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On the homotopy groups of *p*-completed classifying spaces

Abstract. Among the generalizations of Serre's theorem on the homotopy groups of a finite complex we isolate the one proposed by Dwyer and Wilkerson. Even though the spaces they consider must be 2-connected, we show that it can be used to both recover known results and obtain new theorems about *p*-completed classifying spaces.

Introduction

In 1953 Serre proved in his celebrated paper [24] that a simply connected finite CW-complex has infinitely many non-trivial homotopy groups. He conjectured that it should actually have infinitely many non-trivial homotopy groups with 2-torsion, which was proved by McGibbon and Neisendorfer in 1983 [20] by using Miller's solution [21] of the Sullivan conjecture. They show this phenomenon holds for any simply connected CW-complex with finite mod 2 cohomology, replacing thereby the geometric finiteness condition by a purely algebraic one. Later, in 1986, Lannes and Schwartz [16] were able to relax the finiteness condition to *locally finite* mod *p* cohomology, i.e. the cohomology is a direct limit of finite unstable modules over the Steenrod algebra. So they proved Serre's conjecture for "Miller spaces", that is, 1-connected spaces X for which the space of pointed maps from $B\mathbb{Z}/p$ to X is contractible.

In 1990, Dwyer and Wilkerson [12] show the following generalization of Serre's conjecture. Let X be a 2-connected CW-complex of finite type with non-trivial mod *p* cohomology, and such that the module of indecomposable elements in $H^*(X; \mathbb{F}_p)$ is locally finite. Then, infinitely many homotopy groups of X contain *p*-torsion. This *new* algebraic condition obviously includes the previous ones, namely spaces with finite or locally finite mod *p* cohomology. Moreover, their condition enables to study spaces with finitely generated mod *p* cohomology, because the module of indecomposable elements is then finite.

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The cost in the Dwyer-Wilkerson theorem is that one has to work with 2-connected spaces. As they say, "the example of $\mathbb{C}P^\infty$ shows that it would not be enough to assume that X is 1-connected". This is basically the only simply connected Postnikov piece with locally finite module of indecomposable elements, compare with Grodal's results [13, Theorem 1.2] and [14, Theorem 4.2].

Theorem 2.3. *Let X be a p-complete space such that $H^*(X; \mathbb{F}_p)$ is of finite type. Assume that the module of indecomposable elements $QH^*(X; \mathbb{F}_p)$ is locally finite. Then one of the following properties is satisfied:*

(1) X is aspherical,
(2) $X\langle 1\rangle$ is equivalent to a finite product of copies of $K(\mathbb{Z}_p^\wedge, 2)$,
(3) X has infinitely many homotopy groups with p-torsion.

In the last case the space ΩX has infinitely many non-trivial k-invariants.

In particular, this implies the Lannes-Schwartz result, see Corollary 2.4, and in fact Theorem 2.3 can even be applied to understand spaces having a non-trivial fundamental group, such as classifying spaces of discrete groups. A very exciting problem in homotopy theory is to determine the behavior of the p-completion of classifying spaces. When G is a finite group, Levi proves in [17] that either $(BG)_p^\wedge$ is again an Eilenberg-Mac Lane space or it has infinitely many non-trivial homotopy groups. Later, Bastardas and Descheemaeker discovered that the same phenomenon holds for any virtually nilpotent group. This is done in [2] for torsion free groups and the general case is solved in [1]. We show that all these results can be deduced from Theorem 2.3 and we obtain the same statement for certain quasi p-perfect groups of finite virtual mod p cohomology and also for the new concept of p-local finite group, due to Broto, Levi, and Oliver [8].

Theorem 4.1. *Let X be the classifying space of a member of the following four families:*

(1) finite groups,
(2) p-local finite groups,
(3) finitely generated virtually nilpotent groups,
(4) quasi p-perfect groups of finite virtual mod p cohomology.

Then the p-completion of X is either aspherical or it has infinitely many homotopy groups with p-torsion. In this case the space $\Omega(X_p^\wedge)$ has infinitely many non-trivial k-invariants.

1. Local loop spaces

The grounding result for this paper is the equivalence between the algebraic condition that the module of indecomposable elements $QH^*(X; \mathbb{F}_p)$ be locally finite and the topological one that the loop space ΩX is $B\mathbb{Z}/p$-local. The proof of [12, Theorem 3.2] is done at the prime 2 for 1-connected spaces. Although this is probably well-known to the experts, we give here an alternative proof for this result that includes arbitrary connected spaces.

Lemma 1.1. *Let X be a p-complete, connected space such that $H^*(X; \mathbb{F}_p)$ is of finite type. Then $QH^*(X; \mathbb{F}_p)$ is locally finite if and only if ΩX is $B\mathbb{Z}/p$-local.*

Proof. By [23, Proposition 3.9.7 and 6.4.5] $QH^*(X; \mathbb{F}_p)$ is locally finite if and only if $T_V(H^*(X; \mathbb{F}_p))_{T_V(c)} \cong H^*(X; \mathbb{F}_p)$ for any elementary abelian p-group V. By [15, Proposition 3.4.4], this is so if and only if Lannes' T functor computes the cohomology of $\mathrm{map}(BV, X)_c$. Therefore, the above isomorphism can be restated by saying that $\mathrm{map}(BV, X)_c \simeq X$, i.e. ΩX is $B\mathbb{Z}/p$-local.   □

Observe that many interesting spaces verify the condition that the module of indecomposable elements is locally finite. Let us mention one class of examples taken from Lannes, [15]. Recall that a group verifies *virtually* a certain property if it admits a subgroup of finite index which verifies the property.

