$ are finitely generated as algebras over the Steenrod algebra, then there exists an integer n such that F, E, and B are all $\Sigma^n B\mathbb{Z}/p$-local.*

Proof. Since both $H^*(F)$ and $H^*(B)$ are finitely generated as algebras over $\mathcal{A}_p$, the modules of indecomposable elements $QH^*(F)$ and $QH^*(B)$ are finitely generated $\mathcal{A}_p$-modules. Therefore, [CCS07, Lemma 7.1], they belong to some stage $\mathcal{U}_{n-1}$ of the Krull filtration. By [CCS07, Theorem 5.3], we know that both F and B are $\Sigma^n B\mathbb{Z}/p$-local spaces. Since $\Sigma^n B\mathbb{Z}/p$-localization preserves fibrations whose base space is local (see [Far96, Corollary 3.D.3]), it follows that E is also $\Sigma^n B\mathbb{Z}/p$-local. ∎

Proof of Theorem 4.1. Since $H^*(B)$ is finitely generated as algebra over $\mathcal{A}_p$, we know from [CCS07, Theorem 7.3] that there is an H-fibration $B' \to B \to B''$, where B'' has finite mod p cohomology and the fiber B' is an H-Postnikov piece whose homotopy groups are finite direct sums of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$. Since the theorem is true for such Postnikov pieces by Proposition 3.5, Lemma 3.3 shows then that it is enough to prove the theorem when $H^*(B)$ is finite.

In that case, B is a $B\mathbb{Z}/p$-local space (by Miller's solution to the Sullivan conjecture, [Mil84]). By [Far96, Corollary 3.D.3], we have a diagram of horizontal fibrations:

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \| \\ P_{B\mathbb{Z}/p}F & \longrightarrow & P_{B\mathbb{Z}/p}E & \longrightarrow & B \end{array}$$

The mod p cohomology $H^*(P_{B\mathbb{Z}/p}F)$ is finite by [CCS07, Theorem 7.2] and hence so is $H^*(P_{B\mathbb{Z}/p}E)$ by an easy Serre spectral sequence argument. Moreover, since $H^*(E)$ is of finite type, we see from [CCS07, Proposition 1.1] that $T_V H^*(E)$ is of finite type for any V if and only if $H^*(\text{map}_*(BV, E))$ is so. But since B is $B\mathbb{Z}/p$-local, $\text{map}_*(BV, E) \simeq \text{map}_*(BV, F)$ and the mod p cohomology of this pointed mapping space, which is isomorphic to $T_V H*(F)$, is of finite type because it is finitely generated as algebra over $\mathcal{A}_p$ (by exactness of T_V). By Lemma 4.4 there exists an integer n such that $E \simeq P_{\Sigma^n B\mathbb{Z}/p}E$. We can thus apply Lemma 4.3 to conclude that $H^*(E)$ is finitely generated as algebra over the Steenrod algebra. ∎

Corollary 4.5. *Consider an H-space X with finite mod p cohomology. Then the mod p cohomology of its n-connected cover $X\langle n\rangle$ is finitely generated as algebra over $\mathcal{A}_p$. Moreover, $QH^*X\langle n\rangle$ belongs to $\mathcal{U}_{n-2}$.*

Proof. Consider the H-fibration $\Omega(X[n]) \to X\langle n\rangle \to X$. The fiber is an H-Postnikov piece of finite type and the cohomology of the base is finite. Hence Theorem 4.1 applies. The statement about the Krull filtration follows from by [CCS07, Theorem 5.3], because $\Omega^{n-1}(X\langle n\rangle)$ is $B\mathbb{Z}/p$-local. ∎

This can be seen as the mirror result of [CCSb], where we proved that any H-space with finitely generated cohomology as algebra over the Steenrod algebra is an n-connected cover of an H-space with finite mod p cohomology, up to a finite number of homotopy groups.

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