WREATH PRODUCTS AND REPRESENTATIONS OF p-LOCAL FINITE GROUPS

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Abstract. Given two finite p-local finite groups and a fusion preserving morphism between their Sylow subgroups, we study the question of extending it to a continuous map between the classifying spaces. The results depend on the construction of the wreath product of p-local finite groups which is also used to study p-local permutation representations.

1. Introduction

The concept of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ was introduced in [7] by Broto, Levi and Oliver and a short exposition is given in §2. It consists of a finite p-group S and two categories $\mathcal{F}$ and $\mathcal{L}$ whose objects are subgroups of S. This structure is suitable for studying p-completed classifying spaces of finite groups whose Sylow p-subgroup is S. Every finite group has an associated p-local finite group [7, Proposition 1.3, page 786] but the converse is not true.

In this paper we study maps between classifying spaces of p-local finite groups. Suppose that $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are p-local finite groups. Given a group homomorphism $\rho\colon S \to S'$ it is natural to ask if $B\rho\colon BS \to BS'$ can be extended, up to homotopy, to a map $\tilde{f}\colon |\mathcal{L}|^\wedge_p \to |\mathcal{L}'|^\wedge_p$ such that the following square is homotopy commutative where Θ and Θ' are the natural maps described in §2

$$\begin{array}{ccc} BS & \xrightarrow{\Theta} & |\mathcal{L}|^\wedge_p \\ {\scriptstyle B\rho}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ BS' & \xrightarrow[\Theta']{} & |\mathcal{L}'|^\wedge_p . \end{array}$$

Recall that given fusion systems $\mathcal{F}$ and $\mathcal{F}'$ on S and S' respectively, a homomorphism $\psi\colon S \to S'$ is called *fusion preserving* if for every $\varphi \in \mathcal{F}(P, Q)$ there exists some $\varphi' \in \mathcal{F}'(\psi(P), \psi(Q))$ such that $\psi \circ \varphi = \varphi' \circ \psi$. Ragnarsson shows in [19] that stably, namely in the homotopy category of spectra, $\tilde{f}$ in the diagram above exists if and only if ρ is fusion preserving. Unstably this is unknown.

The content of Theorem 1.3 below is that $\tilde{f}$ exists provided the target $\mathcal{L}'$ is replaced with its wreath product with some symmetric group Σ_n, a construction which we now describe.

Let X be a space, then $G \leq \Sigma_n$ acts on X^n by permuting the factors. The *wreath product* of X with G, denoted $X \wr G$, is the homotopy orbit space $(X^n)_{hG}$ (see Definition 3.4). This construction is equipped with a map $\Delta\colon X \to X \wr G$ which factors through the diagonal map $X \to X^n$. For example, we prove in 3.6

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below that if X is an Eilenberg-MacLane space $K(H,1)$ then there is a homotopy equivalence $X \wr G \simeq K(H \wr G, 1)$ such that Δ is induced by the diagonal inclusion $H \leq H \wr G$.

1.1. **Theorem.** *Fix a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ where $S \neq 1$. Let K be a subgroup of Σ_n and let S' be a Sylow p-subgroup of $S \wr K$. Then there exists a p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ which is equipped with a homotopy equivalence $|\mathcal{L}| \wr K \simeq |\mathcal{L}'|$ such that the composition*

$$BS' \xrightarrow{B incl} B(S \wr K) \simeq (BS) \wr K \xrightarrow{\Theta \wr K} |\mathcal{L}| \wr K \simeq |\mathcal{L}'|$$

is homotopic to the natural map $\Theta' \colon BS' \to |\mathcal{L}'|$. Moreover, $(S', \mathcal{F}', \mathcal{L}')$ satisfying these properties is unique up to an isomorphism of p-local finite groups.

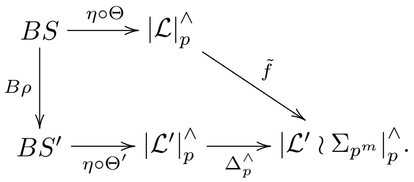
In Remark 5.3 we show that when Theorem 1.1 is applied to a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ of a finite group G then $(S', \mathcal{F}', \mathcal{L}')$ is the p-local finite group of $G \wr K$.

We prove Theorem 1.1 in §5 which is highly technical, however, the remainder of the paper is completely independent of it.

1.2. **Definition.** We call the p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ in the theorem above the *wreath product* of $(S, \mathcal{F}, \mathcal{L})$ with K and denote its fusion system and linking system by $\mathcal{F} \wr K$ and $\mathcal{L} \wr K$ respectively. Let $\Delta \colon |\mathcal{L}| \to |\mathcal{L}| \wr K \simeq |\mathcal{L}'|$ denote the diagonal inclusion followed by the homotopy equivalence in Theorem 1.1.

If $S = 1$ we cannot apply Theorem 1.1, but in this case $|\mathcal{L}| = *$ and we choose $(S', \mathcal{F}', \mathcal{L}')$ to be the p-local finite group associated to K and the map $\Delta \colon |\mathcal{L}| \to |\mathcal{L}'|$ is any map $* \to |\mathcal{L}'|$.

1.3. **Theorem.** *Let $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ be p-local finite groups and suppose that $\rho \colon S \to S'$ is a fusion preserving homomorphism. Then there exists some $m \geq 0$ and a map $\tilde{f} \colon |\mathcal{L}|^\wedge_p \to |\mathcal{L}' \wr \Sigma_{p^m}|^\wedge_p$ such that the diagram below commutes up to homotopy*



A *permutation representation* of a finite group G is a homomorphism $\rho \colon G \to \Sigma_n$. The *rank* of ρ is n. In this paper we shall call ρ simply a "representation". Clearly G acts on itself by left (or right) translations giving rise to Cayley's embedding

$$\mathrm{reg}_G \colon G \to \Sigma_{|G|}$$

which is called the *regular permutation representation* of G.

Two representations $\rho_1, \rho_2 \colon G \to \Sigma_n$ are *equivalent* if they are conjugate in Σ_n, that is, if they differ by an inner automorphism of Σ_n. The set of equivalence classes of representations of G of rank n is denoted $\mathrm{Rep}_n(G)$. The inclusions of subgroups $\Sigma_n \times \Sigma_m \leq \Sigma_{n+m}$ and $\Sigma_n \times \Sigma_m \leq \Sigma_{nm}$ obtained by taking the disjoint union and the product of the sets $[n] = \{1, \ldots, n\}$ and $[m] = \{1, \ldots, m\}$ give rise to commutative, associative and unital binary operations $+$ and $\times$ on the set $\coprod_{n \geq 0} \mathrm{Rep}_n(G)$. We shall write $k \cdot \rho$ for the k-fold sum $\rho + \cdots + \rho$.

A classical result which goes back to Hurewicz states that the classifying space functor induces a bijection

$$\mathrm{Rep}_n(G) \approx [BG, B\Sigma_n], \qquad (\rho \mapsto B\rho).$$

When the target is p-completed, a theorem of Dwyer and Zabrodsky [12] shows that there is also a bijection $\mathrm{Rep}_n(P) \approx [BP, (B\Sigma_n)^\wedge_p]$ when P is a p-group. Therefore, given a map $f \colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$, f admits a representation $\rho \colon S \to \Sigma_n$, unique up to equivalence, which renders the following square homotopy commutative

$$\begin{array}{ccc} BS & \xrightarrow{\Theta} & |\mathcal{L}| \\ {\scriptstyle B\rho}\downarrow & & \downarrow{\scriptstyle f} \\ B\Sigma_n & \xrightarrow[\eta]{} & (B\Sigma_n)^\wedge_p. \end{array}$$

1.4. **Definition.** A *permutation representation* of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is a homotopy class of maps $f \colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$. We say that f is *S-regular* if $n = m \cdot |S|$ for some $m \geq 0$ and ρ in the diagram above is equivalent to $m \cdot \mathrm{reg}_S$.

We shall deduce from Theorem 1.3 the following result which is a p-local form of Cayley's theorem. Recall from [6, Definition 2.2] that a map $f \colon X \to Y$ of spaces is a *homotopy monomorphism at p* if $H^*(X; \mathbb{F}_p)$ is a finitely generated module over $H^*(Y; \mathbb{F}_p)$ via f^*.

1.5. **Theorem.** *Every p-local finite group $(S, \mathcal{F}, \mathcal{L})$ admits an S-regular permutation representation $f \colon |\mathcal{L}| \to (B\Sigma_{p^m})^\wedge_p$ which is a homotopy monomorphism at p.*

The reason we didn't define permutation representations as maps $|\mathcal{L}| \to B\Sigma_n$ (without p-completing the target) is that in general there is little hope to expect to find "interesting" such maps. For example, the nerve of the linking system of the Solomon p-local finite group, constructed by Levi and Oliver in [14], was shown to be simply connected in [10] and therefore [21, Theorem 8.1.11] implies that $[|\mathcal{L}_{\mathrm{Sol}}|, B\Sigma_n] = *$. In particular, the restriction of any $f \colon |\mathcal{L}_{\mathrm{Sol}}| \to B\Sigma_n$ to BS via Θ is induced by the trivial representation $\rho \colon S \to \Sigma_n$.

Let $\mathcal{F}$ be a fusion system on S. A representation $\rho \colon S \to \Sigma_n$ is called *$\mathcal{F}$-invariant* if for every $P \leq S$ and every $\varphi \in \mathcal{F}(P, S)$ the representations $\rho|_P$ and $\rho \circ \varphi$ of P are equivalent. Let $\mathrm{Rep}_n(\mathcal{F})$ denote the set of all the equivalence classes of the $\mathcal{F}$-invariant representations of S of rank n. The inclusions $\Sigma_m \times \Sigma_n \leq \Sigma_{m+n}$ and $\Sigma_m \times \Sigma_n \leq \Sigma_{mn}$ render the sets $\coprod_{n \geq 0} \mathrm{Rep}_n(\mathcal{F})$ with commutative, associative and unital binary operations $+$ and $\times$ such that $+$ is distributive over $\times$.

More generally, the set of *representations at p of rank n of a space* X is $\mathrm{Rep}_n(X) = [X, (B\Sigma_n)^\wedge_p]$. Since $(B\Sigma_m)^\wedge_p \times (B\Sigma_n)^\wedge_p \simeq (B(\Sigma_m \times \Sigma_n))^\wedge_p$ (see [3, Theorem I.7.2]), the maps $(B(\Sigma_m \times \Sigma_n))^\wedge_p \to (B\Sigma_{m+n})^\wedge_p$ and $(B(\Sigma_m \times \Sigma_n))^\wedge_p \to (B\Sigma_{mn})^\wedge_p$ induced by the inclusions equip $\coprod_{n \geq 0} \mathrm{Rep}_n(X)$ with commutative and associative binary operations $+$ and $\times$ such that $+$ is distributive over $\times$.

Given $(S, \mathcal{F}, \mathcal{L})$ we let $\mathrm{Rep}_n(\mathcal{L})$ denote $\mathrm{Rep}_n(|\mathcal{L}|)$.

1.6. **Definition.** The *ring* $\mathrm{Rep}(\mathcal{L})$ *of the virtual permutation representations* of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the Grothendieck group completion of the commutative monoid $(\coprod_{n \geq 0} \mathrm{Rep}_n(\mathcal{L}), +)$.

The *ring* $\operatorname{Rep}(\mathcal{F})$ *of the virtual* $\mathcal{F}$*-invariant representations of* S of a saturated fusion system $\mathcal{F}$ on S is the Grothendieck group completion of the commutative monoid $(\coprod_{n\geq 0} \operatorname{Rep}_n(\mathcal{F}), +)$.

Clearly $\operatorname{Rep}(\mathcal{F})$ is a subring of $\operatorname{Rep}(S)$. In §8 we will construct a ring homomorphism $\Phi\colon \operatorname{Rep}(\mathcal{L}) \to \operatorname{Rep}(\mathcal{F})$ which sends a map $f\colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ to the representation $\rho\colon S \to \Sigma_n$ such that $f \circ \Theta \simeq \eta \circ B\rho$ as in Definition 1.4. We shall also see that $\operatorname{reg}_S\colon S \to \Sigma_{|S|}$ generates an ideal $\operatorname{Rep}^{\mathrm{reg}}(\mathcal{F})$ in $\operatorname{Rep}(\mathcal{F})$ whose underlying group is isomorphic to $\mathbb{Z}$.

The idea behind the next definition is that if H is a subgroup of index n in a finite group G then $\operatorname{reg}_G|_H \simeq n \cdot \operatorname{reg}_H$. Therefore the image of the restriction map $\operatorname{Rep}(G) \to \operatorname{Rep}(H)$ intersects $\operatorname{Rep}^{\mathrm{reg}}(H) := \{k \cdot \operatorname{reg}_H\}_{k\in\mathbb{Z}}$ in a subgroup of index divisible by n.

1.7. **Definition.** The lower index of S in $\mathcal{L}$ denoted $\operatorname{Lind}(\mathcal{L}\colon S)$ is the index of $\operatorname{Im}(\Phi) \cap \operatorname{Rep}^{\mathrm{reg}}(\mathcal{F})$ in $\operatorname{Rep}^{\mathrm{reg}}(\mathcal{F})$.

We will prove in Lemma 8.5 that $\operatorname{Lind}(\mathcal{L}\colon S)$ is a p-power. We conjecture that it is always equal to 1. A partial result is the theorem below.

1.8. **Theorem.** *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a p-local finite group. Then* $\operatorname{Lind}(\mathcal{L}\colon S) = 1$ *if either*

(1) $(S, \mathcal{F}, \mathcal{L})$ *is associated with a finite group.*
(2) $(S, \mathcal{F}, \mathcal{L})$ *is one of the exotic examples in* [20] *or in* [7] *or in* [8].

2. Preliminaries on p-local finite groups

We start with the notion of a saturated fusion system which is due to Puig [17] (see also [7]).

2.1. **Definition.** A fusion system $\mathcal{F}$ on a finite p-group S is a category whose objects are the subgroups of S and the set of morphisms $\mathcal{F}(P, Q)$ between two subgroups P, Q, satisfies the following conditions:

(a) $\mathcal{F}(P, Q)$ consists of group monomorphisms and contains the set $\operatorname{Hom}_S(P, Q)$ of all the homomorphisms $c_s\colon P \to Q$ which are induced by conjugation by elements $s \in S$.
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

In a fusion system $\mathcal{F}$ over a p-group S, we say that two subgroups $P, Q \leq S$ are $\mathcal{F}$-conjugate if there is an isomorphism between them in $\mathcal{F}$. Let $\operatorname{Syl}_p(G)$ the set of the Sylow p-subgroups of a group G. Given $P \leq G$ and $g \in G$, $c_g \in \operatorname{Hom}(P, G)$ is the monomorphism $c_g(x) = gxg^{-1}$. We write $\operatorname{Out}_\mathcal{F}(P) = \operatorname{Aut}_\mathcal{F}(P)/\operatorname{Inn}(P)$.

2.2. **Definition.** Let $\mathcal{F}$ be a fusion system on a p-group S. A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P. A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.

A fusion system $\mathcal{F}$ on S is *saturated* if:

(I) Each fully normalized subgroup $P \leq S$ is fully centralized and $\operatorname{Aut}_S(P) \in \operatorname{Syl}_p(\operatorname{Aut}_\mathcal{F}(P))$.

(II) For $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$ set
$$N_\varphi = \{g \in N_S(P) | \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\}.$$
If $\varphi(P)$ is fully centralized then there is $\bar{\varphi} \in \mathcal{F}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

2.3. **Definition.** Let $\mathcal{F}$ be a fusion system on a p-group S. A subgroup $P \leq S$ is $\mathcal{F}$-*centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers. A subgroup $P \leq S$ is $\mathcal{F}$-*radical* if $\mathrm{Out}_\mathcal{F}(P)$ has no non-trivial normal p-subgroup.

2.4. **Definition.** [7] Let $\mathcal{F}$ be a fusion system on a p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor $\pi \colon \mathcal{L} \longrightarrow \mathcal{F}^c$ and monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_\mathcal{L}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions:

(A) π is the identity on objects. For each pair of objects $P, Q \in \mathcal{L}$, the action of $Z(P)$ on $\mathcal{L}(P, Q)$ via precomposition and $\delta_P \colon P \to \mathrm{Aut}_\mathcal{L}(P)$ is free and π induces a bijection $\mathcal{L}(P, Q)/Z(P) \xrightarrow{\cong} \mathcal{F}(P, Q)$.
(B) If $P \leq S$ is $\mathcal{F}$-centric then $\pi(\delta_P(g)) = c_g \in \mathrm{Aut}_\mathcal{F}(P)$ for all $g \in P$.
(C) For each $f \in \mathcal{L}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)} \downarrow & & \downarrow {\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow[f]{} & Q \end{array} \quad .$$

A *p-local finite group* $(S, \mathcal{F}, \mathcal{L})$ consists of a saturated fusion systems $\mathcal{F}$ on S together with an associated linking system.

2.5. **Remark.** For $P, Q \leq S$, let $N_S(P, Q)$ denote the set of the elements $s \in S$ such that $sPs^{-1} \leq Q$. In [7, Proposition 1.11] it is shown that $(S, \mathcal{F}, \mathcal{L})$ can be equipped with injections $\delta_{P,Q} \colon N_S(P, Q) \to \mathcal{L}(P, Q)$ where $P, Q \leq S$ are $\mathcal{F}$-centric such that $\delta_{P,P}$ extends the monomorphisms $\delta_P \colon P \to \mathrm{Aut}_\mathcal{L}(P)$. We denote $\delta_{P,Q}(s)$ by $\hat{s} \in \mathcal{L}(P, Q)$. The construction of the $\delta_{P,Q}$'s has the property that $\hat{s_1} \circ \hat{s_2} = \widehat{s_1 s_2}$. Also, if $P \leq Q$ we write ι_P^Q for $\delta_{P,Q}(1)$. This gives a choice of lifts in $\mathcal{L}$ for the inclusion of $\mathcal{F}$-centric subgroups in $\mathcal{F}$. This choice is "compatible" in the sense that $\iota_Q^R \circ \iota_P^Q = \iota_P^R$.

2.6. **Remark.** Every morphism in $\mathcal{L}$ is both a monomorphism and an epimorphism (but not necessarily an isomorphism). This is shown in [7, remarks after Lemma 1.10] and [4, Corollary 3.10]. We shall use this fact repeatedly throughout.

The orbit category of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is denoted by $\mathcal{O}(\mathcal{F})$. This is the category whose objects are the subgroups of S and whose morphisms are
$$\mathcal{O}(\mathcal{F})(P, Q) = \mathrm{Rep}_\mathcal{F}(P, Q) \overset{def}{=} \mathrm{Inn}(Q) \setminus \mathcal{F}(P, Q).$$
Also, $\mathcal{O}(\mathcal{F}^c)$ is the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of S.

2.7. **Proposition.** [7, Proposition 2.2] *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. There exists a functor $\widetilde{B} \colon \mathcal{O}(\mathcal{F}^c) \to$ **Top** which is isomorphic in the homotopy category of spaces to the functor $P \mapsto BP$, and such that there is a homotopy equivalence*
$$\underset{\mathcal{O}(\mathcal{F}^c)}{\mathrm{hocolim}}\, \widetilde{B} \xrightarrow{\simeq} |\mathcal{L}|.$$

2.8. **Notation.** For a finite group G, let $\mathcal{B}G$ denote the category with one object $\bullet_G$ and G as its set of automorphisms. For an $\mathcal{F}$-centric $P \leq S$ the monomorphism δ_P gives rise to a functor $\mathcal{B}P \to \mathcal{L}$ which, by abuse of notation, we denote by δ_P. For $P = S$, upon taking nerves of categories, we obtain a map

$$\Theta\colon BS \to |\mathcal{L}|$$

and we write $\Theta|_{BQ}$ for $\Theta \circ B\mathrm{incl}_Q^S$.

If Q is $\mathcal{F}$-centric, then the natural isomorphism of functors in Proposition 2.7 shows that $\Theta|_{BQ}$ is homotopic to $BQ \simeq \tilde{B}(Q) \to \mathrm{hocolim}_{\mathcal{O}(\mathcal{F}^c)}\tilde{B} = |\mathcal{L}|$. Therefore, for any $\mathcal{F}$-centric $Q \leq S$ and any morphism $\rho\colon Q \to S$ in $\mathcal{F}$ we have $\Theta \circ B\rho \simeq \Theta|_{BQ}$. In particular, $\Theta|_{BQ'} \circ B\psi \simeq \Theta|_{BQ}$ for any $\psi \in \mathrm{Iso}_{\mathcal{F}}(Q, Q')$. It follows from Alperin's fusion theorem for saturated fusion systems [7, Theorem A.10] that:

2.9. **Proposition.** *For any $Q, Q' \leq S$ and any $\rho \in \mathcal{F}(Q, Q')$ there is a homotopy equivalence $\Theta|_{BQ'} \circ B\rho \simeq \Theta|_{BQ}$.*

2.10. **Notation.** Given a map $f\colon X \to Y$ of spaces, let $\mathrm{map}^f(X, Y)$ denote the path component of f in $\mathrm{map}(X, Y)$. By convention f is the basepoint of this space.

The following proposition on mapping spaces will be needed in §7.

2.11. **Proposition.** *Fix a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and let P be a finite p-group. Given a homomorphism $\rho\colon P \to S$, set $Q = \rho(P) \leq S$. Then:*

(a) *There is a homotopy equivalence*

$$\mathrm{map}^{\eta\circ\Theta\circ B\rho}(BP, |\mathcal{L}|_p^\wedge) \simeq \mathrm{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge),$$

and this space is the p-completed classifying space of a p-local finite group.

(b) *After p-completion, the map*

$$\mathrm{map}^{\Theta|_{BQ}}(BQ, |\mathcal{L}|) \xrightarrow{\eta_*} \mathrm{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge).$$

induces a split surjection on homotopy groups.

Proof. (a) First of all, we can choose a fully centralized subgroup $Q' \leq S$ in $\mathcal{F}$ and an isomorphism $\psi\colon Q \to Q'$ in $\mathcal{F}$. Let $\rho'\colon P \to S$ denote the composition $P \xrightarrow{\rho} Q \xrightarrow{\psi} Q' \leq S$. By Proposition 2.9 observe that

$$\Theta|_{BQ} \simeq \Theta|_{BQ'} \circ B\psi. \tag{1}$$

Hence, $\Theta \circ B\rho \simeq \Theta \circ B\rho'$. It follows from [7, Theorem 6.3] that there are homotopy equivalences

$$\mathrm{map}^{\eta\circ\Theta\circ B\rho}(BP, |\mathcal{L}|_p^\wedge) \simeq \mathrm{map}^{\eta\circ\Theta\circ B\rho'}(BP, |\mathcal{L}|_p^\wedge) \simeq \\ \mathrm{map}^{\eta\circ\Theta|_{BQ'}}(BQ', |\mathcal{L}|_p^\wedge) \simeq \mathrm{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge)$$

where the first equivalence is implied by equation (1) and the third one follows since $B\psi\colon BQ \to BQ'$ is a homotopy equivalence. Also by [7, Theorem 6.3], this space is homotopy equivalent to the classifying space of a p-local finite group $|C_{\mathcal{L}}(Q')|_p^\wedge$.

(b) We can assume from (1), by replacing Q with Q' if necessary, that Q is fully centralised in $\mathcal{F}$. In [7, pp. 822] a functor

$$\Gamma\colon C_{\mathcal{L}}(Q) \times \mathcal{B}Q \to \mathcal{L}$$

is constructed where $C_{\mathcal{L}}(Q)$ is the centraliser linking system [7, Definition 2.4] of Q in $\mathcal{F}$. By p-completing the geometric realisation of Γ and taking adjoints we obtain a commutative square in which the bottom row is a homotopy equivalence by [7, Theorem 6.3]

$$\begin{array}{ccc} |C_{\mathcal{L}}(Q)| & \xrightarrow{|\Gamma|^{\#}} & \operatorname{map}^{\Theta|_{BQ}}(BQ, |\mathcal{L}|) \\ \eta \big\downarrow & & \big\downarrow \eta_* \\ |C_{\mathcal{L}}(Q)|^\wedge_p & \underset{\simeq}{\xrightarrow{(|\Gamma|^\wedge_p)^{\#}}} & \operatorname{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|^\wedge_p). \end{array} \tag{2}$$

Since $|C_{\mathcal{L}}(Q)|$ is p-good by [7, Proposition 1.12], upon p-completion of the diagram (2), we see that the vertical arrow on the left becomes an equivalence and therefore the composition $(\eta_*)^\wedge_p \circ (|\Gamma|^{\#})^\wedge_p$ is a homotopy equivalence. In particular $(\eta_*)^\wedge_p$ is split surjective on homotopy groups. □

We end this section with a description of the product of p-local finite groups.

2.12**.** Let $\mathcal{F}_i$ be a saturated fusion system on a finite p-group S_i for $i = 1, \dots, n$. Define $S = \prod_{i=1}^n S_i$ and consider the product category $\prod_{i=1}^n \mathcal{F}_i$. Its objects are the subgroups of S of the form $\prod_i P_i$ where $P_i \le S_i$, and morphisms have the form $\prod_i P_i \xrightarrow{\prod_i \varphi_i} \prod_i Q_i$ where $\varphi_i \in \mathcal{F}_i(P_i, Q_i)$.

2.13. **Notation.** For $P \le S = \prod_{i=1}^n S_i$, we denote by $P^{(i)}$ the image of P under the projection $p^{(i)} \colon S \to S_i$. Clearly $P \le \prod_{i=1}^n P^{(i)}$.

Let $\mathcal{F}$ be the fusion system on S generated by $\prod_i \mathcal{F}_i$. Thus, every morphism $\varphi \in \mathcal{F}(P,Q)$ is given by the restriction of a morphism $\prod_i P^{(i)} \xrightarrow{\prod_i \varphi_i} \prod_i Q^{(i)}$ in $\prod_i \mathcal{F}_i$. The φ_i's are unique in the sense that they are completely determined by φ because $p^{(i)}|_P \colon P \to P^{(i)}$ are by definition surjective and $p^{(i)}|_Q \circ \varphi = \varphi_i \circ p^{(i)}|_P$. We see that $\varphi \mapsto (\varphi_i)_{i=1}^n$ induces an inclusion $\mathcal{F}(P,Q) \subseteq \prod_i \mathcal{F}_i(P^{(i)}, Q^{(i)})$. In particular, $\prod_i \mathcal{F}_i$ is a full subcategory of $\mathcal{F}$.

We shall write $\times_{i=1}^n \mathcal{F}_i$ for the fusion system $\mathcal{F}$ just defined and we call it the product fusion system of the $\mathcal{F}_i$'s.

2.14. **Lemma.** *With the notation above, $(S, \mathcal{F})$ is a saturated fusion system. If $P \le S$ is $\mathcal{F}$-centric then all the groups $P^{(i)}$ are $\mathcal{F}_i$-centric for $i = 1, \dots, n$.*

The assignment $P \mapsto \prod_i P^{(i)}$ and the inclusions $\mathcal{F}(P,Q) \subseteq \prod_i \mathcal{F}_i(P^{(i)}, Q^{(i)})$ give rise to a functor $r \colon \mathcal{F}^c \to \prod_i \mathcal{F}_i^c$ which is a retract of the inclusion $\prod_i \mathcal{F}_i^c \subseteq \mathcal{F}^c$.

Proof. In [7, Lemma 1.5] it is proven that $\mathcal{F} = \times_i \mathcal{F}_i$ is a saturated fusion system on S.

The assignments $P \mapsto \prod_i P^{(i)}$ and $\varphi \mapsto \prod \varphi_i$ give rise to a functor $r \colon \mathcal{F} \to \prod_i \mathcal{F}_i$ which by inspection is a retraction to the inclusion $j \colon \prod_i \mathcal{F}_i \to \mathcal{F}$. It remains to show that j and r restrict to $\prod_i \mathcal{F}_i^c$ and $\mathcal{F}^c$.

Observe that $C_S(P) = \prod_i C_{S_i}(P^{(i)})$ for any $P \le S$. If P is $\mathcal{F}$-centric then

$$\prod_{i=1}^n C_{S_i}(P^{(i)}) = C_S(P) \le P \le \prod_{i=1}^n P^{(i)}. \tag{1}$$

Therefore $C_{S_i}(P^{(i)}) \le P^{(i)}$ for all i. Now, if Q_i are $\mathcal{F}_i$-conjugate to $P^{(i)}$ via isomorphisms $\varphi_i \in \mathcal{F}_i(P^{(i)}, Q_i)$ then $(\varphi_1 \times \ldots \times \varphi_n)|_P$ is an $\mathcal{F}$-isomorphism onto some $Q \le S$ such that $Q^{(i)} = Q_i$. By definition Q is also $\mathcal{F}$-centric and applying (1) to Q we obtain that $C_{S_i}(Q_i) \le Q_i$ for all i. We deduce that $P^{(i)}$ are $\mathcal{F}_i$-centric.

Assume now that $P_i \leq S_i$ are $\mathcal{F}_i$-centric for all $i = 1, \dots, n$. Then $P = \prod_i P_i$ is $\mathcal{F}$-centric because if Q is $\mathcal{F}$-conjugate to P then it has the form $\prod_i Q_i$ where Q_i are $\mathcal{F}_i$-conjugate to P_i and therefore $C_S(Q) = \prod_i C_{S_i}(Q_i) \leq Q$. $\square$

While the construction of the product of saturated fusion systems appears in [7], we were not able to find a construction of the product of p-local finite groups in the literature.

2.15. **Definition.** Let $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ be p-local finite groups for $i = 1, \dots, n$. Their product $\times_{i=1}^n (S_i, \mathcal{F}_i, \mathcal{L}_i)$ is the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ where $S = \prod_{i=1}^n S_i$ and $\mathcal{F} = \times_{i=1}^n \mathcal{F}_i$. The centric linking system $\mathcal{L} = \times_{i=1}^n \mathcal{L}_i$ is defined as the following pullback of small categories where r is defined in Lemma 2.14

$$\begin{array}{ccc} \times_{i=1}^n \mathcal{L}_i & \xrightarrow{r_{\mathcal{L}}} & \prod_{i=1}^n \mathcal{L}_i \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle \prod_{i=1}^n \pi_i} \\ (\times_{i=1}^n \mathcal{F}_i)^c & \xrightarrow[r]{} & \prod_{i=1}^n \mathcal{F}_i^c. \end{array}$$

The functor $\pi\colon \mathcal{L} \to \mathcal{F}$ is defined by the pullback and the monomorphisms $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ are defined by the compositions

$$P \leq \prod_i P^{(i)} \xrightarrow{\prod_i \delta_{P^{(i)}}} \prod_i \mathrm{Aut}_{\mathcal{L}_i}(P^{(i)}).$$

We need to prove that axioms (A)-(C) of Definition 2.4 hold.

Proof. We first note that for any $\mathcal{F}$-centric subgroups $P, Q \leq S$ the set $\mathcal{L}(P,Q)$ is the pullback in

(1)
$$\begin{array}{ccc} \mathcal{L}(P,Q) & \hookrightarrow & \prod_{i=1}^n \mathcal{L}_i(P^{(i)}, Q^{(i)}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle \prod_i \pi} \\ \times_{i=1}^n \mathcal{F}_i(P,Q) & \xhookrightarrow{r} & \prod_{i=1}^n \mathcal{F}_i(P^{(i)}, Q^{(i)}). \end{array}$$

We start by proving that the monomorphisms δ_P are well-defined. That is, given $g = (g_i) \in P \leq S$ where P is $\mathcal{F}$-centric, $\prod_i \delta_{P^{(i)}}(g_i) \in \mathrm{Aut}_{\mathcal{L}}(P)$. The pullback diagram (1) shows that it is enough to check that $\prod \pi_i(\delta_{P^{(i)}}(g_i)) \in r((\times_{i=1}^n \mathcal{F}_i)^c)$. It follows from the fact that $\pi_i(\delta_{P^{(i)}}(g_i)) = c_{g_i} \in \mathrm{Aut}_{\mathcal{F}_i}(P^{(i)})$ and $r(c_g) = \prod c_{g_i}$. This also shows that axiom (B) holds since $\pi(\delta_P(g)) = \prod \pi_i(\delta_{P^{(i)}}(g_i))|_P = c_g|_P$.

We continue to prove that $(S, \mathcal{F}, \mathcal{L})$ satisfies axioms (A) and (C). It follows from the definition that π is the identity on objects. Observe that $\prod_i C_{S_i}(P^{(i)})$ acts transitively and freely on the fibre of the right-hand arrow in (1) because axiom (A) holds in $(S_i, \mathcal{F}_i, \mathcal{L}_i)$. Now, axiom (A) for $(S, \mathcal{F}, \mathcal{L})$ follows from the fact that $C_S(P) = \prod_i C_{S_i}(P^{(i)})$ and that diagram (1) is a pullback square so the fibres of the vertical arrows are isomorphic.

Finally, axiom (C) for $(S, \mathcal{F}, \mathcal{L})$ follows by applying axiom (C) to each component of a morphism $f \in \mathcal{L}(P,Q)$ and each $g \in P \leq \prod_i P^{(i)}$. $\square$

2.16. **Remark.** A choice of compatible lifts for inclusion $\{\iota_{P_i}^{Q_i}\}$ in every $\mathcal{L}_i$ (see 2.5) gives rise to a choice $\{\iota_P^Q\}$ of compatible lifts for the inclusions in $(S, \mathcal{F}, \mathcal{L})$ where $\iota_P^Q = (\iota_{P^{(i)}}^{Q^{(i)}})_{i=1}^n$.

2.17. **Proposition.** *Given p-local finite groups $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ for $i = 1, \dots, n$, the category $\prod_i \mathcal{L}_i$ is a full subcategory of $\times_i \mathcal{L}_i$ and the inclusion $j \colon \prod_i \mathcal{L}_i \to \times_i \mathcal{L}_i$ induces a homotopy equivalence on nerves. In particular, $\prod_{i=1}^n |\mathcal{L}_i| \simeq | \times_{i=1}^n \mathcal{L}_i|$.*

Proof. Set $\mathcal{L} = \times_{i=1}^n \mathcal{L}_i$. The category $\prod_i \mathcal{L}_i$ is a full subcategory of $\mathcal{L}$ by Definition 2.15 and the fact that $\prod_i \mathcal{F}_i$ is a full subcategory of $\times_i \mathcal{F}_i$. The assignment $P \mapsto \prod_i P^{(i)}$ and the inclusion $\mathcal{L}(P, Q) \subseteq \prod_{i=1}^n \mathcal{L}_i(P^{(i)}, Q^{(i)})$ give rise to a functor $r_{\mathcal{L}} \colon \mathcal{L} \to \prod_{i=1}^n \mathcal{L}_i$ (see the pullback diagram in Definition 2.15) which is a retract to the inclusion j by Lemma 2.14. Also there is a natural transformation $\mathrm{Id} \to j \circ r$ which is defined on an object $P \in \mathcal{L}$ by $\iota_P^{r(P)} \colon P \to r(P) = \prod_{i=1}^n P^{(i)}$ (see Remark 2.16). This shows that $|r|$ is a homotopy inverse to $|j| \colon \prod_i |\mathcal{L}_i| \to |\mathcal{L}|$. □

2.18. **Remark.** Given a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, Definition 2.15 allows us to consider its n-fold product with itself denoted $(S^{\times n}, \mathcal{F}^{\times n}, \mathcal{L}^{\times n})$. By construction, the action of the symmetric group Σ_n on $S^{\times n}$ extends to an action on the fusion system $\mathcal{F}^{\times n}$ and the linking system $\mathcal{L}^{\times n}$ by permuting the factors. Moreover, the functor $\pi \colon \mathcal{L}^{\times n} \to \mathcal{F}^{\times n}$ and the distinguished monomorphisms $\delta_P \colon P \to \mathrm{Aut}_{\mathcal{L}^{\times n}}(P)$ for every $\mathcal{F}^{\times n}$-centric $P \le S^{\times n}$ are Σ_n-equivariant from the construction in Definition 2.15. Therefore, also the inclusion $\mathcal{B}S^{\times n} \xrightarrow{\delta_{S^{\times n}}} \mathcal{B}\mathrm{Aut}_{\mathcal{L}^{\times n}}(S^{\times n}) \to \mathcal{L}^{\times n}$ is Σ_n-equivariant and so is the induced map $\Theta \colon BS^{\times n} \to |\mathcal{L}^{\times n}| \simeq |\mathcal{L}|^{\times n}$.

The choice of ι_P^Q in $\mathcal{L}^{\times n}$ made in Remark 2.16 is easily seen to be invariant under the action of Σ_n as well.

Finally, the functor j and the homotopy equivalence in Proposition 2.17 are also equivariant with respect to the action of Σ_n by permuting coordinates.

3. The wreath product of spaces

Let G be a finite group and X a G-space. The Borel construction X_{hG} is the orbit space of $EG \times X$ where EG is a contractible space on which G acts freely on the right. Recall from 2.8 that $\mathcal{B}G$ is the small category with one object and G as a morphism set. Then X can be viewed as a functor $X \colon \mathcal{B}G \to Top$ and the Borel construction is a model for $\mathrm{hocolim}_{\mathcal{B}G} X$. There is a natural map $X_{hG} \to X/G$ to the orbit space of X induced by the map $EG \to *$.

A standard model for EG is given by the nerve of the category $\mathcal{E}G$ whose object set is G and there exists a unique morphism between any two objects. This construction is natural so that if $H \le G$ then EH is an H-subspace of EG. Moreover, the identity element of G renders EG with a natural choice of a basepoint (which is not invariant under G.) This basepoint provides an augmentation map $\kappa(X) \colon X \to X_{hG}$ which fits into a fibration sequence

$$X \xrightarrow{\kappa(X)} X_{hG} \to BG. \tag{3.1}$$

A fixed point $x \in X$ corresponds to a G-map $* \to X$ and gives rise to a section $s \colon BG \to X_{hG}$ for this fibration.

If $N \lhd G$ then $EG \times_N X$ is a model for X_{hN} on which G/N acts freely in a natural way. As a consequence we obtain a composite homotopy equivalence

$$(X_{hN})_{hG/N} \xrightarrow{\simeq} (EG \times_N X)_{hG/N} \xrightarrow{\simeq} (EG \times_N X)/_{\frac{G}{N}} = EG \times_G X = X_{hG}. \tag{3.2}$$

Moreover, note that $(EG \times_N X)/_{\frac{G}{N}} = EG \times_G X = X_{hG}$ and that the composition in the bottom row of the following commutative diagram is by inspection equal to the map $\kappa\colon X \to X_{hG}$

$$\begin{array}{ccccccc}
X & \xrightarrow{\kappa} & X_{hN} = EN \times_N X & \xrightarrow{\kappa} & (EN \times_N X)_{hG/N} & & \\
\| & & \downarrow{\simeq} & & i \downarrow{\simeq} & \searrow^{\simeq} & \\
X & \xrightarrow[\kappa]{} & EG \times_N X & \xrightarrow[\kappa]{} & (EG \times_N X)_{hG/N} & \xrightarrow[\simeq]{\pi} & (EG \times_N X)/_{\frac{G}{N}} = X_{hG}.
\end{array}$$

This shows that

(3.3) $\qquad X \xrightarrow{\kappa} X_{hN} \xrightarrow{\kappa} (X_{hN})_{hG/N} \xrightarrow[\simeq]{\pi\circ i} X_{hG} \qquad$ is equal to $\qquad X \xrightarrow{\kappa} X_{hG}.$

3.4. **Definition.** The wreath product of a space X with a subgroup G of Σ_k is the space

$$X \wr G := (X^{\times k})_{hG}$$

where G acts by permuting the factors of $X^{\times k}$. The diagonal map $\Delta_X\colon X \to X^{\times k}$ and $\kappa\colon X^{\times k} \to X \wr G$ give rise to a natural map

$$\Delta(X)\colon X \to X \wr G.$$

We shall use a left normed notation for iteration of the wreath product construction. That is, by convention, $X \wr G_1 \wr G_2 \wr \cdots \wr G_n$ denotes $(\cdots((X \wr G_1) \wr G_2) \wr \cdots) \wr G_n$.

3.5. **Proposition.** *Given permutation groups $G_i \leq \Sigma_{k_i}$ where $i = 1, \ldots, n$, there is a homotopy equivalence*

$$\alpha_n\colon X \wr G_1 \wr G_2 \wr \cdots \wr G_n \xrightarrow{\simeq} X \wr (G_1 \wr G_2 \wr \cdots \wr G_n)$$

which is natural in X. Moreover, the composition

$$X \xrightarrow{\Delta} X \wr G_1 \xrightarrow{\Delta} (X \wr G_1) \wr G_2 \xrightarrow{\Delta} \cdots \xrightarrow{\Delta} X \wr G_1 \wr G_2 \wr \cdots \wr G_n \xrightarrow[\simeq]{\alpha_n} X \wr (G_1 \wr G_2 \wr \cdots \wr G_n)$$

is homotopic to $\Delta\colon X \to X \wr (G_1 \wr G_2 \wr \cdots \wr G_n)$.

Proof. We start with $n = 2$. Define $G = G_1 \wr G_2$ and set $N = G_1^{\times k_2}$. Since $EN = (EG_1)^{\times k_2}$, we obtain a homeomorphism

$$((X^{\times k_1})_{hG_1})^{\times k_2} \cong (X^{\times k_1 k_2})_{hN}$$

which is Σ_{k_2}-equivariant and where N acts on $\prod_{k_2} X^{\times k_1}$ via k_2 copies of the action of G_1 on $X^{\times k_1}$. Clearly $G/N \cong G_2 \leq \Sigma_{k_2}$ acts on this space by permuting the factors and the homotopy equivalence $\alpha_2\colon X \wr G_1 \wr G_2 \simeq X \wr (G_1 \wr G_2)$ is defined with the aid of (3.2) by

$$(((X^{\times k_1})_{hG_1})^{\times k_2})_{hG_2} = ((X^{\times k_1 k_2})_{hN})_{hG/N} \xrightarrow{\simeq} (X^{\times k_1 k_2})_{hG}.$$

Furthermore the triangle below commutes by (3.3)

(1)
$$\begin{array}{ccccc}
X & \xrightarrow{\Delta(x)} & X \wr G_1 & \xrightarrow{\Delta(X \wr G_1)} & X \wr G_1 \wr G_2 \\
& \searrow_{\Delta(X)} & & & \downarrow{\alpha_2} \\
& & & & X \wr (G_1 \wr G_2).
\end{array}$$

We define α_n for $n \geq 2$ inductively by the composition

$$X \wr G_1 \wr \cdots \wr G_n \xrightarrow{\alpha_{n-1} \wr G_n} X \wr (G_1 \wr \cdots \wr G_{n-1}) \wr G_n \xrightarrow{\alpha_2} X \wr (G_1 \wr \cdots \wr G_n).$$

Consider the following commutative diagram where the triangle on the left commutes by induction hypothesis

$$
\begin{array}{ccccccc}
 & & X \wr G_1 \wr \cdots \wr G_{n-1} & \xrightarrow{\Delta} & X \wr G_1 \wr \cdots \wr G_n & & \\
 & \nearrow^{\Delta\circ\cdots\circ\Delta} & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_{n-1}\wr G_n} & \searrow^{\alpha_n} & \\
X & \xrightarrow[\Delta]{} & X \wr (G_1 \wr \cdots \wr G_{n-1}) & \xrightarrow[\Delta]{} & X \wr (G_1 \wr \cdots \wr G_{n-1}) \wr G_n & \xrightarrow[\alpha_2]{\simeq} & X \wr (G_1 \wr \cdots \wr G_n).
\end{array}
$$

The property of α_n stated in the proposition follows from (1) applied to the composition at the bottom row of this diagram. □

3.6. **Remark.** Clearly Σ_k fixes all the points in the image of the diagonal map $X \to X^k$. If $X \neq \emptyset$, then the fibre sequence (3.1) $X^k \to X \wr G \to BG$ splits for any $G \leq \Sigma_k$ and the long exact sequence in homotopy groups gives rise to isomorphisms

$$
\begin{aligned}
\pi_1(X \wr G) &\cong (\pi_1 X) \wr G && \text{and} \\
\pi_i(X \wr G) &\cong (\pi_i X)^k && \text{for all } i \geq 2.
\end{aligned}
$$

Moreover, $\kappa\colon X^k \to X \wr G$ induces inclusions $\prod_k \pi_* X \leq \pi_*(X \wr G)$ on which $G \leq \pi_1(X \wr G)$ acts on higher homotopy groups by permuting the factors.

In particular, if $X = BH$ for a discrete group H, there is a homotopy equivalence $(BH) \wr G \simeq B(H \wr G)$ and $\Delta\colon BH \to (BH) \wr G \simeq B(H \wr G)$ is homotopic to the map induced by the diagonal inclusion $H \leq H \wr G$.

Let Y be a G-space. For any space X, $\mathrm{map}(X, Y)$ becomes a G-space, and the evaluation map $X \times \mathrm{map}(X, Y) \xrightarrow{\mathrm{ev}} Y$ is clearly G-equivariant. Therefore it gives rise to a map $\mathrm{ev}_{hG}\colon X \times \mathrm{map}(X, Y)_{hG} \to Y_{hG}$ whose adjoint is denoted

$$(\mathrm{ev}_{hG})^{\#}\colon \mathrm{map}(X, Y)_{hG} \to \mathrm{map}(X, Y_{hG}).$$

If the component $\mathrm{map}^f(X, Y)$ of some $f\colon X \to Y$ is invariant under the G-action then inspection of the adjunction shows that $(\mathrm{ev}_{hG})^{\#}$ restricts to

$$(\mathrm{ev}_{hG})^{\#}\colon \mathrm{map}^f(X, Y)_{hG} \to \mathrm{map}^{\kappa(Y)\circ f}(X, Y_{hG}).$$

Moreover, the composite

$$\mathrm{map}^f(X, Y) \xrightarrow{\kappa} \mathrm{map}^f(X, Y)_{hG} \xrightarrow{(\mathrm{ev}_{hG})^{\#}} \mathrm{map}^{\kappa\circ f}(X, Y_{hG}) \tag{3.7}$$

coincides with the natural map induced by $Y \xrightarrow{\kappa(Y)} Y_{hG}$ when applying $\mathrm{map}(X, -)$.

3.8. **Proposition.** *Fix a map $f\colon A \to X$ and $G \leq \Sigma_k$. Denote the adjoint of*

$$A \times (\mathrm{map}^f(A, X) \wr G) = A \times \mathrm{map}^{\Delta_X \circ f}(A, X^k)_{hG} \xrightarrow{\mathrm{ev}_{hG}} (X^k)_{hG} = X \wr G$$

by $\gamma\colon \mathrm{map}^f(A, X) \wr G \to \mathrm{map}^{\Delta(X)\circ f}(A, X \wr G)$. Then:

(a) *The triangle*

$$
\begin{array}{ccc}
\mathrm{map}^f(A, X) & & \\
\downarrow{\scriptstyle \Delta} & \searrow^{\mathrm{map}(A, \Delta(X))} & \\
\mathrm{map}^f(A, X) \wr G & \xrightarrow[\gamma]{} & \mathrm{map}^{\Delta(X)\circ f}(A, X \wr G).
\end{array}
$$

is commutative.

(b) *If the natural map $BG \to \mathrm{map}^c(A, BG)$ into the the space of the constant maps induces a homotopy equivalence then γ is a homotopy equivalence.*

Proof. (a) Note that $\prod_k \mathrm{map}^f(A,X) = \mathrm{map}^{\Delta_X\circ f}(A,X^k)$ and that this component is invariant under the action of $G \le \Sigma_k$. The commutativity of the triangle follows from (3.7) and Definition 3.4.

(b) Consider the following ladder in which the rows are fibre sequences and π_* is induced by $X \to *$.

$$\begin{array}{ccccc}
\mathrm{map}^f(A,X)^k & \longrightarrow & \mathrm{map}^f(A,X)\wr G & \longrightarrow & BG \\
{\scriptstyle\mathrm{incl}}\downarrow & & \gamma\downarrow & & \simeq\downarrow{\scriptstyle\mathrm{const}} \\
F & \longrightarrow & \mathrm{map}^{\Delta(X)\circ f}(A, X\wr G) & \xrightarrow[\pi_*]{} & \mathrm{map}^c(A,BG).
\end{array} \tag{1}$$

It commutes because the right hand square commutes as a consequence of the commutativity of the following square and adjunction

$$\begin{array}{ccc}
A\times \mathrm{map}^{\Delta_X\circ f}(A,X^k)_{hG} & \longrightarrow & A\times \mathrm{map}(A,*)_{hG} \\
{\scriptstyle\mathrm{ev}_{hG}}\downarrow & & \downarrow{\scriptstyle\mathrm{proj}=\mathrm{ev}_{hG}} \\
(X^{\times k})_{hG} & \xrightarrow[\pi]{} & *_{hG} = BG.
\end{array}$$

Now, F is a union of path components of $\mathrm{map}(A,X^k)$ because it is the fibre of the fibration $\mathrm{map}(A,X\wr G)\to \mathrm{map}(A,BG)$ over the component of the constant map. Moreover, F clearly contains the component $\mathrm{map}^{\Delta_X\circ f}(A,X^k)$ and inspection of γ shows that the map between the fibres is simply the inclusion. Comparison of the long exact sequences in homotopy of the fibre sequences in (1) shows that F is connected, whence $F = \mathrm{map}^f(A,X)^{\times k}$. Application of the five lemma to the exact sequences in homotopy now yields the result. □

3.9. **Remark.** The hypothesis on A in part (b) of Proposition 3.8 is satisfied by all classifying spaces BK of finite groups since $\mathrm{map}^c(BK,BG)\simeq BG$.

4. Killing homotopy groups

The aim of this section is to study the effect on homotopy groups of the map $X \xrightarrow{\Delta(X)} X\wr\Sigma_k \xrightarrow{\eta} (X\wr\Sigma_k)^\wedge_p$ where $\Delta(X)$ was defined in the last section and η is the p-completion map.

4.1. **Proposition.** *Let X be a pointed space. Then the kernel of $\pi_*X\to\pi_*(X^\wedge_p)$ contains all the elements whose order is prime to p.*

Proof. Let $[\Theta]\in\pi_*(X)$ be an element of order k prime to p. Then the map $\Theta\colon S^n\to X$ factors through the Moore space $M(\mathbb{Z}/k,n)$, which is a nilpotent space with the same mod p homology of a point. It follows that $\eta\circ\Theta\colon S^n\to X^\wedge_p$ factors through $M(\mathbb{Z}/k,n)^\wedge_p\simeq *$ (see [3, Ch. VI.5]), and therefore is nullhomotopic. □

An element of exponent n in a group G is an element whose order divides n. For the proof of the next result, recall that for any space, $\pi_1(X)$ acts on the groups π_*X, see e.g. [21, Corollary 7.3.4] or [23, Ch. III]. We write α^ω for the action of $\omega\in\pi_1X$ on $\alpha\in\pi_nX$.

4.2. **Lemma.** *Fix an integer $n\ge 3$ and a pointed space X. Then the kernel of*

$$\pi_*X \xrightarrow{\Delta(X)_*} \pi_*(X\wr\Sigma_n) \xrightarrow{\eta_*} \pi_*((X\wr\Sigma_n)^\wedge_p)$$

contain all the elements of exponent n in π_*X.

Proof. We recall from Remark 3.6 that

$$\begin{aligned} \pi_1(X \wr \Sigma_n) &= (\pi_1 X) \wr \Sigma_n \\ \pi_i(X \wr \Sigma_n) &= \oplus_n \pi_i X \qquad \text{for } i \geq 2. \end{aligned}$$

Furthermore, $\kappa\colon \prod_n X \to X \wr \Sigma_n$ induces the inclusion $\prod_n \pi_* X \leq \pi_*(X \wr \Sigma_n)$. The section $s\colon B\Sigma_n \to X \wr \Sigma_n$ defined by the fixed point $(*, \dots, *) \in X^n$ induces the inclusion $\Sigma_n \leq \pi_1(X \wr \Sigma_n)$ which acts by permuting the factors of $\pi_*(X^n) \leq \pi_*(X \wr \Sigma_n)$.

Since $n \geq 3$ we can choose elements $\omega_k \in \Sigma_n$ whose order is prime to p and $\omega_k(1) = k$ for all $k = 1, \dots, n$. Indeed, if $p > 2$ we can choose the involutions $\omega_k = (1, k)$. If $p = 2$ we chan choose ω_k to be 3-cycles (note that $n \geq 3$.) In both cases we choose $\omega_1 = \mathrm{id}$.

For every $k = 1, \dots, n$ let $j_k\colon X \to \prod_n X$ denote the inclusion into the kth factor. Note that $\mathrm{diag}\colon X \to X^n$ induces $\mathrm{diag}_*(\theta) = (\theta, \dots, \theta) \in \prod_n \pi_* X$. By inspection of the action of $\omega_k \in \pi_1(X \wr \Sigma_n)$, it follows that for any $\theta \in \pi_i X$, $(\kappa \circ j_k)_*(\theta) = ((\kappa \circ j_1)_*(\theta))^{\omega_k} \in \pi_i(X \wr \Sigma_n)$. Now fix some $\theta \in \pi_i X$ of exponent n. Since $\Delta(X)$ is defined as the composition $X \xrightarrow{\mathrm{diag}} \prod_n X \xrightarrow{\kappa} X \wr \Sigma_n$, we have

$$\Delta(X)_*(\theta) = \prod_{k=1}^{n} (\kappa \circ j_k)_*(\theta) = \prod_{k=1}^{n} ((\kappa \circ j_1)_*(\theta))^{\omega_k}.$$

Now consider the p-completion map $X \wr \Sigma_n \xrightarrow{\eta} (X \wr \Sigma_n)^\wedge_p$ and note that it maps ω_k to the trivial element by Proposition 4.1. By applying η_* and using the naturality of the action of the fundamental group we see that

$$\begin{aligned} (\eta \circ \Delta(X))_*(\theta) = \prod_{k=1}^{n} \eta_*\big(((\kappa \circ j_1)_*(\theta))^{\omega_k}\big) &= \prod_{k=1}^{n} \eta_*\big((\kappa \circ j_1)_*(\theta)\big)^{\eta_*(\omega_k)} \\ &= (\eta_*((\kappa \circ j_1)_*(\theta)))^n = \eta_*((\kappa \circ j_1)_*(\theta^n)) = 0. \end{aligned}$$

□

4.3. **Lemma.** *Fix a map $f\colon X \to Y$ and assume that every element of $\pi_i \mathrm{map}^f(X, Y)$ has exponent k for some $k \geq 3$. Assume further that $\mathrm{map}^{\eta \circ \Delta(Y) \circ f}(X, (Y \wr \Sigma_k)^\wedge_p)$ is p-complete. Then the induced homomorphism*

$$\pi_i \mathrm{map}^f(X, Y) \xrightarrow{\mathrm{map}(X, \eta \circ \Delta(Y))} \pi_i \mathrm{map}^{\eta \circ \Delta(Y) \circ f}(X, (Y \wr \Sigma_k)^\wedge_p)$$

is trivial.

Proof. According to Proposition 3.8(a) the triangle in the diagram below commutes up to homotopy.

$$\begin{array}{ccccc} \mathrm{map}^f(X,Y) & \xrightarrow{\Delta(Y)_*} & \mathrm{map}^{\Delta \circ f}(X, Y \wr \Sigma_k) & \xrightarrow{\eta_*} & \mathrm{map}^{\eta \circ \Delta \circ f}(X, (Y \wr \Sigma_k)^\wedge_p) \\ & \searrow^{\Delta} & \uparrow{\scriptstyle \gamma} & & \vdots \uparrow \\ & & \mathrm{map}^f(X,Y) \wr \Sigma_k & \xrightarrow{\eta} & (\mathrm{map}^f(X,Y) \wr \Sigma_k)^\wedge_p \end{array}$$

Since $\mathrm{map}^{\eta \circ \Delta(Y) \circ f}(X, (Y \wr \Sigma_k)^\wedge_p)$ is p-complete, the map $(\eta_* \circ \gamma)^\wedge_p$ gives rise to a choice of a map for the dotted arrow so that the square is homotopy commutative. We can now apply Lemma 4.2 to the diagonal arrow Δ and the bottom arrow η. □

5. The wreath product of p-local finite groups

Given a finite group G, the space $(BG) \wr \Sigma_k$ is the classifying space of the group $G \wr \Sigma_k$ (see 3.6). In this section we prove an analogous result for p-local finite groups.

Recall from Remark 2.5 that any p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is equipped with functions $\delta_{P,Q} \colon N_S(P,Q) \to \mathcal{L}(P,Q)$, where P, Q are $\mathcal{F}$-centric. We shall denote $\delta_{P,Q}(s)$ by $\hat{s}$. Thus, an element $s \in S$ permutes the set of all morphisms $\mathcal{L}$, by either pre-composition with $\widehat{s^{-1}}$ (i.e. $\varphi \mapsto \varphi \circ \widehat{s^{-1}}$) or by post-composition with $\hat{s}$ (i.e $\varphi \mapsto \hat{s} \circ \varphi$). We obtain an action of S on $\mathcal{L}$ by conjugation of the subgroup $P \leq S$ and by conjugation of morphisms $\varphi \mapsto \hat{s} \circ \varphi \circ \widehat{s^{-1}}$.

5.1. **Definition.** The action of a group G on S is called fusion preserving if the image of $G \xrightarrow{\tau} \mathrm{Aut}(S)$ consists of fusion preserving automorphisms, that is, for every $\varphi \in \mathcal{F}(P,Q)$ and every $g \in G$ the composition $\tau_g \circ \varphi \circ \tau_g^{-1}$ belongs to $\mathcal{F}(\tau_g(P), \tau_g(Q))$.

In this section we prove Theorem 5.2 which is a variant of [4, Theorem 4.6]. While condition (2) of Theorem 5.2 offers some simplifications, we relax the assumption imposed in [4] that G is a finite p-group. The main idea of the proof remains the same but some new arguments were also needed and therefore we decided to present a complete proof of Theorem 5.2.

5.2. **Theorem.** *Let G be a finite group which acts on the centric linking system $\mathcal{L}_0$ of a p-local finite group $(S_0, \mathcal{F}_0, \mathcal{L}_0)$. The action of $g \in G$ on $\varphi \in \mathcal{L}_0$ is denoted by $\varphi \mapsto g \cdot \varphi \cdot g^{-1}$. Assume that $S_0 \lhd G$ and let S be a Sylow p-subgroup of G. Assume further that:*

1. $\mathrm{Aut}_G(S_0)$ *acts via fusion preserving automorphisms.*
2. *For any $g \in G$, if $c_g \in \mathcal{F}_0(P_0, Q_0)$ for $\mathcal{F}_0$-centric subgroups $P_0, Q_0 \leq S_0$, then $g \in S_0$.*
3. *The action of G on $\mathcal{L}_0$ extends the action of S_0 on $\mathcal{L}_0$ by conjugation.*
4. *The monomorphism $\delta_{S_0} \colon S_0 \to \mathrm{Aut}_{\mathcal{L}_0}(S_0)$ is G-equivariant.*
5. *The projection $\pi_0 \colon \mathcal{L}_0 \to \mathcal{F}_0$ is G-equivariant, that is $\pi_0(g \cdot \varphi \cdot g^{-1}) = c_g \circ \pi_0(\varphi) \circ c_{g^{-1}}$.*
6. *There is a compatible choice of lifts of inclusions in $\mathcal{L}_0$ such that for any $g \in G$ and every inclusion of $\mathcal{F}_0$-centric subgroups $P_0 \leq Q_0$, we have $g \cdot \iota_{P_0}^{Q_0} \cdot g^{-1} = \iota_{{}^gP_0}^{{}^gQ_0}$.*

Then, there exists a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ with the following properties:

(a) *There are inclusions $\mathcal{F}_0 \subseteq \mathcal{F}$, $\mathcal{F}_0^c \subseteq \mathcal{F}^c$ and $\mathcal{L}_0 \subseteq \mathcal{L}$ in such a way that the distinguished monomorphisms δ_P in $\mathcal{L}$ extend the ones in $\mathcal{L}_0$. The map $i \colon |\mathcal{L}_0| \to |\mathcal{L}|$ induced by the inclusion fits in a homotopy fibre sequence*

$$|\mathcal{L}_0| \xrightarrow{i} |\mathcal{L}| \to B(G/S_0).$$

Moreover, if S_0 has a complement K in G, that is $G = S_0 \rtimes K$, then:

(b) *There is a homotopy equivalence $|\mathcal{L}_0|_{hK} \xrightarrow{\simeq} |\mathcal{L}|$ such that the composition $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$ is homotopic to $|\mathcal{L}_0| \xrightarrow{i} |\mathcal{L}|$ and such that $\Theta \colon BS \to |\mathcal{L}|$ is homotopic to the composition*

$$BS \xrightarrow{Bincl} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|.$$

(c) *Up to isomorphism $(S, \mathcal{F}, \mathcal{L})$ is the unique p-local finite group with the properties in (b).*

As a corollary we obtain the proof of Theorem 1.1 in the Introduction.

Proof of Theorem 1.1. By Remark 2.18 there is an action of Σ_n on the n-fold product $(S_0, \mathcal{F}_0, \mathcal{L}_0) = (S^{\times n}, \mathcal{F}^{\times n}, \mathcal{L}^{\times n})$ by permuting the factors.

The action of S_0 on $\mathcal{L}_0$ by conjugation clearly extends to an action of $S_0 \rtimes \Sigma_n$ because $S_0 = S^{\times n}$ acts on every coordinate of $\mathcal{L}_0 = \mathcal{L}^{\times n}$ and Σ_n acts by permuting the factors of $\mathcal{L}_0$ and the factors of $S_0 = S^{\times n}$. Set $G = S \wr K = S_0 \rtimes K$. We shall now show that the action of G on $\mathcal{L}_0$ satisfies hypotheses (1)-(6) of Theorem 5.2.

Hypothesis (1) is clearly satisfied because K acts on S_0 by permuting the factors which is an automorphism of $\mathcal{F}_0 = \mathcal{F}^{\times n}$. Hypothesis (3) holds by the definition of the action of $G = S_0 \rtimes K$ on $\mathcal{L}_0$. Hypothesis (4) holds for similar reasons since $K \leq \Sigma_n$ acts on $P_0 \leq S_0$ and on $\mathrm{Aut}_{\mathcal{F}_0}(P_0) \leq \prod_i \mathrm{Aut}_{\mathcal{F}}(P_0^{(i)})$ by permuting the factors (see Definition 2.15) where $P_0^{(i)}$ is the image of P_0 under the projection $p^i \colon S^{\times n} \to S$ to the ith factor. For hypothesis (5) note that $\pi \colon \mathcal{L}_0 \to \mathcal{F}_0$ is Σ_n-equivariant and it is also S_0-equivariant since $\pi(\hat{s}) = c_s$ for any $s \in S$. Hypothesis (6) holds as we indicated above for the choice of the morphisms $\{\iota_{P_0}^{Q_0}\}$ which we described in Remarks 2.16 and 2.18.

It remains to check hypothesis (2). Fix an $\mathcal{F}_0$-centric subgroup $P_0 \leq S_0$ and let $P_0^{(i)}$ be defined as above (see 2.13). Since $P_0^{(i)}$ are $\mathcal{F}$-centric for $i = 1, \dots, n$ by Lemma 2.14 and $S \neq 1$, it follows that $P_0^{(i)} \neq 1$ whence $Z(P_0^{(i)}) \neq 1$ for all $i = 1, \dots, n$. Also note that $\prod_i Z(P_0^{(i)}) = \prod_i C_S(P_0^{(i)}) = C_{S_0}(P_0) \leq P_0$ because P_0 is $\mathcal{F}_0$-centric. Fix some $g = (s_1, \dots, s_n; \sigma) \in G = S \wr K$ and assume that $g \notin S_0$, namely $\sigma \neq 1$. Without loss of generality we can assume that $\sigma(1) = 2$. Choose $1 \neq z_1 \in C_S(P_0^{(1)})$ and consider $(z_1, 1, \dots, 1; \mathrm{id}) \in \prod_{i=1}^n Z(P_0^i) \leq P_0$. Then

$$\begin{aligned} c_g((z_1, 1, \dots, 1; \mathrm{id})) &= (s_1, \dots, s_n; \sigma)(z_1, 1, \dots, 1; \mathrm{id})(s_{\sigma^{-1}(1)}^{-1}, \dots, s_{\sigma^{-1}(n)}^{-1}; \sigma^{-1}) \\ &= (1, s_2 z_1 s_2^{-1}, 1, \dots, 1; \mathrm{id}). \end{aligned}$$

Therefore $c_g \notin \mathcal{F}_0(P_0, S_0)$ because it cannot be a restriction of a morphism in $\prod_n \mathcal{F}$.

Now we apply Theorem 5.2(b) to conclude that there exists a p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ with $(|\mathcal{L}_0|)_{hK} \simeq |\mathcal{L}'|$ such that

$$BS' \xrightarrow{B\mathrm{incl}} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}'| \tag{1}$$

is homotopic to $\Theta' \colon BS' \to |\mathcal{L}'|$. Also observe that the horizontal arrows in

$$\begin{array}{ccc} (BS)^{\times n} & = & BS_0 \\ {\scriptstyle \Theta^{\times n}} \downarrow & & \downarrow {\scriptstyle \Theta_0} \\ |\mathcal{L}|^{\times n} & \xrightarrow[\simeq]{} & |\mathcal{L}_0| \end{array}$$

form a Σ_n-equivariant map of the vertical arrows. It follows that the composite in (1) is homotopic to the map

$$BS' \xrightarrow{B\mathrm{incl}} BG \simeq (BS) \wr K \xrightarrow{\Theta \wr K} |\mathcal{L}| \wr K \simeq |\mathcal{L}'|.$$

which is therefore homotopic to $\Theta' \colon BS' \to |\mathcal{L}'|$. Finally, the uniqueness of $(S', \mathcal{F}', \mathcal{L}')$ with this property is guaranteed by part (c) of Theorem 5.2. $\square$

5.3. **Remark.** If the p-local finite group in Theorem 1.1 is associated with a finite group G then $(S', \mathcal{F}', \mathcal{L}')$ satisfies $|\mathcal{L}'|^\wedge_p \simeq (|\mathcal{L}|^\wedge_p \wr K)^\wedge_p \simeq (BG^\wedge_p \wr K)^\wedge_p \simeq B(G \wr K)^\wedge_p$. Those equivalences follow from the Serre spectral sequence associated to $|\mathcal{L}|^n \times_K EK$ and [3, Lemma I.5.5] since the spaces involved are p-good ([7, Proposition 1.12]).

In the remainder of this section we will prove Theorem 5.2. **From now on, the hypotheses and notation of Theorem 5.2 are in force.** The construction of $(S, \mathcal{F}, \mathcal{L})$ will be obtained in a sequence of steps which we describe now in 5.4–5.17. These statements will be proved after the proof of Theorem 5.2 which succeeds them.

5.4. **Definition.** Let $\mathcal{H}_0$ denote the set of all the $\mathcal{F}_0$-centric subgroups of S_0. Fix once and for all a Sylow p-subgroup S of G and for every $P \leq S$ let P_0 denote $P \cap S_0$.

5.5. **Lemma.** *The action of G on the set of all subgroups of S_0 by conjugation restricts to an action on the set $\mathcal{H}_0$.*

5.6. **Definition.** Let $\mathcal{F}_1$ be the fusion system on S_0 generated by $\mathcal{F}_0$ and $\mathrm{Aut}_G(S_0)$. Define a category $\mathcal{L}_1$ whose object set is $\mathcal{H}_0$ and

$$\mathrm{Mor}(\mathcal{L}_1) = \Big(\coprod_{P_0, Q_0 \in \mathcal{H}_0} G \times \mathcal{L}_0(P_0, Q_0) \Big) / \Big\{ (gs, \varphi) \sim (g, \hat{s} \circ \varphi) \ ; \ s \in S_0 \Big\}.$$

The morphisms set $\mathcal{L}_1(P_0, Q_0)$ where $P_0, Q_0 \in \mathcal{H}_0$ consists of the equivalence classes $[g : \varphi]$ such that $g \in G$ and $\varphi \in \mathcal{L}_0(P_0, Q_0^g)$. Composition is given by the formula

$$[g : \varphi] \circ [h : \psi] = [gh : (h^{-1} \varphi h) \circ \psi],$$

and identities are the elements of the form $[1 : \mathrm{id}_{P_0}]$.

Define a functor $\pi_1 \colon \mathcal{L}_1 \to \mathcal{F}_1$ which is the identity on the set of objects and

$$\pi_1([g : \varphi]) = c_g \circ \pi_0(\varphi).$$

We also define functions $\hat{\delta}_{P_0, Q_0} \colon N_G(P_0, Q_0) \to \mathcal{L}_1(P_0, Q_0)$ by $g \mapsto [g : \iota_{P_0}^{Q_0^g}]$ and denote the image of g by $\hat{g}$.

After showing that $\mathcal{L}_1$ is well defined we will prove the following properties.

5.7. **Lemma.** *The category $\mathcal{L}_1$ satisfies the following properties:*

(a) *There is an inclusion functor $j \colon \mathcal{L}_0 \to \mathcal{L}_1$ which is the identity on objects and $\varphi \mapsto [1 : \varphi]$ on morphisms.*

(b) *Every morphism in $\mathcal{L}_1$ has the form $\hat{g} \circ \varphi$ where φ is a morphism in $\mathcal{L}_0 \subseteq \mathcal{L}_1$. If $\varphi \in \mathcal{L}_0(P_0, Q_0)$ and $x \in N_G(P_0)$, then $\varphi \circ \hat{x} = \hat{x} \circ (x^{-1} \varphi x)$.*

(c) *There is a homotopy fibre sequence*

$$|\mathcal{L}_0| \xrightarrow{|j|} |\mathcal{L}_1| \to B(G/S_0).$$

If S_0 admits a complement K in G then there is a homotopy equivalence $|\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$ such that the composition $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$ is homotopic to the map induced by the inclusion j. Moreover, the composite

$$BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$$

is homotopic to the map $BG \to |\mathcal{L}_1|$ induced by the functor $k \colon \mathcal{B}G \to \mathcal{L}_1$ with $k(\bullet_G) = S_0$ and $k(g) = [g : 1_{S_0}]$.

The next step in our construction is to define the following category.

5.8. **Definition.** Define a category $\mathcal{L}_2$ whose object set is

$$\mathcal{H} = \{P \le S \ : \ P_0 \in \mathcal{H}_0\}$$

and whose morphism sets are defined by

$$\mathcal{L}_2(P,Q) = \{\psi \in \mathcal{L}_1(P_0,Q_0) \ : \ \forall x \in P \exists y \in Q(\psi \circ \hat{x} = \hat{y} \circ \psi)\}.$$

By construction $\mathcal{L}_2(P,Q) \subseteq \mathcal{L}_1(P_0,Q_0)$ and composition of morphisms is obtained by composing them in $\mathcal{L}_1$. Identities id_P have the form $[1 : \mathrm{id}_{P_0}]$. Also define maps $\hat{\delta}_{P,Q} \colon N_G(P,Q) \to \mathcal{L}_2(P,Q)$ by $g \mapsto [g : \iota_{P_0}^{Q_0^g}]$ and denote the image of g by $\hat{g}$.

The main properties of the category $\mathcal{L}_2$ and its relation to the previously defined $\mathcal{L}_1$ are contained in next two lemmas.

5.9. **Lemma.** *The category $\mathcal{L}_1$ is the full subcategory of $\mathcal{L}_2$ on the objects $\mathcal{H}_0$ and the inclusion $j \colon \mathcal{L}_1 \to \mathcal{L}_2$ induces a homotopy equivalence on nerves.*

5.10. **Lemma.** *The category $\mathcal{L}_2$ satisfies the following properties:*

(a) *For every morphism $\psi \in \mathcal{L}_2(P,Q)$ there exists a unique group monomorphism $\pi_2(\psi) \colon P \to Q$ which satisfies $\psi \circ \hat{x} = \widehat{\pi_2(\psi)(x)} \circ \psi$ in $\mathcal{L}_2$ for all $x \in P$. Moreover, $\pi_2(\psi)|_{P_0} = \pi_1(\psi)$.*

(b) *π_2 carries identities to identities and $\pi_2(\lambda) \circ \pi_2(\psi) = \pi_2(\lambda \circ \psi)$ for every $P \xrightarrow{\psi} Q \xrightarrow{\lambda} R$ in $\mathcal{L}_2$.*

(c) *For every $\hat{g} \in \mathcal{L}_2(P,Q)$ with $g \in N_G(P,Q)$, we have $\pi_2(\hat{g}) = c_g$.*

(d) *Given $\psi \in \mathcal{L}_2(P,Q)$, if $\pi_2(\psi)$ is an isomorphism of groups then ψ is an isomorphism in $\mathcal{L}_2$.*

Lemma 5.10 justifies the following definition.

5.11. **Definition.** Let $\mathcal{F}_2$ be the category whose object set is $\mathcal{H}$ (see Definition 5.8) and whose morphism sets $\mathcal{F}_2(P,Q)$ are the set of group monomorphisms $\pi_2(\mathcal{L}_2(P,Q))$ defined by Lemma 5.10. By the properties shown in this lemma, there results a projection functor $\pi_2 \colon \mathcal{L}_2 \to \mathcal{F}_2$ which is the identity on objects.

5.12. **Lemma.** *The category $\mathcal{F}_2$ satisfies the following properties:*

(a) *For every $P, Q \in \mathcal{H}$, $\mathrm{Hom}_G(P,Q) \subseteq \mathcal{F}_2(P,Q)$. In particular, $\mathcal{F}_2$ contains all the inclusions $P \le Q$ of groups in $\mathcal{H}$.*

(b) *Every morphism in $\mathcal{F}_2$ factors as an isomorphism in $\mathcal{F}_2$ followed by an inclusion. In particular, every isomorphism of groups $f \colon P \to Q$ in $\mathcal{F}_2$ is an isomorphism in $\mathcal{F}_2$.*

Thus, $\mathcal{F}_2$ falls short of being a fusion system on S only because its set of objects $\mathcal{H}$ need not contain all the subgroups of S.

5.13. **Definition.** Let $\mathcal{F}$ denote the fusion system on S generated by $\mathcal{F}_2$.

5.14. **Lemma.** *The fusion system $\mathcal{F}$ over S satisfies the following properties:*

(a) *$\mathcal{F}_2$ is the full subcategory of $\mathcal{F}$ generated by the objects in $\mathcal{H}$.*

(b) *Every $P \in \mathcal{H}$ is $\mathcal{F}$-centric. In particular, $\mathcal{H}_0 \subseteq \mathcal{F}^c$.*

(c) *Every morphism $f \in \mathcal{F}(P,Q)$ restricts to a morphism $f|_{P_0} \in \mathcal{F}(P_0,Q_0)$.*

5.15. **Lemma.** *The functor $\pi_2 \colon \mathcal{L}_2 \to \mathcal{F}$ satisfies all the axioms of a centric linking system on the object set $\mathcal{H}$.*

Finally, the last step in the proof is to show that the fusion system $(S, \mathcal{F})$ defined in 5.13 is saturated and that $\mathcal{L}_2$ can be extended to a unique centric linking system $\mathcal{L}$ associated to $\mathcal{F}$.

5.16. **Lemma.** *$\mathcal{F}$ is a saturated fusion system on S.*

5.17. **Lemma.** *There exists a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ such that $\pi_2 \colon \mathcal{L}_2 \to \mathcal{F}$ is the restriction of $\pi \colon \mathcal{L} \to \mathcal{F}$ and moreover $\hat{\delta}_P \colon P \to \mathrm{Aut}_{\mathcal{L}_2}(P)$ are the distinguished monomorphisms of $(S, \mathcal{F}, \mathcal{L})$ for all $P \in \mathcal{H}$. Moreover, $\mathcal{L}_2$ is a full subcategory of $\mathcal{L}$ and the inclusion $\mathcal{L}_2 \subseteq \mathcal{L}$ induces a homotopy equivalence on nerves.*

Assuming definitions and lemmas 5.4–5.17, we can now prove Theorem 5.2.

Proof of Theorem 5.2. The p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is constructed in Lemma 5.17. Together with Lemma 5.9 we obtain inclusions of full subcategories $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}$ which induce homotopy equivalences on nerves. By Lemma 5.7(c), there results the homotopy fibre sequence of part (a).

Now assume that S_0 has a complement K in G and we prove points (b) and (c). Lemma 5.7(c) shows that there are homotopy equivalences $|\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1| \simeq |\mathcal{L}|$ such that $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$ is homotopic to the map induced by the inclusion $\mathcal{L}_0 \subseteq_j \mathcal{L}_1 \subseteq \mathcal{L}$. Moreover the map

$$BS \xrightarrow{B\mathrm{incl}} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$$

is induced by the functor $\Lambda_0 \colon \mathcal{B}S \to \mathcal{L}$ which sends $\bullet_S$ to S_0 and defined on morphisms by $s \mapsto [s : 1_{S_0}] = \hat{s} \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ (see Lemmas 5.17, 5.7 and Definition 5.8). The map $\Theta \colon BS \to |\mathcal{L}|$ is the realisation of the functor $\Lambda_1 \colon \mathcal{B}S \to \mathcal{B}\mathrm{Aut}_{\mathcal{L}}(S) \to \mathcal{L}$ where $s \mapsto \hat{s} \in \mathrm{Aut}_{\mathcal{L}}(S)$, then the lift of the inclusion $\iota_{S_0}^S \in \mathcal{L}(S_0, S)$ provides a natural transformation $\Lambda_0 \to \Lambda_1$ (note that $\hat{s} \circ \iota_{S_0}^S = \iota_{S_0}^S \circ \hat{s}$ by Remark 2.5). Therefore $|\Lambda_0|$ and $|\Lambda_1|$ are homotopic and the proof of point (b) is complete.

Now assume that $(S, \mathcal{F}', \mathcal{L}')$ is another p-local finite group which satisfies the properties in point (b). Let λ denote the composition $BS \to BG = (BS_0)_{hK} \to |\mathcal{L}_0|_{hK}$. By assumption there is a homotopy commutative diagram

$$\begin{array}{ccccc} & & BS & & \\ & \swarrow^{\Theta} & \downarrow{\scriptstyle\lambda} & {}^{\Theta'}\searrow & \\ |\mathcal{L}| & \xleftarrow[\simeq]{} & |\mathcal{L}_0|_{hK} & \xrightarrow[\simeq]{} & |\mathcal{L}'|. \end{array}$$

The isomorphism of $(S, \mathcal{F}, \mathcal{L})$ and $(S, \mathcal{F}', \mathcal{L}')$ follows from [7, Theorem 7.7] $\square$

The rest of the section is devoted to the proof of statements in 5.5–5.17.

Proof of Lemma 5.5. First of all, observe that $S_0 \lhd G$ so for any $P_0 \in \mathcal{H}_0$ and $g \in G$ we have $C_{S_0}(gP_0g^{-1}) = gC_{S_0}(P_0)g^{-1} = Z(gP_0g^{-1})$ because P_0 is $\mathcal{F}_0$-centric.

Now fix some $P_0 \in \mathcal{H}_0$ and $g \in G$. It follows from hypothesis (1) that every $R_0 \leq S_0$ which is $\mathcal{F}_0$-conjugate to gP_0g^{-1} has the form gQ_0g^{-1} for some $Q_0 \leq S_0$ which is $\mathcal{F}_0$-conjugate to P_0. In particular $Q_0 \in \mathcal{H}_0$. It follows from the calculation above that $C_{S_0}(gP_0g^{-1}) = Z(gP_0g^{-1})$ and that $C_{S_0}(R_0) = Z(gQ_0g^{-1}) = Z(R_0)$. This shows that gP_0g^{-1} is $\mathcal{F}_0$-centric, namely $gP_0g^{-1} \in \mathcal{H}_0$. $\square$

5.18. **Lemma.** *For every $\mathcal{F}_0$-centric $P_0, Q_0 \leq S_0$, every $s \in N_{S_0}(P_0, Q_0)$ and every $g \in G$ we have $g\hat{s}g^{-1} = \widehat{gsg^{-1}}$ as morphisms in $\mathcal{L}_0({}^gP_0, {}^gQ_0)$.*

Proof. Set $R_0 = gQ_0g^{-1}$. It suffices to show that the equality holds after post-composition with $\iota_{R_0}^{S_0}$ because the latter is a monomorphism in $\mathcal{L}_0$ (see Remark 2.6). Note that $\iota_{R_0}^{S_0} = g(\iota_{Q_0}^{S_0})g^{-1}$ by hypothesis (6), therefore using Remark 2.5, we conclude that $\iota_{R_0}^{S_0} \circ g\hat{s}g^{-1} = g\hat{s}g^{-1}$ and $\iota_{R_0}^{S_0} \circ \widehat{gsg^{-1}} = \widehat{gsg^{-1}}$ as morphisms in $\mathcal{L}_0(P_0, S_0)$. We may therefore prove the equality needed in this lemma under the assumption that $Q_0 = S_0$.

Remark 2.5 shows that $\hat{s}\colon P_0 \to S_0$ is equal to $\delta_{S_0}(s) \circ \iota_{P_0}^{S_0}$, which together with hypothesis (6) and the fact that $\iota_{gP_0g^{-1}}^{S_0}$ is an epimorphism in $\mathcal{L}_0$ imply that it suffices to prove (5.18) when $P_0 = S_0$. But this is hypothesis (4) of Theorem 5.2. ∎

Proof of Definition 5.6. By Lemma 5.5 if $Q_0 \in \mathcal{H}_0$ then $Q_0^g \in \mathcal{H}_0$ for any $g \in G$. This shows that pairs $[g : \varphi]$ where $\varphi \in \mathcal{L}_0(P_0, Q_0^g)$ are well defined and that, moreover, every element $[g : \varphi]$ in $\mathrm{Mor}(\mathcal{L}_1)$ has this form. The verification that the formula for composition of morphisms is well defined is identical to the one in [4, Theorem 4.6]. Specifically, for any $g_0, h_0 \in S_0$

$$[gg_0 : \varphi] \circ [hh_0 : \psi] = [gg_0hh_0 : (h_0^{-1}h^{-1}\varphi hh_0) \circ \psi] = \qquad \text{by hypothesis (3)}$$
$$[gg_0h : (h^{-1}\varphi h) \circ \hat{h_0} \circ \psi] = [gh : \widehat{h^{-1}g_0h} \circ (h^{-1}\varphi h) \circ \hat{h_0} \circ \psi] = \qquad \text{by Lemma 5.18}$$
$$[gh : h^{-1}(\hat{g_0} \circ \varphi)h \circ \hat{h_0} \circ \psi] = [g : \hat{g_0} \circ \varphi] \circ [h : \hat{h_0} \circ \psi].$$

Associativity is straightforward as well as checking that $[1 : 1_{P_0}]$ are identity morphisms $P_0 \to P_0$.

It is evident from the definition that π_1 maps identity morphisms in $\mathcal{L}_1$ to identities in $\mathcal{F}_1$. It also respects compositions by the following calculation which uses hypothesis (5) in the third equality

$$\begin{aligned}\pi_1([g : \varphi]) \circ \pi_1([h : \psi]) &= c_g \circ \pi_0(\varphi) \circ c_h \circ \pi_0(\psi)\\ &= c_{gh} \circ (c_{h^{-1}} \circ \pi_0(\varphi) \circ c_h) \circ \pi_0(\psi) = c_{gh} \circ \pi_0(h^{-1}\varphi h) \circ \pi_0(\psi)\\ &= c_{gh} \circ \pi_0(h^{-1}\varphi h \circ \psi) = \pi_1([gh : h^{-1}\varphi h \circ \psi]) = \pi_1([g : \varphi] \circ [h : \psi]).\end{aligned}$$

∎

Proof of Lemma 5.7. (a) By Definition 5.6 we have $[1 : \varphi] \circ [1 : \varphi'] = [1 : \varphi \circ \varphi']$ so j is clearly associative and unital. It is an inclusion functor because $[1 : \varphi] = [1 : \varphi']$ if and only if $\varphi = \varphi'$ by the definition of morphisms in $\mathcal{L}_1$.

(b) Clearly, every morphism ψ in $\mathcal{L}_1$ has the form $[g : \varphi] = [g : 1] \circ [1 : \varphi] = \hat{g} \circ \varphi$. Given φ and x as in the statement, by Definition 5.6

$$\varphi \circ \hat{x} = [1 : \varphi] \circ [x : 1] = [x : x^{-1}\varphi x] = [x : 1_{Q_0^x}] \circ [1 : x^{-1}\varphi x] = \hat{x} \circ x^{-1}\varphi x.$$

(c) Set $\bar{G} = G/S_0$ and denote its elements by $\bar{g} = gS_0$. There is a functor $\Pi\colon \mathcal{L}_1 \to \mathcal{B}(\bar{G})$ which sends every object of $\mathcal{L}_1$ to $\bullet_{\bar{G}}$ and maps $[g : \varphi] \mapsto \bar{g}$. This assignment is evidently well defined and functorial by the constructions of $\mathcal{L}_1$ in Definition 5.6.

Now, consider the comma category $(\bullet_{\bar{G}} \downarrow \Pi)$. Its objects are pairs $(\bar{g}, P_0)$ and morphisms $(\bar{g}, P_0) \to (\bar{h}, Q_0)$ are morphisms $[x : \lambda] \in \mathcal{L}_1(P_0, Q_0)$ such that $\bar{x} = \bar{h}\bar{g}^{-1}$. We can easily check that $\hat{g}\colon P_0^g \to P_0$ provides an isomorphism $(\bar{e}, P_0^g) \to (\bar{g}, P_0)$ in $(\bullet_{\bar{G}} \downarrow \Pi)$. Therefore, the set of objects of the form $(\bar{e}, P_0)$ form a skeletal full subcategory of $(\bullet_{\bar{G}} \downarrow \Pi)$, that is, it contains an element from every isomorphism class of objects. This subcategory is clearly isomorphic to $\mathcal{L}_0$ and moreover the composition $\mathcal{L}_0 \subseteq (\bullet_{\bar{G}} \downarrow \Pi) \to \mathcal{L}_1$ is the inclusion j in part (a).

Moreover, any morphism $\bar{g} \in \mathcal{B}\bar{G}$ clearly induces an automorphism of the category $(\bullet_{\bar{G}} \downarrow \Pi)$. Therefore, Quillen's theorem B [18] applies in this situation to show that $|(\bullet_{\bar{G}} \downarrow \Pi)| \to |\mathcal{L}_1| \to |\mathcal{B}(G/S_0)|$ is a homotopy fibre sequence. Finally, using the homotopy equivalence $|j|$ we obtain the homotopy fibre sequence $|\mathcal{L}_0| \xrightarrow{|j|} |\mathcal{L}_1| \xrightarrow{|\Pi|} BG/S_0$.

Now suppose that S_0 has a complement K in G. Recall that G acts on the category $\mathcal{L}_0$ and we view the restriction of this action to K as a functor $\mathcal{B}K \to \mathbf{Cat}$. Let $\mathrm{Tr}_K(\mathcal{L}_0)$ denote the transporter category (or Grothendieck construction) of this functor; See e.g. [22]. The object set of $\mathrm{Tr}_K(\mathcal{L}_0)$ is $\mathcal{H}_0$, and the morphisms $P_0 \to Q_0$ are pairs (k, φ) where $\varphi \in \mathcal{L}_0({}^kP_0, Q_0)$. Composition is given by the following formula: $(k_2, \varphi_2) \circ (k_1, \varphi_1) = (k_2k_1, \varphi_2 \circ k_2\varphi_1k_2^{-1})$. Define a functor $\Phi\colon \mathrm{Tr}_K(\mathcal{L}_0) \to \mathcal{L}_1$ which is the identity on objects and

$$\Phi\colon \mathrm{Tr}_K(\mathcal{L}_0)(P_0, Q_0) \to \mathcal{L}_1(P_0, Q_0) \qquad \text{is defined by } (k, \varphi) \mapsto [k : k^{-1}\varphi k].$$

It is clear that $\Phi(1, \mathrm{id}) = [1 : \mathrm{id}]$ and for any pair of composable morphisms (k_2, φ_2) and (k_2, φ_2) in $\mathrm{Tr}_K(\mathcal{L}_0)$,

$$\begin{aligned}\Phi(k_2, \varphi_2) \circ \Phi(k_1, \varphi_1) &= [k_2 : k_2^{-1}\varphi_2k_2] \circ [k_1 : k_1^{-1}\varphi_1k_1] \\ &= [k_2k_1 : k_1^{-1}k_2^{-1}\varphi_2k_2k_1 \circ k_1^{-1}\varphi_1k_1] = \Phi(k_2k_1, \varphi_2 \circ k_2\varphi_1k_2^{-1}).\end{aligned}$$

By definition Φ is bijective on the object set. We will show now that it is bijective on morphism sets. For any morphism $\psi = [g : \varphi] \in \mathcal{L}_1(P_0, Q_0)$ there is a unique $k \in K \cap gS_0$, hence $\psi = [k : \varphi']$ for a unique $k \in K$ and a unique $\varphi' \in \mathcal{L}_0(P_0, {}^{k^{-1}}Q_0)$. Then $(k, k\varphi'k^{-1}) \in \mathrm{Tr}_K(\mathcal{L}_0)(P_0, Q_0)$ is a preimage of $[k : \varphi']$ under Φ. In fact, it is unique because $K \cap S_0 = 1$.

Thomason [22] constructed a homotopy equivalence $|\mathcal{L}_0|_{hK} \xrightarrow{\beta} |\mathrm{Tr}_K(\mathcal{L}_0)|$ such that $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathrm{Tr}_K(\mathcal{L}_0)|$ is homotopic to the map induced by the inclusion $\mathcal{L}_0 \subseteq \mathrm{Tr}_K(\mathcal{L}_0)$ via $\varphi \mapsto [\bar{e} : \varphi]$. Furthermore, by inspection Φ carries the subcategory of $\mathcal{L}_0$ in $\mathrm{Tr}_K(\mathcal{L}_0)$ onto $\mathcal{L}_0 \subseteq \mathcal{L}_1$ via the identity map. We deduce that $|\Phi| \circ \beta$ is a homotopy equivalence $|\mathcal{L}_0|_{hK} \to |\mathcal{L}_1|$ whose composition with $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK}$ is homotopic to the map induced by the inclusion $j\colon \mathcal{L}_0 \to \mathcal{L}_1$.

To complete the proof we now consider the subcategory $\mathcal{B}S_0$ of $\mathcal{B}\mathrm{Aut}_{\mathcal{L}_0}(S_0) \subseteq \mathcal{L}_0$ via the monomorphism $\delta_{S_0}\colon S_0 \to \mathrm{Aut}_{\mathcal{L}_0}(S_0)$ and observe that it is invariant under the action of K by Lemma 5.18. Thus, there is an inclusion of subcategories $\mathrm{Tr}_K \mathcal{B}S_0 \subseteq \mathrm{Tr}_K \mathcal{L}_0$ induced by $\mathrm{Tr}_K(\delta_{S_0})$. By inspection there is an isomorphism of categories $\mathrm{Tr}_K \mathcal{B}S_0 \cong \mathcal{B}G$ via the functor $(k, s) \mapsto sk$ such that he composition

$$\mathcal{B}G \cong \mathrm{Tr}_K(\mathcal{B}S_0) \subseteq \mathrm{Tr}_K(\mathcal{L}_0) \xrightarrow{\Phi} \mathcal{L}_1$$

is the functor which sends $\bullet_G$ to S_0 and $g \mapsto [g : 1] \in \mathrm{Aut}_{\mathcal{L}_1}(S_0)$. □

Here are more properties of $\mathcal{L}_1$ that we will need later.

5.19. **Lemma.** *The category $\mathcal{L}_1$ satisfies the following properties:*

(a) *For every $P_0, Q_0, R_0 \in \mathcal{H}_0$ and every $g \in N_G(P_0, Q_0)$ and $h \in N_G(Q_0, R_0)$ the equality $\hat{h} \circ \hat{g} = \widehat{hg}$ holds in $\mathcal{L}_1$.*

(b) *Fix $P_0, Q_0 \in \mathcal{H}_0$ and $\psi \in \mathcal{L}_1(P_0, Q_0)$. Then, for every $x \in N_G(P_0)$ there exists at most one $y \in N_G(Q_0)$ such that $\psi \circ \hat{x} = \hat{y} \circ \psi$. In this case*

$y = gxg^{-1}s_0$ *for a unique* $s_0 \in S_0$*. Moreover, if* $x \in P_0$ *then* $y = \pi_1(\varphi)(x)$ *satisfies* $\psi \circ \hat{x} = \hat{y} \circ \psi$.

(c) *Every morphism* $\hat{g} \in \mathcal{L}_1(P_0, Q_0)$ *is both a monomorphism and an epimorphism.*

(d) *Fix* $\psi \in \mathcal{L}_1(P_0, Q_0)$ *such that* $\pi_1(\psi)(P_0) \leq R_0$ *for some* $R_0 \leq Q_0$*. Then there exists* $\lambda \in \mathcal{L}_1(P_0, R_0)$ *such that* $\psi = \iota \circ \lambda$ *where* $\iota = \hat{e} \in \mathcal{L}_1(R_0, Q_0)$.

(e) *If* $\pi_1(\psi) = \pi_1(\psi')$ *where* $\psi, \psi' \in \mathcal{L}_1(P_0, Q_0)$ *then* $\psi' = \psi \circ \hat{z}$ *for a unique* $z \in Z(P_0)$.

(f) *Fix* $P_0 \in \mathcal{H}_0$ *and set* $H := \{g \in G \mid gP_0g^{-1}$ *is* $\mathcal{F}_0$*-conjugate to* $P_0\}$*. Then* H *is a subgroup of* G *which contains* S_0 *and* $|\mathrm{Aut}_{\mathcal{L}_1}(P_0) : \mathrm{Aut}_{\mathcal{L}_0}(P_0)| = |H : S_0|$.

Proof. (a) From Definition 5.6, there are equalities $\hat{h} \circ \hat{g} = [h : \iota_{Q_0}^{R_0^h}] \circ [g : \iota_{P_0}^{Q_0^g}] = [hg : \iota_{Q_0^g}^{R_0^{hg}} \circ \iota_{P_0}^{Q_0^g}] = [hg : \iota_{P_0}^{R_0^{hg}}] = \widehat{hg}$.

(b) By Definition 5.6, ψ has the form $[g : \varphi]$ for some $g \in G$ and $\varphi \in \mathcal{L}_0(P_0, Q_0^g)$. If y exists then, again by Definition 5.6,

$$\hat{y} \circ \psi = [y : 1] \circ [g : \varphi] = [yg : \varphi],$$
$$\psi \circ \hat{x} = [g : \varphi] \circ [x : 1] = [gx : x^{-1}\varphi x].$$

Since $\psi \circ \hat{x} = \hat{y} \circ \psi$ in $\mathcal{L}_1$, there exists some $s \in S_0$ such that

$$\text{(i) } yg = gxs \qquad \text{and} \qquad \text{(ii) } \varphi = \widehat{s^{-1}} \circ (x^{-1}\varphi x).$$

Note that $x^{-1}\varphi x$ is an epimorphism in $\mathcal{L}_0$ (Remark 2.6) so the morphism $\widehat{s^{-1}} \in \mathrm{Iso}_{\mathcal{L}_0}(Q_0^{gx}, Q_0^g)$ which solves equation (ii) must be unique, hence s is unique. Set $s_0 = gsg^{-1}$. Then $s_0 \in S_0$ because $S_0 \lhd G$ and $y = gxsg^{-1} = gxg^{-1} \cdot s_0$.

If $x \in P_0$ then axiom (C) satisfied by the linking system $\mathcal{L}_0$ (see Definition 2.4) implies that

$$\psi \circ \hat{x} = [g : \varphi] \circ [x : 1] = [gx : \widehat{x^{-1}} \circ \varphi \circ \hat{x}] = [g : \varphi \circ \hat{x}] =$$
$$= [g : \widehat{\pi_0(\varphi)(x)} \circ \varphi] = [c_g(\pi_0(\varphi)(x)) \cdot g : \varphi] = c_g(\widehat{\pi_0(\varphi)(x)}) \cdot g \circ \psi.$$

(c) By inspection, every $\hat{g} \in \mathcal{L}_1(P_0, Q_0)$ has the form $\iota \circ \hat{g}$ where $\hat{g} \in \mathcal{L}_1(P_0, {}^gP_0)$ and $\iota = \hat{e} \in \mathcal{L}_1({}^gP_0, Q_0)$. Since $\hat{g}$ in this factorisation is clearly an isomorphism, it suffices to prove the result for ι of the form $\hat{e} = [e : \iota_{P_0}^{Q_0}]$.

Assume that $[h : \varphi], [h' : \varphi'] \in \mathcal{L}_1(R_0, P_0)$ satisfy $\iota \circ [h : \varphi] = \iota \circ [h' : \varphi']$. Since

$$\iota \circ [h : \varphi] = [1 : \iota_{P_0}^{Q_0}] \circ [h : \varphi] = [h : \iota_{P_0^h}^{Q_0^h} \circ \varphi]$$

and similarly $\iota \circ [h' : \varphi'] = [h' : \iota_{P_0^{h'}}^{Q_0^{h'}} \circ \varphi']$, we see from the definition that there exists some $s \in S_0$ such that $h' = hs$ and

$$\iota_{P_0^{h'}}^{Q_0^{h'}} \circ \varphi' = \widehat{s^{-1}} \circ \iota_{P_0}^{Q_0} = \iota_{P_0^{hs}}^{Q_0^{hs}} \circ \widehat{s^{-1}} \circ \varphi \qquad \text{in } \mathcal{L}_0.$$

Since $\iota_{P_0^{h'}}^{Q_0^{h'}}$ is a monomorphism in $\mathcal{L}_0$ it follows that $\varphi' = \widehat{s^{-1}} \circ \varphi$ and therefore $[h' : \varphi'] = [hs : \widehat{s^{-1}} \circ \varphi] = [h : \varphi]$. This shows that ι is a monomorphism.

Now assume that the morphisms $[h : \varphi], [h' : \varphi'] \in \mathcal{L}_1(Q_0, R_0)$ are such that $[h : \varphi] \circ \iota = [h' : \varphi'] \circ \iota$. Then

$$[h : \varphi \circ \iota_{P_0}^{Q_0}] = [h' : \varphi' \circ \iota_{P_0}^{Q_0}]$$

and it follows from the definition that there exists some $s \in S_0$ such that $h' = hs$ and $\varphi' \circ \iota_{P_0}^{Q_0} = \widehat{s^{-1}} \circ \varphi \circ \iota_{P_0}^{Q_0}$. Since $\iota_{P_0}^{Q_0}$ is an epimorphism in $\mathcal{L}_0$ we obtain that $[h' : \varphi'] = [hs : \widehat{s^{-1}} \circ \varphi] = [h : \varphi]$. Therefore ι is an epimorphism.

(d) Write $\psi = [g : \varphi]$ for some $\varphi \in \mathcal{L}_0(P_0, Q_0^g)$. Note that $\pi_1(\psi) = c_g \circ \pi_0(\varphi)$ so $\pi_0(\varphi)(P_0) = R_0^g$. Since $\mathcal{L}_0$ is a linking system, [7, Lemma 1.10] implies that we can factor φ as $P_0 \xrightarrow{\bar{\varphi}} R_0^g \xrightarrow{\iota_{R_0^g}^{Q_0^g}} Q_0^g$. We shall now consider $\lambda \in \mathcal{L}_1(P_0, R_0)$ defined by $\lambda = [g : \bar{\varphi}]$. By hypothesis (6)

$$\iota \circ \lambda = [e : \iota_{R_0}^{Q_0}] \circ [g : \bar{\varphi}] = [g : \iota_{R_0^g}^{Q_0^g} \circ \bar{\varphi}] = [g : \varphi] = \psi.$$

(e) Write $\psi = [g : \varphi]$ and $\psi' = [g' : \varphi']$ in $\mathcal{L}_1(P_0, Q_0)$. By assumption and Definition 5.6 we see that $c_g \circ \pi_0(\varphi) = c_{g'} \circ \pi_0(\varphi')$, whence $\pi_0(\varphi) = c_{g^{-1}g'} \circ \pi_0(\varphi')$. Since $\pi_0(\varphi), \pi_0(\varphi') \in \mathcal{F}_0$, we obtain that $c_{g^{-1}g'} \in \mathcal{F}_0(Q_0^{g'}, Q_0^g)$. Then hypothesis (2) implies that $g^{-1}g' \in S_0$.

Denote $\varphi'' = \widehat{gg^{-1}} \circ \varphi'$ and $s = g^{-1}g' \in S_0$. Then $\psi' = [gs : \varphi'] = [g : \varphi'']$ and $\pi_1(\psi) = \pi_1(\psi')$ reads $c_g \circ \pi_0(\varphi) = c_g \circ \pi_0(\varphi'')$. In particular $\pi_0(\varphi) = \pi_0(\varphi'')$ and the axioms of $\mathcal{L}_0$ guarantee the existence of a unique $z \in Z(P_0)$ such that $\varphi'' = \varphi \circ \hat{z}$. It now follows that $\psi' = [g : \varphi''] = [g : \varphi \circ \hat{z}] = \psi \circ \hat{z}$. Finally, the element $z \in Z(P_0)$ is unique because

$$\psi \circ \hat{z} = [g : \varphi] \circ [z : 1] = [g : \varphi] \circ [1 : \hat{z}] = [g : \varphi \circ \hat{z}],$$

That is, if $\psi \circ \hat{z} = \psi \circ \hat{z'}$ then by Definition 5.6 we see that $\varphi \circ \hat{z} = \varphi \circ \hat{z'}$ and therefore $z = z'$ because φ is a monomorphism in $\mathcal{L}_0$ and $\delta_{P_0} \colon P_0 \to \mathrm{Aut}_{\mathcal{L}_0}(P_0)$ is a monomorphism of groups.

(f) By hypothesis (1) if Q_0 is $\mathcal{F}_0$-conjugate to Q_0' then gQ_0g^{-1} is $\mathcal{F}_0$-conjugate to $gQ_0'g^{-1}$ for any $g \in G$. This implies that H is a subgroup of G and it contains S_0 because $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$.

Let $g_1, \cdots, g_n$ be representatives for the cosets of S_0 in H. By Definition 5.6 every element $\psi \in \mathrm{Aut}_{\mathcal{L}_1}(P_0)$ can be described as $\psi = [g_i : \varphi]$ by a unique pair (g_i, φ) for some $i = 1, \ldots, n$ where $\varphi \in \mathcal{L}_0(P_0, {}^{g_i}P_0)$. Also note that $|\mathcal{L}_0(P_0, {}^{g_i}P_0)| = |\mathrm{Aut}_{\mathcal{L}_0}(P_0)|$ because ${}^{g_i}P_0$ is $\mathcal{F}_0$-conjugate to P_0. This shows that $|\mathrm{Aut}_{\mathcal{L}_1}(P_0)| = n \cdot |\mathrm{Aut}_{\mathcal{L}_0}(P_0)| = |H : S_0| \cdot |\mathrm{Aut}_{\mathcal{L}_0}(P_0)|$. □

We now turn to the study of the properties of the category $\mathcal{L}_2$.

Proof of Definition 5.8. If $\psi \in \mathcal{L}_2(P, Q)$ and $\rho \in \mathcal{L}_2(Q, R)$, we leave it as an easy exercise for the reader to check that $\rho \circ \psi \in \mathcal{L}_1(P_0, R_0)$ belongs to $\mathcal{L}_2(P, R)$. Thus, composition of morphisms in $\mathcal{L}_2$ is well defined. It is easily seen to be unital and associative because this is the case in $\mathcal{L}_1$.

Since $S_0 \lhd G$ it follows that $N_G(P, Q) \subseteq N_G(P_0, Q_0)$, $N_G(P) \leq N_G(P_0)$ and $N_G(Q) \leq N_G(Q_0)$. Now fix some $g \in N_G(P, Q)$ and $x \in P$ and set $y = gxg^{-1} \in Q$. It follows from Lemma 5.19(a) that $\hat{g} \circ \hat{x} = \widehat{gx} = \widehat{yg} = \hat{y} \circ \hat{g}$. Therefore $\hat{g} \in \mathcal{L}_2(P, Q)$. □

Proof of Lemma 5.9. By construction $\mathcal{L}_2(P_0, Q_0) \subseteq \mathcal{L}_1(P_0, Q_0)$ for any $P_0, Q_0 \in \mathcal{H}_0$. For every $x \in P_0$ and every $\psi = [g : \varphi] \in \mathcal{L}_1(P_0, Q_0)$ it follows from Lemma 5.19(b) that $\psi \circ \hat{x} = \hat{y} \circ \psi$ in $\mathcal{L}_1$ where $y = \pi_1(\psi)(x) \in Q_0$. Therefore $\psi \in \mathcal{L}_2(P_0, Q_0)$ and we conclude that $\mathcal{L}_1(P_0, Q_0) = \mathcal{L}_2(P_0, Q_0)$.

The inclusion functor $j\colon \mathcal{L}_1 \to \mathcal{L}_2$ has a left inverse $r\colon \mathcal{L}_2 \to \mathcal{L}_1$ which maps an object P to P_0 and maps morphisms via the inclusions $\mathcal{L}_2(P, Q) \subseteq \mathcal{L}_1(P_0, Q_0)$. Observe that $r \circ j = \mathrm{Id}_{\mathcal{L}_1}$ because $\mathcal{L}_2(P_0, Q_0) = \mathcal{L}_1(P_0, Q_0)$.

By Lemma 5.19(b) we see that $\mathcal{L}_2(P_0, P)$ contains $[e : 1_{P_0}] = \hat{e}$. These morphisms define a natural transformation $j \circ r \to \mathrm{Id}$. This is because we recall that $[e : 1_{P_0}]$ and $[e : 1_{Q_0}]$ are the identities of P_0 and Q_0 in $\mathcal{L}_1$ and for any $\psi \in \mathcal{L}_2(P, Q) \subseteq \mathcal{L}_1(P_0, Q_0)$

$$\psi \circ [e : 1_{P_0}] = [e : 1_{Q_0}] \circ \psi.$$

Then it follows that j and r yield homotopy equivalences on nerves. □

Proof of Lemma 5.10. (a) By Definition 5.8, for every $x \in P$ there exists some $y \in Q$ such that $\psi \circ \hat{x} = \hat{y} \circ \psi$. Since $P \leq N_G(P_0)$ and $Q \leq N_G(Q_0)$, Lemma 5.19(b) implies that y is unique. There results a well defined function $\pi_2(\psi)\colon P \to Q$. In addition, since $\hat{x}$ and $\hat{y} = \widehat{\pi_2(\psi)(x)}$ are morphisms in $\mathcal{L}_2$ (see Definition 5.8) and $\mathcal{L}_2(P, Q) \subseteq \mathcal{L}_1(P_0, Q_0)$ we deduce that the equation $\psi \circ \hat{x} = \widehat{\pi_2(\psi)(x)} \circ \psi$ holds in $\mathcal{L}_2$ and moreover $\pi_2(\psi)\colon P \to Q$ is the unique function that satisfies this equality for all $x \in P$. The fact that $\pi_2(\psi)|_{P_0} = \pi_1(\psi)$ follows from the last assertion in Lemma 5.19(b).

We claim that $\pi_2(\psi)\colon P \to Q$ is a group monomorphism. For $x, x' \in P$, let $y = \pi_2(\psi)(x)$ and $y' = \pi_2(\psi)(x')$. Then, in $\mathcal{L}_1$,

$$\psi \circ \widehat{xx'} = \psi \circ \hat{x} \circ \hat{x'} = \hat{y} \circ \psi \circ \hat{x'} = \hat{y} \circ \hat{y'} \circ \psi = \widehat{yy'} \circ \psi.$$

This shows that $\pi_2(\psi)$ is a homomorphism. If $x \in \ker \pi_2(\psi)$ then $\psi \circ \hat{x} = \hat{1} \circ \psi$ so Lemma 5.19(b) with $y = 1$ shows that $x \in P \cap S_0 = P_0$. But $1 = \pi_2(\psi)(x) = \pi_2(\psi)|_{P_0}(x) = c_g \circ \pi_0(\varphi)(x)$ so $x \in \ker \pi_0(\varphi) = 1$. It follows then that $\ker\big(\pi_2(\psi)\big) = 1$.

(b) Clearly $\pi_2([e : 1_{P_0}]) = \mathrm{Id}_{P_0}$. Now given $P \xrightarrow{\psi} Q \xrightarrow{\lambda} R$ in $\mathcal{L}_2$, set $y = \pi_2(\psi)(x)$ and $z = \pi_2(\lambda)(y)$. Then $\psi \circ \hat{x} = \hat{y} \circ \psi$ and $\lambda \circ \hat{y} = \hat{z} \circ \lambda$ so $\lambda \circ \psi \circ \hat{x} = \hat{z} \circ \lambda \circ \psi$ whence, by the uniqueness statement in Lemma 5.19(b), we conclude that $z = \pi_2(\lambda \circ \psi)(x)$.

(c) This follows from Lemma 5.19(a) because for any $x \in P$ we have $\hat{g} \circ \hat{x} = \widehat{gx} = \widehat{c_g(x)g} = \widehat{c_g(x)} \circ \hat{g}$ in $\mathcal{L}_1$ so $\pi_2(\hat{g}) = c_g$.

(d) Observe that $\pi_2(\psi)(P_0) = \pi_1(\psi)(P_0) \leq Q_0$ by part (a). Since $\pi_2(\psi)\colon P \to Q$ is an isomorphism, for every $y_0 \in Q_0 \leq Q$ there exists some $x \in P$ such that $\pi_2(\psi)(x) = y_0$, namely $\psi \circ \hat{x} = \widehat{y_0} \circ \psi$. By Lemma 5.19(b) we know that $y_0 = gxg^{-1}$ mod S_0 and since $S_0 \lhd G$ we deduce that $x \in S_0 \cap P = P_0$. This shows that $\pi_2(\psi)(P_0) = Q_0$ and therefore $\pi_1(\psi)$ is an isomorphism of groups.

Write $\psi = [g : \varphi]$. Since $\pi_1(\psi)$ is an isomorphism, $\varphi \in \mathcal{L}_0(P_0, Q_0^g)$ is an isomorphism and therefore ψ is an isomorphism in $\mathcal{L}_1$ whose inverse $\psi^{-1} \in \mathcal{L}_1(Q_0, P_0)$ is $[g^{-1} : g\varphi^{-1}g^{-1}]$. To check that ψ^{-1} is a morphism in $\mathcal{L}_2(Q, P)$ consider some $y \in Q$. Since $\pi_2(\psi)$ is an isomorphism there exists $x \in P$ such that $\psi \circ \hat{x} = \widehat{y^{-1}} \circ \psi$ in $\mathcal{L}_1$. Since these morphisms are invertible in $\mathcal{L}_1$ we see that $\widehat{x^{-1}} \circ \psi^{-1} = \psi \circ \hat{y}$. This shows that ψ^{-1} is an inverse to ψ in $\mathcal{L}_2$. □

For later use we also need the following technical lemma.

5.20. **Lemma.** *Fix some $P \in \mathcal{H}$ and consider $N_S(P_0)$ as a subgroup of $\mathrm{Aut}_{\mathcal{L}_1}(P_0)$ via $\hat{\delta}_{P_0,P_0} : x \mapsto \hat{x}$. Let Q be a subgroup of $N_S(P_0)$ and assume that $Q = \psi P \psi^{-1}$ for some $\psi \in \mathrm{Aut}_{\mathcal{L}_1}(P_0)$. Then $P_0 = Q_0$ and ψ is an isomorphism in $\mathcal{L}_2$ from P to Q.*

Proof. Recall from Lemma 5.9 that $\mathrm{Aut}_{\mathcal{L}_1}(P_0) = \mathrm{Aut}_{\mathcal{L}_2}(P_0)$. For $x \in P_0$ set $y = \psi x \psi^{-1} \in Q$. Thus $\psi \circ \hat{x} = \hat{y} \circ \psi$ and by Definition 5.11, $y = \pi_2(\psi)(x) \in P_0$. This shows that $P_0 = \psi P_0 \psi^{-1}$ and, in particular, $P_0 \leq Q_0$. Moreover $P_0 \lhd Q$ because $P_0 \lhd P$.

Since $P_0 \leq Q_0$ we may consider $\iota := \hat{e} \in \mathcal{L}_1(P_0, Q_0)$ where $e \in G$ is the identity element, and define $\lambda = \iota \circ \psi \in \mathcal{L}_1(P_0, Q_0)$. For every $x \in P$ set $y = \psi x \psi^{-1}$. By definition $y \in Q$ which normalises Q_0 and P_0 so Lemma 5.19(a) implies

$$\lambda \circ \hat{x} = \iota \circ \psi \circ \hat{x} = \iota \hat{y} \circ \psi = \hat{y} \circ \hat{e} \circ \psi = \hat{y} \circ \psi.$$

We conclude from Definition 5.8 that $\lambda \in \mathcal{L}_2(P, Q)$. Furthermore, $\pi_2(\lambda)$ is an isomorphism because it is a monomorphism by Lemma 5.10(a) and $|P| = |Q|$. Lemma 5.10(d) now shows that λ is an isomorphism in $\mathcal{L}_2$ and, in particular, it is an isomorphism of the objects P_0 and Q_0 in $\mathcal{L}_1$. In particular $|P_0| = |Q_0|$ and therefore $\lambda = \psi$. □

Proof of Lemma 5.12. (a) This is immediate from Lemma 5.10(c). By taking $e \in N_G(P, Q)$ for any inclusion $P \leq Q$ in $\mathcal{H}$ we obtain $\mathrm{incl}_P^Q \in \mathcal{F}_2(P, Q)$.

(b) Fix a homomorphism $f : P \to Q$ in $\mathcal{F}_2$ and set $R = f(P)$. Note that by Lemma 5.10(a)

$$f(P_0) = \pi_2(\psi)|_{P_0}(P_0) = \pi_1(\psi)(P_0) \leq Q_0.$$

Therefore $f(P_0) \leq Q_0 \cap R \leq S_0 \cap R = R_0$. Also $R_0 = S_0 \cap R \leq S_0 \cap Q = Q_0$. Now, by definition $\psi \in \mathcal{L}_1(P_0, Q_0)$ and Lemma 5.19(d) asserts that in $\mathcal{L}_1$ we can write $\psi = \iota \circ \lambda$ where $\lambda \in \mathcal{L}_1(P_0, R_0)$ and $\iota = \hat{e} \in \mathcal{L}_1(R_0, Q_0)$.

We now claim that $\lambda \in \mathcal{L}_2(P, R)$. To check this, we fix some $x \in P$. By definition $y = f(x) \in R$ satisfies $\psi \circ \hat{x} = \hat{y} \circ \psi$ in $\mathcal{L}_1$. Equivalently $\iota \circ \lambda \circ \hat{x} = \hat{y} \circ \iota \circ \lambda$. Now, $y \in R \leq N_G(R_0)$ and also $y \in Q \leq N_G(Q_0)$, so Lemma 5.19(a) implies that

$$\iota \circ \lambda \circ \hat{x} = \iota \circ \hat{y} \circ \lambda.$$

Lemma 5.19(c) implies that ι is a monomorphism in $\mathcal{L}_1$ so $\lambda \circ \hat{x} = \hat{y} \circ \lambda$ in $\mathcal{L}_1$. This shows that $\lambda \in \mathcal{L}_2(P, R)$ as needed, and that moreover $\psi = \iota \circ \lambda$ in $\mathcal{L}_2$ because ι is in $\mathcal{L}_2$ as well. In particular, by parts (b) and (c) of Lemma 5.10, we obtain that

$$f = \pi_2(\psi) = \mathrm{incl}_R^Q \circ \pi_2(\lambda).$$

From this equality it follows that $\pi_2(\lambda)$ is an isomorphism of groups because $|P| = |R|$. Moreover, Lemma 5.10(d) implies that λ is an isomorphism in $\mathcal{L}_2$ and therefore $\pi_2(\lambda)$ is an isomorphism in $\mathcal{F}_2$. This completes the proof. □

5.21. **Lemma.** *Consider $P \leq S$ such that $P_0 \in \mathcal{H}_0$. Then $C_G(P) = C_{S_0}(P) = Z(P_0)^P$ where P acts on $Z(P_0)$ by conjugation.*

Proof. If $g \in C_G(P)$ then $c_g|_{P_0} = \mathrm{id}_{P_0} \in \mathrm{Aut}_{\mathcal{F}_0}(P_0)$. By hypothesis (2), $g \in S_0$, and it follows that $C_G(P) = C_{S_0}(P)$. Now, $C_{S_0}(P) \leq C_{S_0}(P_0) = Z(P_0)$ because P_0 is $\mathcal{F}_0$-centric. Therefore, $C_G(P) = C_{Z(P_0)}(P) = Z(P_0)^P$. □

Proof of Lemma 5.14. (a) Clearly $\mathcal{H}$ is closed to taking supergroups because $\mathcal{H}_0$ is closed to taking supergroups in S_0. Since $\mathcal{F}$ is generated by inclusions and restriction of homomorphisms in $\mathcal{F}_2$, Lemma 5.12 shows that for any $P, Q \in \mathcal{H}$ the inclusion $\mathcal{F}_2(P,Q) \subseteq \mathcal{F}(P,Q)$ is an equality.

(b) By definition $P_0 \in \mathcal{H}_0$. By Lemma 5.21, $C_S(P) = Z(P_0)^P \leq P$. Assume that Q is $\mathcal{F}$-conjugated to P. By part (a) there exists some $\psi \in \mathcal{L}_2(P,Q)$ such that $\pi_2(\psi)(P) = Q$. Parts (a) and (d) of Lemma 5.10 imply that ψ is an isomorphism in $\mathcal{L}_2$. From Definition 5.8 it is clear that ψ is an isomorphism in $\mathcal{L}_1(P_0, Q_0)$ and in particular $Q_0 \in \mathcal{H}_0$, namely Q_0 is $\mathcal{F}_0$-centric. It follows from Lemma 5.21 that $C_S(Q) = Z(Q_0)^Q \cong Z(P_0)^P$, whence P is $\mathcal{F}$-centric.

(c) For any $f \in \mathcal{F}(P,Q)$ where $P, Q \in \mathcal{H}$, part (a) implies that $f = \pi_2(\psi)$ for some $\psi \in \mathcal{L}_2(P,Q) \subseteq \mathcal{L}_2(P_0,Q_0)$. The result follows from Lemma 5.10(a) which shows that $f|_{P_0} = \pi_1(\psi)$ whose image is contained in Q_0 by Definition 5.6. □

Proof of Lemma 5.15. The monomorphisms $\delta_P \colon P \to \mathrm{Aut}_{\mathcal{L}_2}(P)$ are the restrictions of the maps $\hat{\delta}_{P,Q} \colon N_G(P,Q) \to \mathcal{L}_2(P,Q)$, i.e. $\delta_P(g) = [g : 1_{P_0}]$.

To verify axiom (A) in [7, Definition 1.7], see also 2.4, we need to show that for any $P, Q \in \mathcal{H}$ the set $\pi_2^{-1}(f)$ where $f \in \mathcal{F}(P,Q)$ admit a transitive free action of $C_S(P)$ via $\delta_P \colon N_S(P) \to \mathrm{Aut}_{\mathcal{L}_2}(P)$. Note that $\mathcal{F}(P,Q) = \mathcal{F}_2(P,Q)$ by Lemma 5.14. Consider $\psi, \psi' \in \mathcal{L}_2(P,Q)$ such that $\pi_2(\psi) = \pi_2(\psi')$ and recall that $\psi, \psi' \in \mathcal{L}_1(P_0,Q_0)$. By restriction to P_0, Lemma 5.10(a) shows that $\pi_1(\psi) = \pi_1(\psi')$. Lemma 5.19(f) shows that there exists $z \in Z(P_0)$ such that $\psi' = \psi \circ \hat{z}$ in $\mathcal{L}_1$. Note that $\hat{z} \in \mathrm{Aut}_{\mathcal{L}_2}(P_0)$ by Definition 5.6 so the equality $\psi' = \psi \circ \hat{z}$ also holds in $\mathcal{L}_2$. Furthermore, Lemma 5.19(c) implies that

$$\pi_2(\psi) = \pi_2(\psi') = \pi_2(\psi \circ \hat{z}) = \pi_2(\psi) \circ c_z.$$

As a consequence $z \in C_S(P)$ and we conclude that $C_S(P)$ acts transitively on the fibres of $\pi_2 \colon \mathcal{L}_2(P,Q) \to \mathcal{F}(P,Q)$. The action is free by Lemma 5.21 and the uniqueness assertion in Lemma 5.19(f).

Axiom (B) holds by Lemma 5.10(c). To verify axiom (C) we fix a morphism $\psi \in \mathcal{L}_2(P,Q)$ and an element $g \in P$. Set $f = \pi_2(\psi) \in \mathcal{F}(P,Q)$. By the definition of the morphisms in $\mathcal{L}_2$, see Lemma 5.10(a) we have $\psi \circ \hat{g} = \widehat{f(g)} \circ \psi$, which is what we need. □

Notation. We shall write $P \simeq_{\mathcal{F}} Q$ for the statement that $P, Q \leq S$ are $\mathcal{F}$-conjugate.

Clearly S_0 acts on $\mathcal{H}_0$ by conjugation and $[P_0]_{S_0}$ denotes the orbit of P_0, i.e. the conjugacy class. By Lemma 5.5, G acts on $\mathcal{H}_0$ as well. Since G acts via fusion preserving automorphisms, it also acts on the set $\mathcal{H}_0/\mathcal{F}_0$ of the $\mathcal{F}_0$-conjugacy classes of the subgroups $P_0 \in \mathcal{H}_0$ which we denote $[P_0]_{\mathcal{F}_0}$. The stabiliser of $[P_0]_{\mathcal{F}_0}$ under this action of G is denoted, as usual, by $G_{[P_0]_{\mathcal{F}_0}}$. Now, $G_{[P_0]_{\mathcal{F}_0}}$ acts on the set $[P_0]_{\mathcal{F}_0}$. Clearly, $S_0 \leq G_{[P_0]_{\mathcal{F}_0}}$ because $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$. Moreover, since $S_0 \lhd G$, this action induces an action of $G_{[P_0]_{\mathcal{F}_0}}$ on the set $\mathcal{P}$ of all the S_0-conjugacy classes of the subgroups of S_0 that are $\mathcal{F}_0$-conjugate to P_0.

5.22. **Lemma.** *For every $P \in \mathcal{H}$ there exist $\bar{P}, P' \in \mathcal{H}$ such that*

(a) *$\bar{P} = {}^aP$ for some $a \in G$ and $\bar{P} \simeq_{\mathcal{F}} P'$, whence $P \simeq_{\mathcal{F}} P'$, and*
(b) *P'_0 is fully $\mathcal{F}_0$-normalised and $P'_0 \simeq_{\mathcal{F}_0} \bar{P}_0$.*

In addition, $\bar{S} := N_S(P_0')S_0$ is a Sylow p-subgroup of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ and $\bar{S}/S_0$ fixes the S_0-conjugacy class $[P_0']_{S_0}$.

Proof. The argument follows the one in the proof of step 3 in [4, Theorem 4.6].

Clearly $S_0 \cdot P \leq G_{[P_0]_{\mathcal{F}_0}}$ because $P \leq N_G(P_0)$ and $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$. Choose $S' \in \mathrm{Syl}_p(G_{[P_0]_{\mathcal{F}_0}})$ which contains $S_0 \cdot P$. By Sylow's theorems, there exists some $a \in G$ such that $S' = G_{[P_0]_{\mathcal{F}_0}} \cap S^a$. Set $\bar{P} = {}^aP$ and observe that

$$\bar{P} = {}^aP \leq {}^a(G_{[P_0]_{\mathcal{F}_0}} \cap S^a) \leq S.$$

Also $\bar{P}_0 = {}^aP_0 \in \mathcal{H}_0$ by Lemma 5.5, so $\bar{P} \in \mathcal{H}$. In addition, $G_{[\bar{P}_0]_{\mathcal{F}_0}} = {}^a(G_{[P_0]_{\mathcal{F}_0}})$. It follows that

$$\bar{S} := S \cap G_{[\bar{P}_0]_{\mathcal{F}_0}} = {}^a(S') \in \mathrm{Syl}_p(G_{[\bar{P}_0]_{\mathcal{F}_0}}).$$

Consider now the set $\mathcal{P}_{fn}$ of all the S_0-conjugacy classes of the fully $\mathcal{F}_0$-normalised subgroups $R \leq S_0$ which are $\mathcal{F}_0$-conjugate to $\bar{P}_0$. Since G normalises S_0 and it is fusion preserving, it carries fully $\mathcal{F}_0$-normalised subgroups of S_0 to ones, and therefore $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ acts on $\mathcal{P}_{fn}$.

We now restrict the action of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ on $\mathcal{P}_{fn}$ to $\bar{S}$. By [4, Proposition 1.16] we know that $|\mathcal{P}_{fn}| \neq 0 \mod p$. Therefore $\bar{S}/S_0$ must have some fixed point $[R_0]_{S_0}$. Thus, R_0 is fully $\mathcal{F}_0$-normalised and is $\mathcal{F}_0$-conjugate to $\bar{P}_0$. Recall that $\bar{S} \leq S$. For every $g \in \bar{S}$ we have $gR_0g^{-1} \simeq_{S_0} R_0$ so $\bar{S} \leq N_S(R_0)S_0$. On the other hand $S_0N_S(R_0) \leq G_{[R_0]_{\mathcal{F}_0}} = G_{[\bar{P}_0]_{\mathcal{F}_0}}$ and $\bar{S}$ is a Sylow p-subgroup of the latter group, hence

$$\bar{S} = S_0 \cdot N_S(R_0).$$

It remains to find some $P' \in \mathcal{H}$ such that $P' \simeq_{\mathcal{F}} \bar{P}$ and such that $P_0' = R_0$. Now, since $\bar{P} \leq \bar{S}$, it must stabilise $[R_0]_{S_0}$. We conclude that $\bar{P}/\bar{P}_0$ acts on

$$X := \{[f] \in \mathrm{Rep}_{\mathcal{F}_0}(\bar{P}_0, S_0) \ : \ \mathrm{Im}\, f \text{ is } S_0\text{-conjugate to } R_0\}$$

via $[f_0] \mapsto [c_g \circ f_0 \circ c_{g^{-1}}]$. Clearly X is not empty because by construction $\bar{P}_0 \simeq_{\mathcal{F}_0} R_0$. Choose some $f \in \mathcal{F}_0(\bar{P}_0, R_0)$. Then every element of X has the form $[\alpha \circ f]$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}_0}(R_0)$. Moreover $[\alpha \circ f] = [\beta \circ f]$ if and only if $\alpha^{-1}\beta \in \mathrm{Aut}_{S_0}(R_0)$. Therefore

$$|X| = \frac{|\mathrm{Aut}_{\mathcal{F}_0}(R_0)|}{|\mathrm{Aut}_{S_0}(R_0)|} \neq 0 \mod p$$

because R_0 is fully $\mathcal{F}_0$-normalised. Since $\bar{P}$ is a finite p-group, there is some $[f_0] \in X^{\bar{P}}$ where $f_0 \in \mathcal{F}_0(\bar{P}_0, S_0)$ and $\mathrm{Im}\, f_0 = R_0$. Let $\psi_0 \in \mathcal{L}_0(\bar{P}_0, S_0)$ be a lift of f_0.

Recall from Lemma 5.7(a) that we may consider ψ_0 as a morphism in $\mathcal{L}_1(\bar{P}_0, S_0)$ via an inclusion $\mathcal{L}_0 \subseteq \mathcal{L}_1$. Fix some $x \in \bar{P}$. Since $\bar{P}$ fixes $[f_0]$, there exists some $s \in S_0$ such that

$$c_x^{-1} \circ f_0 \circ c_x = c_s \circ f_0.$$

Lifting to $\mathcal{L}_0$ and using hypothesis (5), we see that there exists a unique $z \in C_{S_0}(\bar{P}_0) = Z(\bar{P}_0)$ such that

$$x^{-1}\psi_0 x = \hat{s} \circ \psi_0 \circ \hat{z} = \widehat{sf_0(z)} \circ \psi_0 \qquad \text{in } \mathcal{L}_0. \tag{1}$$

Set $y := xsf_0(z)$ and note that $y \in \bar{P} \cdot S_0 \cdot Z(R_0) \leq S$. Lemma 5.7(c), equation (1) and Remark 2.5 imply that

$$\psi_0 \circ \hat{x} = \hat{x} \circ (x^{-1}\psi_0 x) = \hat{x} \circ \widehat{sf_0(z)}\psi_0 = \hat{y} \circ \psi_0.$$

Therefore, by definition, $\psi_0 \in \mathcal{L}_2(\bar{P}, S)$. Consider $f = \pi_2(\psi_0) \in \mathcal{F}(\bar{P}, S)$ and set $P' = f(\bar{P})$. By Lemmas 5.14(a) and 5.12(b), f restricts to an isomorphism $f\colon \bar{P} \to P'$ in $\mathcal{F}$. By Lemma 5.10(a) and Lemma 5.7(a) we see that $f|_{\bar{P}_0} = \pi_0(\psi_0) = f_0 \in \mathcal{F}_0(\bar{P}_0, R_0)$. Since $f \in \mathcal{F}(\bar{P}, P')$ is an isomorphism we deduce from Lemma 5.14(c) that $P'_0 = f(\bar{P}_0) = R_0$. This completes the proof since f is an $\mathcal{F}$-isomorphism between $\bar{P}$ and P' which restricts to an $\mathcal{F}_0$-isomorphism f_0 between $\bar{P}_0$ and $R_0 = P'_0$. □

5.23. **Lemma.** [4, Step 4] *If $P \le S$ is $\mathcal{F}$-centric but $P \notin \mathcal{H}$, then there exists $P' \le S$ which is $\mathcal{F}$-conjugate to P such that*

$$\mathrm{Out}_S(P') \cap O_p(\mathrm{Out}_{\mathcal{F}}(P')) \ne 1.$$

Proof. The argument is almost repeated from step 4 in the proof of [4, Theorem 4.6], but we include it for completeness. Consider $\bar{P}$ and P' as in Lemma 5.22. Note that $\bar{P} \notin \mathcal{H}$ because $P \notin \mathcal{H}$, namely $P_0 \notin \mathcal{H}_0$, so $\bar{P}_0 \notin \mathcal{H}_0$ by Lemma 5.5.

the action of G is $\mathcal{F}_0$-preserving. As a consequence $P'_0 \notin \mathcal{H}_0$ because $\bar{P}_0 \simeq_{\mathcal{F}_0} P'_0$. Since P'_0 is fully $\mathcal{F}_0$-normalised, it is fully $\mathcal{F}_0$-centralised and since it is not $\mathcal{F}_0$-centric, we deduce that $C_{S_0}(P'_0) \not\le P'_0$.

Since P' normalises S_0 and P'_0 it acts on $C_{S_0}(P'_0)P'_0/P'_0$ by conjugation leaving a non-identity subgroup QP'_0/P'_0 fixed where $Q \le C_{S_0}(P'_0)$ and $Q \not\le P'_0$. Thus, $[P', Q] \le P'_0$ and in particular $Q \le N_S(P')$. If $x \in Q \setminus P'_0$ then $1 \ne [c_x] \in \mathrm{Out}(P')$ because P' is $\mathcal{F}$-centric so $C_S(P') \le P'$ and $Q \setminus P' = Q \setminus P'_0$. Lemma 5.14(c) shows that restriction $\varphi \mapsto \varphi|_{P'_0}$ induces a homomorphism

$$\mathrm{Aut}_{\mathcal{F}}(P') \xrightarrow{\text{rest}} \mathrm{Aut}_{\mathcal{F}}(P'_0).$$

Let $\mathrm{Aut}_{\mathcal{F}}(P'; P'_0)$ denote its kernel and observe that it contains c_x because Q centralises P'_0. Also observe that c_x induces a trivial homomorphism on P'/P'_0 because $[P', Q] \le P'_0$. Thus, c_x is a non-trivial element in the kernel of

$$\mathrm{Aut}_{\mathcal{F}}(P'; P'_0) \xrightarrow{\text{proj}} \mathrm{Aut}(P'/P'_0)$$

which is a p-group by [4, Proposition 1.15]. This shows that c_x is an element of $O_p(\mathrm{Aut}_{\mathcal{F}}(P'; P'_0))$ which is a characteristic subgroup of $\mathrm{Aut}_{\mathcal{F}}(P'; P'_0) \lhd \mathrm{Aut}_{\mathcal{F}}(P')$. Hence, $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P'))$. Since $\mathrm{Aut}_{\mathcal{F}}(P') \to \mathrm{Out}_{\mathcal{F}}(P')$ is an epimorphism and $[c_x] \ne 1$, we see that $O_p(\mathrm{Out}_{\mathcal{F}}(P')) \cap \mathrm{Out}_S(P') \ne 1$. □

Proof of 5.16. We will apply [5, Theorem 2.2] to the collection $\mathcal{H}$ of objects in $\mathcal{F}$. The condition (*) in that theorem has been verified in Lemma 5.23 so, for the proof of the saturation of $\mathcal{F}$ it remains to check conditions (I) and (II) of saturation in [7, Definition 1.2], see also 2.2 for the elements of $\mathcal{H}$. The argument is again present in [4] with some changes.

Condition I. Fix $P \in \mathcal{H}$ which is fully $\mathcal{F}$-normalised. We have to show that it is fully $\mathcal{F}$-centralised and that $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. By Lemma 5.14(b) we know that P is $\mathcal{F}$-centric and in particular fully $\mathcal{F}$-centralised.

Consider $\bar{P}$ and P' as in Lemma 5.22. Recall that $\bar{S} = N_S(P'_0)S_0$ is a Sylow p-subgroup of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$. Lemma 5.7(a) shows that $\mathrm{Aut}_{\mathcal{L}_0}(\bar{P}_0) \le \mathrm{Aut}_{\mathcal{L}_1}(\bar{P}_0)$ and by Lemma 5.19(g)

$$(1) \qquad |\mathrm{Aut}_{\mathcal{L}_1}(P'_0) : \mathrm{Aut}_{\mathcal{L}_0}(P'_0)| = |G_{[\bar{P}_0]_{\mathcal{F}_0}} : S_0|.$$

By definition $N_{S_0}(P_0') = S_0 \cap N_S(P_0')$ so

$$|N_S(P_0')/N_{S_0}(P_0')| = |N_S(P_0')S_0/S_0| = |\bar{S}/S_0|. \tag{2}$$

Now, P_0' is fully $\mathcal{F}_0$-normalised and is $\mathcal{F}_0$-centric so

$$|\mathrm{Aut}_{\mathcal{L}_0}(P_0') : N_{S_0}(P_0')| \neq 0 \mod p. \tag{3}$$

Since $|G_{[P_0']_{\mathcal{F}_0}} : \bar{S}| \neq 0 \mod p$, we deduce from (1), (2) and (3) that

$$|\mathrm{Aut}_{\mathcal{L}_1}(P_0') : N_S(P_0')| = \frac{|\mathrm{Aut}_{\mathcal{L}_1}(P_0')|}{|\mathrm{Aut}_{\mathcal{L}_0}(P_0')|} \cdot \frac{|\mathrm{Aut}_{\mathcal{L}_0}(P_0')|}{|N_{S_0}(P_0')|} \cdot \frac{|N_{S_0}(P_0')|}{|N_S(P_0')|} \neq 0 \mod p,$$

namely $N_S(P_0') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}_1}(P_0'))$.

Fix $\psi \in \mathrm{Aut}_{\mathcal{L}_1}(P_0')$ such that

$$\psi^{-1} N_S(P_0')\psi \ \supseteq R \in \mathrm{Syl}_p(N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P')) \tag{4}$$

and set

$$P'' = \psi P' \psi^{-1} \leq N_S(P_0').$$

Lemma 5.20 shows that $P_0' = P_0''$ and that $\psi \in \mathcal{L}_2(P', P'')$ is an isomorphism. In particular, P'' is $\mathcal{F}$-conjugate to P', hence also to P because $P' = {}^aP$ for some $a \in G$ and $\hat{a} \in \mathcal{L}_2(P, P')$ is an isomorphism. We now claim that

(i) $\mathrm{Aut}_{\mathcal{L}_2}(P'') = N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P'')$ and (ii) $N_S(P'') = N_{N_S(P_0)}(P'')$.

Clearly (i) follows from the definition of the morphisms in $\mathcal{L}_2$ because

$$\begin{aligned} \lambda \in \mathrm{Aut}_{\mathcal{L}_2}(P'') &\iff \forall x \in P'' \, \exists y \in P'' (\lambda \circ \hat{x} \circ \lambda^{-1} = \hat{y}) \\ &\iff \lambda \in N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P''). \end{aligned}$$

For (ii), note that $P'' \subseteq N_S(P_0') \subseteq \mathrm{Aut}_{\mathcal{L}_1}(P_0')$ so by the choice of ψ in equation (4),

$$N_{N_S(P_0')}(P'') = N_S(P_0') \cap N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P'') \in \mathrm{Syl}_p(N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P'')).$$

On the other hand

$$N_{N_S(P_0')}(P'') \leq N_S(P'') \leq N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P''),$$

hence $N_S(P'') = N_{N_S(P_0')}(P''))$. We deduce that $N_S(P'') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}_2}(P'')$. Finally, $\mathrm{Aut}_{\mathcal{L}_2}(P) \cong \mathrm{Aut}_{\mathcal{L}_2}(P'')$ because P'' and P are isomorphic in $\mathcal{L}_2$ (via $\psi \circ \hat{a}$). Also, $|N_S(P)| \geq |N_S(P'')|$ because P is fully $\mathcal{F}$-normalised. Therefore $N_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}_2}(P))$ and Lemma 5.15 implies that $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

Condition II. Fix $P \in \mathcal{H}$ and $\varphi \in \mathcal{F}(P, S)$. Definition 5.11 and part (a) of Lemma 5.14 show that $\varphi(P) \in \mathcal{H}$ and part (b) of this lemma shows that $\varphi(P)$ is $\mathcal{F}$-centric and in particular it is fully $\mathcal{F}$-centralised. We have to prove that φ extends to some $\psi \in \mathcal{F}(N_\varphi, S)$ where

$$N_\varphi = \{g \in N_S(P) \ : \ \varphi \circ c_g = c_s \circ \varphi \text{ for some } s \in S\}.$$

Note that $s \in N_S(\mathrm{Im}\,\varphi)$ in this definition. Set, for convenience $Q = N_\varphi$. We observe that

$$Q \leq N_S(Q_0) \qquad \text{and} \qquad Q \leq N_S(P) \leq N_S(P_0). \tag{5}$$

Let $\tilde{\varphi} \in \mathcal{L}_2(P, S)$ be a lift for φ, that is $\varphi = \pi_2(\tilde{\varphi})$. By definition, for every $q \in Q$ there exists some $s_q \in S$ such that $\varphi \circ c_q = c_{s_q} \circ \varphi$. Lifting to $\mathcal{L}_2$, we see from

Lemma 5.15 that there exists some $z \in C_S(P) = Z(P)$ such that $\tilde{\varphi} \circ \hat{q} = \hat{s_q} \circ \tilde{\varphi} \circ \hat{z} = \hat{s_q} \circ \widehat{\varphi(z)} \circ \tilde{\varphi}$. Set $y_q = s_q\varphi(z)$, then $y_q \in S$ and

$$\tilde{\varphi} \circ \hat{q} = \hat{y_q} \circ \tilde{\varphi} \qquad \text{in } \mathcal{L}_2. \tag{6}$$

By Definition 5.8 the morphism $\tilde{\varphi}$ is an element in $\mathcal{L}_1(P_0, S_0)$. By Lemma 5.7(c) we see that $\tilde{\varphi} = \hat{g} \circ \tilde{\lambda}$ where $g \in G$ and $\tilde{\lambda} \in \mathcal{L}_0(P_0, S_0)$. Set $\lambda = \pi_0(\tilde{\lambda}) \in \mathcal{F}_0(P_0, S_0)$. From parts (a) and (c) of Lemma 5.10 we see that $\varphi|_{P_0} = \pi_1(\tilde{\varphi}) = \pi_1(\hat{g} \circ \tilde{\lambda}) = c_g \circ \lambda$.

By definition, for every $x \in Q_0$ there exists some $s \in S$ such that

$$\varphi \circ c_x = c_s \circ \varphi \qquad \text{in } \mathcal{F}.$$

By restriction to P_0 we obtain an equality of homomorphisms $P_0 \to S$

$$c_g \circ \lambda \circ c_x = c_s \circ c_g \circ \lambda. \tag{7}$$

By restriction of λ to an isomorphism onto its image we see that

$$c_{g^{-1}sg} = \lambda \circ c_x \circ \lambda^{-1} \in \mathcal{F}_0 \qquad \text{because } x \in Q_0 \leq S_0.$$

Hypothesis (2) implies that $g^{-1}sg \in S_0$ and therefore $s \in S_0$. We can therefore rewrite equation (7) as $\lambda \circ c_x = c_{g^{-1}sg} \circ \lambda$ where $g^{-1}sg \in S_0$. Together with equation (5), this shows that $x \in N_\lambda$ where

$$N_\lambda = \{x \in N_{S_0}(P_0) \ : \ \lambda \circ c_x = c_y \circ \lambda \text{ for some } y \in S_0\}.$$

We deduce that $Q_0 \leq N_\lambda$.

Since P_0 is $\mathcal{F}_0$-centric, so is $\lambda(P_0)$ and in particular it is fully $\mathcal{F}_0$-centralised. Axiom (II) in $\mathcal{F}_0$ enables us to extend $\lambda \in \mathcal{F}_0(P_0, S_0)$ to some $\rho \in \mathcal{F}_0(Q_0, S_0)$. Let $\tilde{\rho}$ be a lift for ρ in $\mathcal{L}_0$. Now, $\lambda = \rho \circ \mathrm{incl}_{P_0}^{Q_0} = \pi_0(\tilde{\rho} \circ \iota_{P_0}^{Q_0})$, so there exists some $z \in Z(P_0) \leq P_0 \leq Q_0$ such that

$$\tilde{\lambda} = \tilde{\rho} \circ \iota_{P_0}^{Q_0} \circ \hat{z} = \tilde{\rho} \circ \hat{z} \circ \iota_{P_0}^{Q_0}.$$

Set $\tilde{\theta} = \tilde{\rho} \circ \hat{z}$ and $\theta = \pi_0(\tilde{\theta})$. Thus, $\tilde{\theta} \in \mathcal{L}_0(Q_0, S_0)$ and $\theta \in \mathcal{F}_0(Q_0, S_0)$ satisfy

$$\tilde{\lambda} = \tilde{\theta} \circ \iota_{P_0}^{Q_0} \qquad \text{and} \qquad \theta|_{P_0} = \lambda$$

because $\pi_0(\tilde{\theta})|_{P_0} = \pi_0(\tilde{\rho} \circ \hat{z})|_{P_0} = \rho \circ c_z|_{P_0} = \rho|_{P_0} = \lambda$.

Recall that we started with a lift $\tilde{\varphi} = \hat{g} \circ \tilde{\lambda}$ for φ. By Lemma 5.7(a) we view $\tilde{\theta}$ as a morphism in $\mathcal{L}_1$ and define

$$\tilde{\psi} := \hat{g} \circ \tilde{\theta} \in \mathcal{L}_1(Q_0, S_0).$$

We now prove that for every $q \in Q$, the element $y_q \in S$ defined in equation (6) satisfies

$$\tilde{\psi} \circ \hat{q} = \hat{y_q} \circ \tilde{\psi} \qquad \text{in } \mathcal{L}_1. \tag{8}$$

Observe that $Q = N_\varphi$ so $P \leq Q$ and in particular $P_0 \leq Q_0$. We shall now consider $\iota := \hat{e} \in \mathcal{L}_1(P_0, Q_0)$ where $e \in N_G(P_0, Q_0)$ is the identity of G. Note that under the inclusion $\mathcal{L}_0 \subseteq \mathcal{L}_1$ in Lemma 5.7(a) we have $\iota = \iota_{P_0}^{Q_0}$. Therefore

$$\tilde{\psi} \circ \iota = \hat{g} \circ \tilde{\theta} \circ \iota_{P_0}^{Q_0} = \hat{g} \circ \tilde{\lambda} = \tilde{\varphi} \qquad \text{in } \mathcal{L}_1.$$

Equation (5), Lemma 5.19(a) and equation (6) imply that in $\mathcal{L}_1$

$$\tilde{\psi} \circ \hat{q} \circ \iota = \tilde{\psi} \circ \hat{q} \circ \hat{e} = \tilde{\psi} \circ \hat{e} \circ \hat{q} = \tilde{\psi} \circ \iota \circ \hat{q} = \tilde{\varphi} \circ \hat{q} = \hat{y_q} \circ \tilde{\varphi} = \hat{y_q} \circ \tilde{\psi} \circ \iota$$

We deduce that equation (8) holds because ι is an epimorphism in $\mathcal{L}_1$ by Lemma 5.19(d). By Definition 5.8 we see that $\psi \in \mathcal{L}_2(Q, S)$. Set $\psi := \pi_2(\tilde{\psi})$. Then

$\psi \in \mathcal{F}_2(Q,S) = \mathcal{F}(Q,S)$ and by Lemma 5.10(c) we see that $\psi|_P = \pi_2(\tilde{\psi} \circ \iota) = \pi_2(\tilde{\varphi}) = \varphi$. This completes the proof. □

Proof of Lemma 5.17. Our notation was chosen in such a way that the argument in [4, Theorem 4.6, Step 7] can be read verbatim and we shall therefore avoid reproducing it. □

6. Maps from a homotopy colimit

Let $\mathcal{C}$ be a small category, and $X \colon \mathcal{C} \to \mathbf{Top}$ be a diagram of spaces over $\mathcal{C}$. The values taken by the functor will be denoted by $X(c)$ and $X(\varphi)$ where $c \in \mathcal{C}$, $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c,c')$. The homotopy colimit of the diagram X is the space

$$\mathrm{hocolim}_{\mathcal{C}} X = \Big(\coprod_{n \geq 0} \coprod_{c_0 \to \cdots \to c_n} X(c_0) \times \Delta^n \Big) / \sim$$

where we divide by the usual face and degeneracy identifications [3, Ch. XII].

We filter the homotopy colimit by using the skeleta of the nerve of $\mathcal{C}$, and we define F_nX to be the image of the union of $X(c) \times \Delta^m$ in $\mathrm{hocolim}_{\mathcal{C}} X$ for all $m \leq n$. Notice that F_0X is just $\coprod_{c \in \mathcal{C}} X(c)$ and F_1X is the union of the mapping cylinders of all $\varphi \in \mathrm{Mor}(\mathcal{C})$. Observe that a map $f_1 \colon F_1X \to Y$ is the same as a set of maps $f_1(c) \colon X(c) \to Y$ together with homotopies $f_1(c') \circ X(\varphi) \simeq f_1(c)$ for every $\varphi \in \mathcal{C}(c,c')$. A set of maps $X(-) \xrightarrow{f(-)} Y$ which admits such homotopies is called a *system of homotopy compatible maps* and it gives rise to an element in the set $\varprojlim_{\mathcal{C}} [X(c), Y]$.

Fix a system of homotopy compatible maps $X(-) \xrightarrow{f(-)} Y$. By the remark above it gives rise to a map $f_1 \colon F_1X \to Y$ where $f_1|_{X(c)} = f(c)$. Wojtkowiak [24] addressed the question whether f_1 can be extended, up to homotopy, to a map $\tilde{f} \colon \mathrm{hocolim}_{\mathcal{C}} X \to Y$. The method is to extend f_1 by induction on the spaces F_nX.

Given a map $\tilde{f}_n \colon F_nX \to Y$ whose restriction to $X(c)$ is homotopic to $f(c)$, Wojtkowiak developed an obstruction theory for extending it to $F_{n+1}X$ without changing it on $F_{n-1}X$. The existence of such an extension depends on the vanishing of a certain obstruction class in $\varprojlim^{n+1} \pi_n(\mathrm{map}^{f(c)}(X(c), Y))$. The extension from F_1X to F_2X involves in general a functor of non-abelian groups, into the category of groups and representations, whose $\varprojlim^2$ term is described in Wojtkowiak's work. Fortunately, if these groups are abelian then the Wojtkowiak's definition of $\varprojlim^2$ coincides with the usual one from homological algebra. Once the map has been extended to F_2X, a choice of homotopies allow to define well-defined functors $\pi_n(\mathrm{map}^{f(c)}(X(c), Y))$ into abelian groups for $n > 1$.

Given two maps $\tilde{f}_1, \tilde{f}_2 \colon \mathrm{hocolim}_{\mathcal{C}} X \to Y$ whose restrictions to $X(c)$ are homotopic to $f(c)$, Wojtkowiak also studies an obstruction theory for the construction of a homotopy $\tilde{f}_1 \simeq \tilde{f}_2$. Clearly, $\tilde{f}_1$ and $\tilde{f}_2$ give rise to a homotopy $\tilde{f}_1|_{F_0X} \overset{H_0}{\simeq} \tilde{f}_2|_{F_0X}$. The idea is to extend the homotopy H_0 inductively to $I \times F_nX$. Given a homotopy $\tilde{f}_1|_{F_{n-1}X} \overset{H_{n-1}}{\simeq} \tilde{f}_2|_{F_{n-1}X}$, the possibility of extending it to a homotopy between the restrictions of $\tilde{f}_1$ and $\tilde{f}_2$ to F_nX without changing its values on $F_{n-2}X$ depends on the vanishing of an obstruction class in $\varprojlim^n \pi_n(\mathrm{map}^{f(c)}(X(c), Y))$.

6.1. **Definition** ([7, Definition 3.3]). Fix a prime p. We say that a small category $\mathcal{C}$ has bounded limits at p if there exists $d \geq 0$ such that every functor $F\colon \mathcal{C} \to \mathbb{Z}_{(p)}$-mod has the property that $\varprojlim_{\mathcal{C}}^{i>d} F = 0$. We call d the *height* of $\mathcal{C}$.

6.2. **Theorem.** *Let $\mathcal{C}$ be a finite category with bounded limits at p of height d and consider a sequence of maps $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_d} Y_{d+1}$ with partial composites $y_i = g_i \circ \cdots \circ g_0\colon Y_0 \to Y_{i+1}$. Given a functor $X\colon \mathcal{C} \to$ **Top** and a system of homotopy compatible maps $f(-)\colon X(-) \to Y_0$, define new systems of homotopy compatible maps $f_i(-) = y_i \circ f(-)\colon X(-) \to Y_{i+1}$ for all $i = 0, \dots, d$. Assume that*

(i) *For every $c \in \mathcal{C}$ and every $i = 1, \dots, d$ the induced map*

$$\pi_i \mathrm{map}^{f_{i-1}(c)}(X(c), Y_i) \xrightarrow{(g_i)_*} \pi_i \mathrm{map}^{f_i(c)}(X(c), Y_{i+1})$$

is the trivial homomorphism between abelian groups.

(ii) *The groups $\pi_{*>0}\mathrm{map}^{f_i(c)}(X(c), Y_i)$ are $\mathbb{Z}_{(p)}$-modules for all $c \in \mathcal{C}$ and all i.*

Then

(a) *There exists map $\tilde{f}\colon \operatorname{hocolim}_{\mathcal{C}} X \to Y_d$ which renders the following square homotopy commutative for all $c \in \mathcal{C}$,*

$$\begin{array}{ccc} X(c) & \xrightarrow{f(c)} & Y_0 \\ {\scriptstyle \iota(c)}\downarrow & & \downarrow{\scriptstyle y_{d-1}} \\ \operatorname{hocolim}_{\mathcal{C}} X & \xrightarrow[\tilde{f}]{} & Y_d. \end{array}$$

(b) *If $\tilde{f}_1, \tilde{f}_2\colon \operatorname{hocolim}_{\mathcal{C}} X \to Y_0$ satisfy $\tilde{f}_1|_{X(c)} \simeq \tilde{f}_2|_{X(c)} \simeq f(c)$ for all $c \in \mathcal{C}$ then the compositions $\operatorname{hocolim}_{\mathcal{C}} X \xrightarrow{\tilde{f}_1, \tilde{f}_2} Y_0 \xrightarrow{y_d} Y_{d+1}$ are homotopic.*

Proof. (a) We shall define by induction maps $\tilde{f}_i\colon F_i X \to Y_i$ for all $i = 1, \dots, d$ such that $\tilde{f}_i|_{X(c)} \simeq f_{i-1}(c)$ for all $c \in \mathcal{C}$.

Note that, by definition of a system of homotopy compatible maps, we can construct a map $\tilde{f}_1\colon F_1 X \to Y_1$. Assume by induction that $\tilde{f}_i\colon F_i X \to Y_i$ with $\tilde{f}_i|_{X(c)} \simeq f_{i-1}$ has been constructed for some $1 \leq i < d$. The obstruction class Θ'_{i+1} for the extension of $\tilde{f}_i$ to $F_{i+1}X$ is mapped by the homomorphism

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i+1} \pi_i \mathrm{map}^{f_{i-1}(c)}(X(c), Y_i) \xrightarrow{(g_i)_*} \varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i+1} \pi_i \mathrm{map}^{f_i(c)}(X(c), Y_{i+1})$$

to the obstruction class Θ_{i+1} for the extension of $g_i \circ \tilde{f}_i$ to $F_{i+1}X$. When $i \geq 1$, by hypothesis (i) the groups are abelian and this homomorphism is trivial, whence $\Theta_{i+1} = 0$. Wojtkowiak's obstruction theory guarantees the existence of a map $\tilde{f}_{i+1}\colon F_{i+1}X \to Y_{i+1}$ which agrees with $g_i \circ \tilde{f}_i$ on $F_{i-1}X$ and such that $\tilde{f}_{i+1}|_{X(c)} \simeq g_i \circ f_{i-1}(c) = f_i(c)$. This completes the induction step.

Hypothesis (ii) and the assumption on $\mathcal{C}$ imply that the groups

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i} \pi_{i-1} \mathrm{map}^{f_{d-1}}(X(c), Y_d)$$

are trivial for all $i \geq d+1$. Thus, the obstructions to the extension of $\tilde{f}_d$ to $F_i X$ where $i > d$ must all vanish. We can therefore construct by induction on $i \geq d+1$ maps $\tilde{f}_i\colon F_i X \to Y_d$ such that $\tilde{f}_i|_{X(c)} \simeq f_{d-1}(c)$ for all $c \in \mathcal{C}$ and such that $\tilde{f}_{i+1}$

agrees with $\tilde{f}_i$ on $F_{i-1}X$. We can finally define $\tilde{f}\colon \operatorname{hocolim}_{\mathcal{C}} X = \bigcup_i F_iX \to Y_d$ with the required properties. In fact, $\tilde{f}|_{F_nX} = \tilde{f}_{n+1}|_{F_nX}$ for all $n > d$.

(b) First, we construct by induction homotopies $y_i \circ \tilde{f}_1|_{F_iX} \overset{H_i}{\simeq} y_i \circ \tilde{f}_2|_{F_iX}$ for all $i = 0, \dots, d$. Recall that $F_0X = \coprod_{c\in\mathcal{C}} X(c)$ and we define H_0 as the sum of the homotopies $y_0 \circ \tilde{f}_1|_{X(c)} \simeq y_0 \circ \tilde{f}_2|_{X(c)}$.

Assume by induction that $H_i\colon y_i\circ\tilde{f}_1|_{F_iX}\simeq y_i\circ\tilde{f}_2|_{F_iX}$ has been constructed where $0 \le i < d$. The obstruction Υ'_i for the extension of H_i to a homotopy $y_i\circ\tilde{f}_1|_{F_{i+1}X} \simeq y_i \circ \tilde{f}_2|_{F_{i+1}X}$ is mapped by the homomorphism

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i+1}\pi_{i+1}\mathrm{map}^{f_i(c)}(X(c), Y_{i+1}) \xrightarrow{(g_{i+1})_*} \varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i+1}\pi_{i+1}\mathrm{map}^{f_{i+1}(c)}(X(c), Y_{i+2})$$

to the obstruction class Υ_i for the extension of $g_{i+1} \circ H_i\colon I \times F_iX \to Y_{i+2}$ to $I \times F_{i+1}X$. This homomorphism is trivial by hypothesis (i). Therefore $\Upsilon_i = 0$, and by Wojtkowiak's theory there is a homotopy $y_{i+1} \circ \tilde{f}_1|_{F_{i+1}X} \overset{H_{i+1}}{\simeq} y_{i+1} \circ \tilde{f}_2|_{F_{i+1}X}$. This completes the induction step.

Now, the hypothesis on $\mathcal{C}$ together with (ii) imply that the groups

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i}\pi_i\mathrm{map}^{f_d(c)}(X(c), Y_{d+1})$$

are trivial for all $i \ge d+1$. We can therefore construct by induction on $i \ge d+1$ homotopies $y_d \circ \tilde{f}_1|_{F_iX} \overset{H_i}{\simeq} y_d \circ \tilde{f}_2|_{F_iX}$ such that H_{i+1} and H_i agree on $I \times F_{i-1}X$. There results a homotopy $y_d \circ \tilde{f}_1 \simeq y_d \circ \tilde{f}_2$. □

7. Maps between p-local finite groups

7.1. **Definition.** Let $(S,\mathcal{F})$ be a fusion system. A map $f\colon BS \to X$ is called $\mathcal{F}$-invariant, if for every $\varphi \in \mathcal{F}(P,S)$ the composition $BP \xrightarrow{B\varphi} BS \xrightarrow{f} X$ is homotopic to $f|_{BP} = f \circ B\mathrm{incl}_P^S$.

7.2. **Example.** Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. The map $\Theta\colon BS \to |\mathcal{L}|$ of 2.8 is $\mathcal{F}$-invariant by Proposition 2.9.

Given a p-local finite group $(S,\mathcal{F},\mathcal{L})$, the question we address in this section is when an $\mathcal{F}$-invariant map $f\colon BS \to X$ can be extended to a map $|\mathcal{L}| \to X$. Here is the main result of this section which uses the constructions in §3.

7.3. **Theorem.** *Let $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ be p-local finite groups and consider an $\mathcal{F}$-invariant map $f\colon BS \to |\mathcal{L}'|^\wedge_p$. Then:*

(a) *There exists $m > 0$ and a map $\tilde{f}\colon |\mathcal{L}| \to (|\mathcal{L}'| \wr \Sigma_{p^m})^\wedge_p$ which renders the following square homotopy commutative*

$$\begin{array}{ccc} BS & \xrightarrow{f} & |\mathcal{L}'|^\wedge_p \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Delta^\wedge_p} \\ |\mathcal{L}| & \xrightarrow[\tilde{f}]{} & (|\mathcal{L}'| \wr \Sigma_{p^m})^\wedge_p \end{array}$$

(b) *There exists $e > 0$ such that for any two maps $\tilde{f}_1, \tilde{f}_2 \colon |\mathcal{L}| \to |\mathcal{L}'|^\wedge_p$ with $\Theta \circ \tilde{f}_1 \simeq \Theta \circ \tilde{f}_2 \simeq f$, the compositions $|\mathcal{L}| \xrightarrow{\tilde{f}_1, \tilde{f}_2} |\mathcal{L}'|^\wedge_p \xrightarrow{\Delta^\wedge_p} (|\mathcal{L}'| \wr \Sigma_{p^e})^\wedge_p$ are homotopic.*

7.4. **Example.** If $f = \Theta \colon BS \to |\mathcal{L}|$ then $\tilde{f}$ can be chosen as the identity on $|\mathcal{L}|^\wedge_p$.

For a finite abelian group A, set $A_{(p)} = A \otimes \mathbb{Z}_{(p)}$; this is the set of p-power order elements in A. The abelianisation of a group G is denoted G_{ab}. The subgroup $O^p(G)$ of a finite group G is the subgroup generated by all the elements of order prime to p: it is the smallest normal subgroup of G whose quotient is a p-group.

7.5. **Proposition.** *Let $H = G \wr \Sigma_k$ where G is a finite group. If $p > 2$ and $k \geq 2$ then $H/O^p(H)$ is a factor group of $(G_{\mathrm{ab}})_{(p)}$. If $p = 2$ and $k \geq 3$ then $H/O^p(H)$ is a factor group of $(G_{\mathrm{ab}})_{(2)} \times C_2$.*

Proof. Write $\bar{H} = H/O^p(H)$ and consider the quotient homomorphism $\pi \colon H \to \bar{H}$. Denote by G_i the ith copy of G in $G^{\times k}$. For any $x \in G$ we shall denote by x_i the image of $x \in G_i$ in H via the inclusion $G^{\times k} \leq H$. Note that x_i and y_j, where $x, y \in G$, commute in H if $i \neq j$.

Assume that $p > 2$ and that $k = 2$. Since Σ_k is generated by involutions then $\Sigma_k \leq O^p(H)$. Also note that H is generated by Σ_k and any one of G_i, hence $\bar{H}$ is generated by any one of the images of G_i under π. Let τ denote $(1,2) \in \Sigma_k$ (note that $k \geq 2$). Since $\tau \in O^p(H)$ we see that for any $x \in G$ we have $\pi(x_1) = \pi(x_1\tau) = \pi(\tau x_2) = \pi(x_2)$. Thus, given elements $\bar{x}, \bar{y} \in \bar{H}$ we can choose preimages x_1 and y_2 and observe that $\bar{x}\bar{y} = \pi(x_1)\pi(y_2) = \pi(x_1 y_2) = \pi(y_2 x_1) = \bar{y}\bar{x}$. This shows that $\bar{H}$ is a commutative factor group of G and since it is a p-group it must be a factor of $(G_{\mathrm{ab}})_{(p)}$.

Now assume that $p = 2$ and that $k \geq 3$. Clearly $A_k \leq O^2(H)$ because A_k is generated by elements of odd order. Since H is generated by Σ_k and any one of the G_i's, it follows that $\bar{H}$ is generated by the image of $\tau = (1,2) \in \Sigma_k$ and by the images of any one of the G_i's. Let σ denote the cycle $(1,2,3) \in A_k$ (note that $k \geq 3$). Note that $\sigma \in O^2(H)$ and that $\sigma^{-1} x_1 \sigma = x_2$ for any $x \in G$. Therefore

$$\pi(x_1) = \pi(x_2). \tag{1}$$

Let $\bar{\tau}$ denote $\pi(\tau)$. Then $\bar{\tau}$ and the element $\bar{x} = \pi(x_1)$ commute in $\bar{H}$ because

$$\bar{x}\bar{\tau} = \pi(x_1)\pi(\tau) = \pi(x_1\tau) = \pi(\tau x_2) = \bar{\tau}\pi(x_2) = \bar{\tau}\pi(x_1) = \bar{\tau}\bar{x}.$$

This shows that $\bar{\tau} \in Z(\bar{H})$ and that $\bar{H}$ is a factor group of $G \times C_2$ because $\bar{H}$ is generated by $\bar{\tau}$ and $\bar{x}$ for all $x \in G$. Now consider $\bar{x}, \bar{y} \in \bar{H}$ where $\bar{x} = \pi(x_1)$ and $\bar{y} = \pi(y_1)$ for some $x, y \in G$. Since $\pi(y_1) = \pi(y_2)$ by (1), we conclude that

$$\bar{x}\bar{y} = \pi(x_1)\pi(y_2) = \pi(x_1 y_2) = \pi(y_2 x_1) = \pi(y_2)\pi(x_1) = \bar{y}\bar{x}.$$

It follows that $\bar{H}$ is an abelian 2-group hence it is a factor group of $(G_{\mathrm{ab}})_{(2)} \times C_2$. □

7.6. **Lemma.** *For any p-local finite group $(S, \mathcal{F}, \mathcal{L})$, $\pi_i(|\mathcal{L}|^\wedge_p)$ are finite p-groups for all $i \geq 1$.*

Proof. The fundamental group $\pi_1(|\mathcal{L}|^\wedge_p)$ is a finite p-group by [4, Theorem B]. Using a Serre class argument (see [21, Ch 9.6, Theorem 15]), we only need to show that the integral homology is finite at each degree. In [19], it is proven that the suspension spectrum $\Sigma^\infty |\mathcal{L}|^\wedge_p$ is a retract of $\Sigma^\infty BS$ all of whose integral homology groups are finite abelian p-groups. □

7.7. **Proposition.** *Fix an integer $k \geq 3$ and let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Given a map $f\colon BP \to |\mathcal{L}|^\wedge_p$, let g denote the composition*

$$BP \xrightarrow{f} |\mathcal{L}|^\wedge_p \xrightarrow{\Delta} |\mathcal{L}|^\wedge_p \wr \Sigma_k \xrightarrow{\eta} (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p.$$

Then all the homotopy groups of $\mathrm{map}^g(BP, (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p)$ *are finite abelian p-groups.*

Proof. If $S = 1$ then $|\mathcal{L}| = *$ hence $(|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p \simeq (B\Sigma_k)^\wedge_p$ and g is null-homotopic. Dwyer-Zabrodsky's result [12] shows that the space under study is homotopy equivalent to $(B\Sigma_k)^\wedge_p$ and the result follows from Proposition 7.5 together with [6, Proposition A.2] and Lemma 7.6.

We shall therefore assume that $S \neq 1$. By [7, Theorem 4.4(a)] f is homotopic to

$$BP \xrightarrow{\rho} BS \xrightarrow{\Theta} |\mathcal{L}| \xrightarrow{\eta} |\mathcal{L}|^\wedge_p$$

for some $\rho\colon P \to S$. There results a diagram in which the bottom row is g, the first square commutes up to homotopy and the other squares commute on the nose

(1)
$$\begin{array}{ccccccc}
BP & \xrightarrow{\Theta\circ B\rho} & |\mathcal{L}| & \xrightarrow{\Delta} & |\mathcal{L}| \wr \Sigma_k & \xrightarrow{\eta} & (|\mathcal{L}| \wr \Sigma_k)^\wedge_p \\
\Big\| & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta\wr\Sigma_k} & & \simeq \downarrow{\scriptstyle (\eta\wr\Sigma_k)^\wedge_p} \\
BP & \xrightarrow[f]{} & |\mathcal{L}|^\wedge_p & \xrightarrow[\Delta]{} & |\mathcal{L}|^\wedge_p \wr \Sigma_k & \xrightarrow[\eta]{} & (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p.
\end{array}$$

Since $|\mathcal{L}|$ is p-good by [7, Proposition 1.12], a Serre spectral sequence argument and [3, Lemma I.5.5] show that the vertical arrow on the right of the diagram is a homotopy equivalence. It follows that

(2)
$$\mathrm{map}^g(BP, (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p) \simeq \mathrm{map}^{\eta\circ\Delta\circ\Theta\circ B\rho}(BP, (|\mathcal{L}| \wr \Sigma_k)^\wedge_p).$$

By Theorem 1.1 there exists a p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ where S' is a Sylow p-subgroup of $S \wr \Sigma_k$ such that there is a homotopy equivalence $\omega\colon |\mathcal{L}| \wr \Sigma_k \xrightarrow{\simeq} |\mathcal{L}'|$ and the composition

$$BS' \xrightarrow{B\mathrm{incl}} B(S \wr \Sigma_k) \simeq (BS) \wr \Sigma_k \xrightarrow{\Theta\wr\Sigma_k} |\mathcal{L}| \wr \Sigma_k \xrightarrow[\simeq]{\omega} |\mathcal{L}'|$$

is homotopic to $\Theta'\colon BS' \to |\mathcal{L}'|$. Moreover, $\Delta\colon BS \to (BS) \wr \Sigma_k$ is induced by the diagonal inclusion $S \leq S \wr \Sigma_k$ which factors through the Sylow subgroup S', and it is therefore homotopic to $BS \xrightarrow{B\mathrm{incl}} BS' \xrightarrow{B\mathrm{incl}} B(S \wr \Sigma_k) \simeq (BS) \wr \Sigma_k$. We therefore have the following homotopy commutative diagram

$$\begin{array}{ccccccccc}
BS & = & & = & BS & \xrightarrow{\Theta} & |\mathcal{L}| & = & |\mathcal{L}| \\
\downarrow{\scriptstyle B\mathrm{incl}} & & & \swarrow{\scriptstyle B\Delta} & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \omega\circ\Delta} \\
BS' & \xrightarrow[B\mathrm{incl}]{} & B(S \wr \Sigma_k) & \simeq & (BS) \wr \Sigma_k & \xrightarrow[\Theta\wr\Sigma_k]{} & |\mathcal{L}| \wr \Sigma_k & \xrightarrow[\simeq]{\omega} & |\mathcal{L}'|,
\end{array}$$

from which it follows that

(3)
$$BS \xrightarrow{\Theta} |\mathcal{L}| \xrightarrow{\Delta} |\mathcal{L}| \wr \Sigma_k \xrightarrow[\simeq]{w} |\mathcal{L}'| \qquad \text{is homotopic to} \qquad BS \xrightarrow{B\mathrm{incl}} BS' \xrightarrow{\Theta'} |\mathcal{L}'|.$$

Since $w_p^\wedge$ is a homotopy equivalence and $w_p^\wedge \circ \eta = \eta \circ w$, Proposition 2.11(a) and (3) imply that the mapping space in (2) is homotopy equivalent to

$$\text{(4)} \qquad \mathrm{map}^{\eta\circ\Theta'|_{BS}\circ B\rho}(BP, |\mathcal{L}'|_p^\wedge) \simeq \mathrm{map}^{\eta\circ\Theta'|_{BQ}}(BQ, |\mathcal{L}'|_p^\wedge)$$

where $Q = \rho(P) \le S'$. Part (b) of Proposition 2.11 shows that the map obtained by applying the p-completion functor to

$$\text{(5)} \qquad \mathrm{map}^{\Theta'|_{BQ}}(BQ, |\mathcal{L}'|) \xrightarrow{\eta_*} \mathrm{map}^{\eta\circ\Theta'|_{BQ}}(BQ, |\mathcal{L}'|_p^\wedge)$$

induces split surjections on homotopy groups. Since $Q \le S \le S'$ then (3) implies that $\Theta'|_{BQ} \simeq w \circ \Delta \circ \Theta|_{BQ}$ and therefore, after p-completion

$$\text{(6)} \qquad \mathrm{map}^{\Delta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}| \wr \Sigma_k) \xrightarrow{\eta_*} \mathrm{map}^{\eta\circ\Delta\circ\Theta|_{BQ}}(BQ, (|\mathcal{L}| \wr \Sigma_k)_p^\wedge)$$

induces split surjections on homotopy groups where by (4) the space on the right is homotopy equivalent to (2). Diagram (1) shows that (6) factors up to homotopy through

$$\text{(7)} \qquad \mathrm{map}^{\Delta\circ\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge \wr \Sigma_k) \xrightarrow{\eta_*} \mathrm{map}^{\eta\circ\Delta\circ\Theta|_{BQ}}(BQ, (|\mathcal{L}| \wr \Sigma_k)_p^\wedge)$$

which in addition must also be surjective on homotopy groups. It remains to show that the homotopy groups of the space on the left are finite abelian p-groups.

Proposition 3.8(b) implies that

$$\text{(8)} \qquad \mathrm{map}^{\Delta\circ\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge \wr \Sigma_k) \simeq \mathrm{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge) \wr \Sigma_k.$$

By Proposition 2.11(a) the space $\mathrm{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge)$ is homotopy equivalent to the p-completed classifying space of a p-local finite group. It is therefore p-complete by [7, Proposition 1.12] and its homotopy groups are finite p-groups by Proposition 7.6, albeit the fundamental group is not necessarily abelian. By Remark 3.6, the homotopy groups of the mapping space in (8) are

$$\pi_1(\mathrm{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge)) \wr \Sigma_k, \qquad \text{and}$$
$$\oplus_k\, \pi_i(\mathrm{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge)) \qquad \text{for } i > 1.$$

Now [3, Proposition VII.4.3] shows that the homotopy groups of the p-completion of (8) are finite p-groups. The fundamental group is abelian by Proposition 7.5 together with [6, Proposition A.2]. □

Proof of Theorem 7.3. First, we assume that $S \ne 1$, or else the result is a triviality. Set $\mathcal{C} = \mathcal{O}(\mathcal{F}^c)$ and recall from [7, Corollary 3.4] that $\mathcal{C}$ is a finite category which has bounded limits at p of height $d \ge 1$.

We shall now construct inductively a sequence of spaces and maps

$$|\mathcal{L}'|_p^\wedge = Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_d} Y_{d+1}$$

together with integers $m_0, m_1, \dots, m_{d+1}$, where $m_i \ge 2$, with the following properties. First, $Y_0 = |\mathcal{L}'|_p^\wedge$. Set $f_i = g_i \circ \cdots \circ g_0 \circ f \colon BS \to Y_{i+1}$ and set $G_i = \Sigma_{p^{m_0}} \wr \Sigma_{p^{m_1}} \wr \cdots \wr \Sigma_{p^{m_{i-1}}}$. Then the following holds for all $i = 0, \dots, d$.

(i) There are homotopy equivalences $\omega_{i+1} \colon Y_{i+1} \simeq (|\mathcal{L}'|_p^\wedge \wr G_{i+1})_p^\wedge$ such that

$$|\mathcal{L}'|_p^\wedge = Y_0 \xrightarrow{g_i\circ\cdots\circ g_0} Y_{i+1} \xrightarrow[\omega_{i+1}]{\simeq} (|\mathcal{L}'|_p^\wedge \wr G_{i+1})_p^\wedge$$

is homotopic to $|\mathcal{L}|_p^\wedge \xrightarrow{\Delta} |\mathcal{L}|_p^\wedge \wr G_{i+1} \xrightarrow{\eta} (|\mathcal{L}|_p^\wedge \wr G_{i+1})_p^\wedge$.

(ii) $\pi_*(\mathrm{map}^{f_i|_{BP}}(BP, Y_{i+1}))$ are finite abelian p-groups for all $P \le S$.
(iii) If $i \ge 1$ then for all $P \le S$ the homomorphism induced by g_i

$$\pi_i \mathrm{map}^{f_{i-1}|_{BP}}(BP, Y_i) \xrightarrow{(g_i)_*} \pi_i \mathrm{map}^{f_i|_{BP}}(BP, Y_{i+1})$$

is trivial.

Let $\mathcal{L}_0 = \mathcal{L}'$ and $Y_0 = |\mathcal{L}_0|^\wedge_p$. We now define by induction on $i \ge 1$ the integers m_{i-1} and maps $Y_{i-1} \xrightarrow{g_{i-1}} Y_i$ with the properties (i)-(iii) above. To begin the induction set $m_0 = 2$ and $Y_1 = (Y_0 \wr \Sigma_{p^2})^\wedge_p$ and set $g_0 = \eta \circ \Delta(Y_0)$. Condition (i) holds directly from this definition, condition (ii) follows from Proposition 7.7 since $p^2 \ge 4$ and condition (iii) holds vacuously since g_0 is not required to satisfy it.

Assume by induction that m_{i-1} and $g_{i-1} \colon Y_{i-1} \to Y_i$ have been defined for some $1 \le i < d+1$ such that (i)-(iii) hold. We construct the next pair $(g_i \colon Y_i \to Y_{i+1}, m_i)$ as follows. Let p^{m_i} be the maximum of p^2 and the exponent of the finite abelian p-group

$$\bigoplus_{P \in \mathcal{O}(\mathcal{F}^c)} \pi_i\big(\mathrm{map}^{f_{i-1}|_{BP}}(BP, Y_i)\big).$$

Define $Y_{i+1} = (Y_i \wr \Sigma_{p^{m_i}})^\wedge_p$ and let $g_i \colon Y_i \to Y_{i+1}$ be the composition

$$Y_i \xrightarrow{\Delta(Y_i)} Y_i \wr \Sigma_{p^{m_i}} \xrightarrow{\eta} (Y_i \wr \Sigma_{p^{m_i}})^\wedge_p.$$

Since $|\mathcal{L}'|$ is p-good by [7, Proposition 1.12], the induction hypothesis (i) on Y_i, a Serre spectral sequence argument together with [3, I.5.5] and Theorem 1.1 show that

$$Y_i \simeq (|\mathcal{L}'|^\wedge_p \wr G_i)^\wedge_p \simeq (|\mathcal{L}'| \wr G_i)^\wedge_p \simeq |\mathcal{L}_i|^\wedge_p$$

for some p-local finite group $(S_i, \mathcal{F}_i, \mathcal{L}_i)$. Condition (ii) for g_i holds by Proposition 7.7 because $Y_{i+1} \simeq (|\mathcal{L}_i|^\wedge_p \wr \Sigma_{p^{m_i}})^\wedge_p$.

Furthermore, all the homotopy groups of $|\mathcal{L}_i|^\wedge_p \wr \Sigma_{p^{m_i}}$ are finite by Proposition 7.6 and Remark 3.6, whence this space is p-good by [3, Ch. VII.4.3]. It follows that Y_{i+1} is p-complete. Condition (iii) holds for $g_i \colon Y_i \to Y_{i+1}$ by Proposition 4.3 and the way that m_i was chosen.

By induction hypothesis there is a homotopy equivalence $w_i : Y_i \to (|\mathcal{L}'|^\wedge_p \wr G_i)^\wedge_p$ which renders the top-left square in the following diagram homotopy commutative.

$$\begin{array}{ccccccc}
|\mathcal{L}'|^\wedge_p & \xrightarrow{g_{i-1}\circ\cdots\circ g_0} & Y_i & \xrightarrow{\Delta} & Y_i \wr \Sigma_{p^{m_i}} & \xrightarrow{\eta} & Y_{i+1} \\
\downarrow{\scriptstyle\Delta} & & \simeq\downarrow{\scriptstyle w_i} & & {\scriptstyle w_i\wr\Sigma_{p^{m_i}}}\downarrow\simeq & & {\scriptstyle (w_i\wr\Sigma_{p^{m_i}})^\wedge_p}\downarrow\simeq \\
|\mathcal{L}'|^\wedge_p \wr G_i & \xrightarrow{\eta} & (|\mathcal{L}'|^\wedge_p \wr G_i)^\wedge_p & \xrightarrow{\Delta} & (|\mathcal{L}'|^\wedge_p \wr G_i)^\wedge_p \wr \Sigma_{p^{m_i}} & \xrightarrow{\eta} & ((|\mathcal{L}'|^\wedge_p \wr G_i)^\wedge_p \wr \Sigma_{p^{m_i}})^\wedge_p \\
& \searrow{\scriptstyle\Delta} & & \nearrow{\scriptstyle \eta\wr\Sigma_{p^{m_i}}} & & & {\scriptstyle (\eta\wr\Sigma_{p^{m_i}})^\wedge_p}\uparrow\simeq \\
& & |\mathcal{L}'|^\wedge_p \wr G_i \wr \Sigma_{p^{m_i}} & & \xrightarrow{\eta} & & (|\mathcal{L}'|^\wedge_p \wr G_i \wr \Sigma_{p^{m_i}})^\wedge_p
\end{array}$$

The remainder of the diagram commutes and the composition $\eta \circ \Delta(Y_i)$ in the first row is by definition g_i. By Theorem 1.1, [7, Proposition 1.12] and [3, Lemma I.5.5], the arrows on the right are homotopy equivalences. Define the equivalence $w_{i+1} \colon Y_{i+1} \to (|\mathcal{L}'|^\wedge_p \wr G_{i+1})^\wedge_p$ as the composition of the equivalences in the right column. Now property (i) follows from this diagram and Proposition 3.5. Also, the diagram above shows that

$$\mathrm{map}^{f_i|_{BP}}(BP, Y_{i+1}) \simeq \mathrm{map}^{\Delta\circ f|_{BP}}(BP, (|\mathcal{L}'|^\wedge_p \wr G_{i+1})^\wedge_p)$$

and property (ii) for f_i holds by Proposition 7.7.

We now consider the functor $\tilde{B}\colon \mathcal{C} \to \mathbf{Top}$ recalled in 2.7. Clearly $f\colon BS \to |\mathcal{L}'|^\wedge_p$ gives rise to a system of homotopy compatible maps $f_0\colon \tilde{B}(-) \to |\mathcal{L}'|^\wedge_p$ in the sense described in Section §6. By applying part (a) of Theorem 6.2 to the compositions $BS \xrightarrow{f_0} Y_0 \xrightarrow{g_0} \cdots \xrightarrow{g_d} Y_{d+1}$ we conclude that there exists a map $\tilde{f}_0\colon |\mathcal{L}| \to Y_d \simeq (|\mathcal{L}'| \wr G_d)^\wedge_p$ whose restriction to BS is homotopic to

$$BS \xrightarrow{f} |\mathcal{L}'|^\wedge_p \xrightarrow{\eta\circ\Delta} (|\mathcal{L}'|^\wedge_p \wr G_d)^\wedge_p. \tag{1}$$

Since $|\mathcal{L}'|$ is p-good by [7, Proposition 1.12], we have the following commutative diagram in which the vertical right arrow is a homotopy equivalence

$$\begin{array}{ccccc} |\mathcal{L}'| & \xrightarrow{\Delta} & |\mathcal{L}'| \wr G_d & \xrightarrow{\eta} & (|\mathcal{L}'| \wr G_d)^\wedge_p \\ \eta\downarrow & & \eta\wr G_d\downarrow & & \simeq\downarrow \eta\wr G_d{}^\wedge_p \\ |\mathcal{L}'|^\wedge_p & \xrightarrow{\Delta} & |\mathcal{L}'|^\wedge_p \wr G_d & \xrightarrow{\eta} & (|\mathcal{L}'|^\wedge_p \wr G_d)^\wedge_p. \end{array}$$

Therefore $Y_d \simeq (|\mathcal{L}'| \wr G_d)^\wedge_p$. From Theorem 1.1 we also see that the spaces on the right of this diagram are p-complete. Applying [3, Proposition II.2.8] we deduce that $\eta\circ\Delta$ in (1) is homotopic to $|\mathcal{L}'|^\wedge_p \xrightarrow{\Delta^\wedge_p} (|\mathcal{L}'| \wr G_d)^\wedge_p$ composed with the equivalence in the right of the diagram. Part (a) of this theorem follows by composition with the map induced by the inclusion $G_d \leq \Sigma_{p^{m_0+\cdots+m_{d-1}}}$.

To prove part (b), we analogously apply part (b) of Theorem 6.2 to deduce that

$$|\mathcal{L}| \overset{\tilde{f}_1}{\underset{\tilde{f}_2}{\rightrightarrows}} |\mathcal{L}'|^\wedge_p \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_d} Y_{d+1} \simeq (|\mathcal{L}'| \wr G_{d+1})^\wedge_p$$

are homotopic. The result now follows by composition with the map induced by the inclusion $G_{d+1} \leq \Sigma_{p^{m_0+\cdots+m_d}}$. □

Proof of Theorem 1.3. The induced map $BS \xrightarrow{B\rho} BS' \xrightarrow{\eta\circ\Theta'} |\mathcal{L}'|^\wedge_p$ is clearly $\mathcal{F}$-invariant because $BS' \to |\mathcal{L}'|^\wedge_p$ is $\mathcal{F}'$-invariant by 7.2 and ρ is fusion preserving. The result is now a direct consequence of Theorem 7.3 and Theorem 1.1. □

We say that $\rho\colon S \to \Sigma_n$ is $\mathcal{F}$-invariant if $\rho|_P$ and $\rho\circ\varphi$ are equivalent representations for every $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$.

7.8. **Proposition.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $\rho\colon S \to \Sigma_n$ be a homomorphism. Then the following statements are equivalent:*

(1) *ρ is $\mathcal{F}$-invariant.*
(2) *$B\rho\colon BS \to B\Sigma_n$ is an $\mathcal{F}$-invariant map.*
(3) *$\eta\circ B\rho\colon BS \to (B\Sigma_n)^\wedge_p$ is an $\mathcal{F}$-invariant map.*

Proof. It follows immediately from Dwyer-Zabrodsky's result [12] which gives rise to bijections $\mathrm{Rep}(P, \Sigma_n) \approx [BP, B\Sigma_n] \underset{\approx}{\xrightarrow{\eta_*}} [BP, (B\Sigma_n)^\wedge_p]$ for all $P \leq S$. □

7.9. **Proposition.** *The regular permutation representation of a finite p-group S induces an $\mathcal{F}$-invariant map $B\operatorname{reg}_S\colon BS \to B\Sigma_{|S|}$ for any fusion system $\mathcal{F}$ on S.*

Proof. By Proposition 7.8, it is enough to check that $\operatorname{reg}_S\colon S \to \Sigma_{|S|}$ is $\mathcal{F}$-invariant. Note that S acts freely on S via $\operatorname{reg}_S\colon S \to \Sigma_{|S|}$, that is all the isotropy subgroups are trivial. In particular, any group monomorphism $\varphi\colon P \to S$ where $P \leq S$ renders S a free P-set via $\operatorname{reg}_S \circ\varphi$. Since any two free P-sets of the same cardinality are equivalent, it follows that $\operatorname{reg}_S|_P$ and $\operatorname{reg}_S\circ\varphi$ are conjugate in Σ_n. □

By Example 7.2 and Proposition 7.8, every map $f\colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ gives rise to an $\mathcal{F}$-invariant representation ρ of S of rank n where $B\rho \simeq f|_{BS}$. Not every $\mathcal{F}$-invariant representation of S arises necessarily in this way. However, next proposition gives a partial answer to that question.

7.10. **Proposition.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*

(a) *Given $\rho \in \operatorname{Rep}_n(\mathcal{F})$, there exists some $k \geq 0$ and an element $\tilde{f} \in \operatorname{Rep}_{p^k n}(\mathcal{L})$ such that $\tilde{f}|_{BS}$ is homotopic to $BS \xrightarrow{B(p^k\cdot\rho)} B\Sigma_{p^k n} \xrightarrow{\eta} (B\Sigma_{p^k n})^\wedge_p$.*

(b) *Consider $f_1, f_2 \in \operatorname{Rep}_n(\mathcal{L})$ such that $f_1|_{BS} \simeq f_2|_{BS}$. Then there exists some $e \geq 0$ such that $p^e \cdot f_1 = p^e \cdot f_2$ in $\operatorname{Rep}_{p^e n}(\mathcal{L})$.*

Proof. Let $(S, \mathcal{F}, \mathcal{L})$ be the p-local finite group associated with Σ_n. Then [7, Proposition 1.12] with a standard Serre spectral sequence argument show that

$$(1)\quad (B\Sigma_n)^\wedge_p \simeq |\mathcal{L}|^\wedge_p \xrightarrow{\Delta^\wedge_p} (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p \simeq ((B\Sigma_n)^\wedge_p \wr \Sigma_k)^\wedge_p \xrightarrow{B\mathrm{incl}^\wedge_p} (B\Sigma_{nk})^\wedge_p \quad \text{and}$$

$$(B\Sigma_n)^\wedge_p \xrightarrow{(B\Delta)^\wedge_p} (B\Sigma_{nk})^\wedge_p$$

where $\Delta\colon \Sigma_n \leq \Sigma_{nk}$ is the diagonal inclusion, are homotopic. Both (a) and (b) follow directly from Proposition 7.8, Theorem 7.3 and (1) taking into account the definition of the operation $+$ in $\coprod_{n\geq 0}\operatorname{Rep}_n(\mathcal{F})$ and $\coprod_{n\geq 0}\operatorname{Rep}_n(\mathcal{L})$. □

Proof of Theorem 1.5. Apply Propositions 7.9 and 7.10(a) to obtain some $f \in \operatorname{Rep}_{p^k\cdot|S|}(\mathcal{L})$ such that $f|_{BS}$ is homotopic to $\eta\circ B(p^k\cdot\operatorname{reg}_S)$, that is, $\Phi(f) = p^k\cdot\operatorname{reg}_S$.

By [6, Lemma 2.3], $H^*(S;\mathbb{F}_p)$ is a finitely generated module over the Noetherian $\mathbb{F}_p$-algebra $H^*(B\Sigma_{p^k\cdot|S|};\mathbb{F}_p)$ via the algebra map $(p^k\cdot\operatorname{reg}_S)^*$. Finally, $H^*(|\mathcal{L}|;\mathbb{F}_p)$ is a submodule of $H^*(S;\mathbb{F}_p)$ by [7, Theorem B] and it is therefore finitely generated. Now apply [6, Lemma 2.3] again to deduce that f is a homotopy monomorphism. □

8. The index of the Sylow subgroup

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $f\colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ be a map. The restriction $f|_{BS} = f\circ\Theta$ is $\mathcal{F}$-invariant by Example 7.2 and is homotopic to $(B\rho)^\wedge_p$ for a unique $\rho \in \operatorname{Rep}(S, \Sigma_n)$ which is $\mathcal{F}$-invariant by Proposition 7.8 and [12]. There results maps $\operatorname{Rep}_n(\mathcal{L}) \to \operatorname{Rep}_n(\mathcal{F})$ which are compatible with the operations $+$ and $\times$ defined in the introduction. They give rise to a ring homomorphism

$$\Phi\colon \operatorname{Rep}(\mathcal{L}) \to \operatorname{Rep}(\mathcal{F}).$$

8.1. **Proposition.** *The abelian groups underlying* $\ker(\Phi)$ *and* $\operatorname{coker}(\Phi)$ *are p-torsion.*

Proof. An element in $\ker(\Phi)$ has the form $f_1 - f_2$ where $f_1, f_2 \in \operatorname{Rep}_n(\mathcal{L})$ for some n and $f_1|_{BS} \simeq f_2|_{BS}$. Proposition 7.10 implies that $p^e \cdot (f_1 - f_2) = 0$ in $\operatorname{Rep}(\mathcal{L})$ and it follows that $\ker(\Phi)$ is p-torsion.

An element of $\operatorname{Rep}(\mathcal{F})$ has the form $\rho_1 - \rho_2$ for some $\rho_1 \in \operatorname{Rep}_{n_1}(\mathcal{F})$ and $\rho_2 \in \operatorname{Rep}_{n_2}(\mathcal{F})$. By Proposition 7.10, the definition of Φ and the definition of the operations $+$ in $\operatorname{Rep}(\mathcal{F})$ and $\operatorname{Rep}(\mathcal{L})$, we see that there exist integers $k_1, k_2 \geq 0$ and representations $f_1 \in \operatorname{Rep}_{p^{k_1} n_1}(\mathcal{L})$ and $f_2 \in \operatorname{Rep}_{p^{k_2} n_2}(\mathcal{L})$ such that $\Phi(f_1) = p^{k_1} \cdot \rho_1$ and $\Phi(f_2) = p^{k_2} \cdot \rho_2$. Then $\omega = p^{k_2} \cdot f_1 - p^{k_1} \cdot f_2$ is an element of $\operatorname{Rep}(\mathcal{L})$ such that $\Phi(\omega) = p^{k_1+k_2}(\rho_1 - \rho_2)$. It follows that $\operatorname{coker}(\Phi)$ is p-torsion. □

By Propositions 7.9 the ring $\operatorname{Rep}(\mathcal{F})$ contains $\operatorname{reg}_S \colon S \to \Sigma_{|S|}$ which generates an (additive) infinite cyclic group $\operatorname{Rep}^{\operatorname{reg}}(\mathcal{F}) := \{n \cdot \operatorname{reg}_S\}_{n \in \mathbb{Z}}$ in $\operatorname{Rep}(\mathcal{F})$. Similarly let $\operatorname{Rep}^{\operatorname{reg}}(\mathcal{L})$ denote the additive subgroup of the ring $\operatorname{Rep}(\mathcal{L})$ generated by all the S-regular representations of $(S, \mathcal{F}, \mathcal{L})$; See Definition 1.4.

It follows directly from the definitions that Φ restricts to a group homomorphism

$$\Phi^{\operatorname{reg}} : \operatorname{Rep}^{\operatorname{reg}}(\mathcal{L}) \to \operatorname{Rep}^{\operatorname{reg}}(\mathcal{F}).$$

8.2. **Corollary.** *The cokernel of* $\Phi^{\operatorname{reg}}$ *is a cyclic p-group. The kernel of* $\Phi^{\operatorname{reg}}$ *is an abelian torsion p-group and* $\operatorname{Rep}^{\operatorname{reg}}(\mathcal{L}) \cong \mathbb{Z} \oplus$ *abelian p-torsion group.*

Proof. This follows from Proposition 8.1 which in particular implies that the image of $\Phi^{\operatorname{reg}}$ is isomorphic to $\mathbb{Z}$, whence it splits off from $\operatorname{Rep}^{\operatorname{reg}}(\mathcal{L})$. □

Given a finite group G there is a natural one-to-one correspondence between equivalence classes of permutation representations $G \to \Sigma_n$ and equivalence classes of G-sets of cardinality n. Sum and products of representations (as described in the introduction) correspond to disjoint unions and products of the associated G-sets. Note that reg_G corresponds to a free G-set with one orbit.

Let us return to discuss $\operatorname{Rep}(\mathcal{F})$. Since the product of a free S-set with any other S-set is again a free set, it follows that $\operatorname{Rep}^{\operatorname{reg}}(\mathcal{F})$ and $\operatorname{Rep}^{\operatorname{reg}}(\mathcal{L})$ are in fact ideals in $\operatorname{Rep}(\mathcal{F})$ and $\operatorname{Rep}(\mathcal{L})$ and that $\Phi^{\operatorname{reg}}$ is a ring homomorphism.

8.3. **Example.** Let $(S, \mathcal{F}, \mathcal{L})$ be the p-local finite group of a finite group G. The restriction of $(B \operatorname{reg}_G)^\wedge_p \colon |\mathcal{L}|^\wedge_p \to (B\Sigma_{|G|})^\wedge_p$ to BS is homotopic to $n \cdot (B \operatorname{reg}_S)^\wedge_p$ where $n = |G \colon S|$ because $\operatorname{reg}_G \colon G \to \Sigma_{|G|}$ renders G a free G-set, whence a free S-set. In particular $(B \operatorname{reg}_G)^\wedge_p \circ \Theta$ is an element in $\operatorname{Rep}^{\operatorname{reg}}(\mathcal{L})$ which is mapped by Φ to $n \cdot \operatorname{reg}_S$. It follows that $|G \colon S| \in \operatorname{Im}(\Phi^{\operatorname{reg}})$, whence $|\operatorname{coker}(\Phi^{\operatorname{reg}})|$ divides $|G \colon S|$.

8.4. **Definition.** Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Define the upper and lower index of S in $\mathcal{L}$ by

$$\operatorname{Uind}(\mathcal{L} \colon S) = |\operatorname{coker}(\Phi^{\operatorname{reg}})|$$
$$\operatorname{Lind}(\mathcal{L} \colon S) = |\operatorname{Rep}^{\operatorname{reg}}(\mathcal{F}) : \operatorname{Rep}^{\operatorname{reg}}(\mathcal{F}) \cap \operatorname{Im}(\Phi)|.$$

Clearly $\operatorname{Lind}(\mathcal{L} \colon S)$ divides $\operatorname{Uind}(\mathcal{L} \colon S)$ because $\operatorname{Im}(\Phi^{\operatorname{reg}}) \leq \operatorname{Im}(\Phi) \cap \operatorname{Rep}^{\operatorname{reg}}(\mathcal{F})$.

8.5. **Lemma.** *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a p-local finite group. Then* $\operatorname{Uind}(\mathcal{L} \colon S)$ *is a p-power. If there exists a permutation representation* $\rho \colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ *such that* $\rho|_{BS} \simeq B(n \cdot \operatorname{reg}_S)$ *with* $n \geq 1$ *prime to p, then* $\operatorname{Uind}(\mathcal{L} \colon S) = 1$, *and in particular also* $\operatorname{Lind}(\mathcal{L} \colon S) = 1$.

Proof. The first statement follows from Corollary 8.2. The existence of ρ shows that $n \in \operatorname{Im}(\Phi^{\operatorname{reg}})$ hence, $\operatorname{Uind}(\mathcal{L} \colon S) = 1$. □

We shall now prove Theorem 1.8. In fact we prove the following stronger result.

8.6. **Theorem.** *Under the hypotheses of Theorem 1.8 we have* $\mathrm{Uind}(\mathcal{L}\colon S) = 1$.

Proof. (1) This follows from Lemma 8.5 and Example 8.3.

(2) Let $\mathcal{C}_n$ be the poset $\{c_0, c_1^i, c_2^i| \quad i = 1, \dots, n\}$ whose only relations are defined by $c_1^i \prec c_0$ and $c_1^i \prec c_2^i$ for all $i = 1, \dots, n$. View $\mathcal{C}_n$ as a small category where $x \prec y$ corresponds to an arrow $x \to y$.

In [16, Section 7], the authors prove that if the longest chain of proper inclusions of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of S has length ≤ 2, then $|\mathcal{L}| \simeq \mathrm{hocolim}_{\mathcal{C}_n} F$ where the functor $F\colon \mathcal{C}_n \to \mathbf{Top}$ has the following properties. The values of F are the classifying spaces of finite groups G_0, G_1^i and G_2^i for $i = 1, \dots, n$ and the maps $F(c_1^i) \to F(c_0)$ and $F(c_1^i) \to F(c_2^i)$ are induced by inclusion of groups $G_1^i \leq G_0$ and $G_1^i \leq G_2^i$. In addition $k_i = |G_2^i\colon G_1^i|$ are prime to p, and S is a subgroup of G_0 of index prime to p. Also, the map $\Theta\colon BS \to |\mathcal{L}|$ factors up to homotopy through $BG_0 \simeq F(c_0) \to \mathrm{hocolim}_{\mathcal{C}_n} F \simeq |\mathcal{L}|$.

Set $k = \prod_1^n k_i$ and $k_0 = |G_0| \cdot k$. Note that k_0 is divisible by $|G_1^i|$ and $|G_2^i|$ for all i because $k_0 = k \cdot |G_0| = k \cdot |G_1^i| \cdot |G_0\colon G_1^i|$ and k_i divides k. Set $\ell_i = k_0/|G_1^i|$ and $m_i = k_0/|G_2^i|$. Consider the following permutation representations for $i = 1, \dots, n$

$$k \cdot \mathrm{reg}_{G_0}\colon G_0 \to \Sigma_{k_0}, \qquad \ell_i \cdot \mathrm{reg}_{G_1^i}\colon G_1^i \to \Sigma_{k_0}, \qquad m_i \cdot \mathrm{reg}_{G_2^i}\colon G_2^i \to \Sigma_{k_0}.$$

Note that $(k \cdot \mathrm{reg}_{G_0})|_{G_1^i}$ and $(m_i \cdot \mathrm{reg}_{G_2^i})|_{G_1^i}$ are equivalent to $\ell_i \cdot \mathrm{reg}_{G_1^i}$ because all of them render the set $\{1, \dots, k_0\}$ a free G_1^i-set with ℓ_i orbits. By taking classifying spaces there results a system of homotopy compatible maps $F \to B\Sigma_{k_0}$. It can be rectified to a system of compatible maps $F \to B\Sigma_{k_0}$ as follows. First, set the maps $F(c_1^i) \to B\Sigma_{k_0}$ to be the composition of $F(c_1^i) \to F(c_0) \to B\Sigma_{k_0}$. Next, replace the maps $F(c_1^i) \to F(c_i^2)$ by cofibrations and change the maps $F(c_2^i) \to B\Sigma_n$ up to homotopy to obtain a system of compatible maps $F \to B\Sigma_{k_0}$.

There results a map $f\colon |\mathcal{L}| \simeq \mathrm{hocolim}\, F \to B\Sigma_{k_0}$ such that $f|_{BS} = f \circ B\iota_S^{G_0} \simeq k \cdot |G_0\colon S| \cdot B\,\mathrm{reg}_S$ where $k \cdot |G_0\colon S|$ is prime to p. By applying Lemma 8.5 we deduce that $\mathrm{Uind}(\mathcal{L}\colon S) = 1$.

Now, all the exotic examples in [7, Examples 9.3, 9.4], [8] and [11] satisfy the condition of [16, Section 7] that chains of proper inclusions of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of S have length ≤ 2. □

8.7. **Conjecture.** *For all p-local finite groups* $\mathrm{Uind}(\mathcal{L}\colon S) = 1$.

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