Proposition 1.2. *[15, p. 203] Let G be a group of virtually finite mod p cohomological dimension. Then $T H^*(G; \mathbb{F}_p) \cong \prod_{\rho \in Rep(\mathbb{Z}/p, G)} H^*(C_G(\rho); \mathbb{F}_p)$. If moreover G has virtually finite mod p cohomology, then there is a weak equivalence of mapping spaces $\mathrm{map}(B\mathbb{Z}/p, BG)_p^\wedge \simeq \mathrm{map}(B\mathbb{Z}/p, (BG)_p^\wedge)$.   □*

The second part of the proposition is a consequence of [15, Proposition 3.4.3] which states that the T functor computes the cohomology of the corresponding mapping space.

Corollary 1.3. *Let G be a group of virtually finite mod p cohomology. Then, the space $\Omega\big((BG)_p^\wedge\big)$ is $B\mathbb{Z}/p$-local.*

Proof. Consider the fibration

$$\mathrm{map}_*(B\mathbb{Z}/p, (BG)_p^\wedge) \to \mathrm{map}(B\mathbb{Z}/p, (BG)_p^\wedge) \to (BG)_p^\wedge.$$

By Proposition 1.2, we know that the total space is equivalent to the p-completed mapping space $\mathrm{map}(B\mathbb{Z}/p, BG)_p^\wedge$, which can be in turn identified with the p-completion of $\coprod BC_G(\rho)$, where the disjoint union is taken over the representations $Rep(\mathbb{Z}/p, G)$, see for example [7, Proposition 7.1]. The base point is given by the trivial representation, i.e. lies in the classifying space of the trivial representation, whose centralizer is G itself. Therefore, looping once the above fibration, we obtain that $\mathrm{map}_*(B\mathbb{Z}/p, \Omega(BG)_p^\wedge)$ is contractible.   □

Example 1.4. A virtually nilpotent group G is by definition an extension of a finite group Q by a nilpotent group N. We notice first that if G is finitely generated one can always assume that N is torsion free since any finitely generated nilpotent group is virtually torsion free. Finitely generated torsion free nilpotent groups have finite cohomological dimension, see for example [9, VIII.2]. We infer from the above corollary that $\Omega(BG)_p^\wedge$ is $B\mathbb{Z}/p$-local for any finitely generated virtually nilpotent group G.

Consider the inclusion $O^p(Q) \to Q$ of the maximal p-perfect subgroup of Q as in [17] and construct the pull-back $G' = \lim(O^p(Q) \to Q \leftarrow G)$. Since the quotient $Q/O^p(Q)$ is a p-group P, the fibration $BO^p(Q) \to BQ \to BP$ is preserved by p-completion, and so is the pull-backed one $BG' \to BG \to BP$. Therefore $(BG)_p^\wedge$ is the total space of a fibration

$$(BG')^\wedge_p \longrightarrow (BG)^\wedge_p \longrightarrow BP$$

where P is a finite p-group and G' has a normal, finitely generated, torsion free, nilpotent subgroup N such that the quotient is p-perfect.

When G is virtually nilpotent, finitely generated, and torsion free, BG is an *infra-nilmanifold* (see [10, Theorem 3.1.3]) so that the cohomology of G itself is finite dimensional. In this case $(BG)^\wedge_p$ is $B\mathbb{Z}/p$-local (as is its loop space of course).

The class of groups of finite virtual cohomological dimension is much larger than the class of virtually nilpotent ones, but we do not know if all these groups are $\mathbb{F}_p$-good, which prevents us from being able to obtain our results in full generality. It is well-known that spaces with p-perfect fundamental group are $\mathbb{F}_p$-good, [5, Proposition 3.2].

Example 1.5. Let G be a p-perfect group of virtually finite mod p cohomology. Then $(BG)^\wedge_p$ is simply connected and $\Omega\big((BG)^\wedge_p\big)$ is $B\mathbb{Z}/p$-local. Examples of such groups are given by the special linear groups $SL_n(\mathbb{Z})$ and the Steinberg groups $St_n(\mathbb{Z})$, which are even perfect groups. The homotopy groups of their p-completed classifying spaces are closely related to the algebraic K-theory groups of $\mathbb{Z}$.

Our next example is a slight generalization. Recall that the *lower p-central series* of a group G is defined inductively by $\Gamma^p_0(G) = G$ and $\Gamma^p_{n+1}(G)$ is generated by elements of form $xyx^{-1}y^{-1}z^p$ for $x \in G$ and $y, z \in \Gamma^p_n(G)$. In particular, a group G is p-perfect if and only if $G = \Gamma^p_1(G)$.

Example 1.6. In analogy with the terminology used by Wagoner in [25] and Loday in [19], we say that a group G is *quasi p-perfect* if the subgroup $\Gamma^p_1(G)$ is p-perfect. This means that G is an extension of an elementary abelian p-group with a p-perfect one. To make sure that BG is $\mathbb{F}_p$-good we impose the following condition:

For any finite set $g_1, \ldots, g_n$ of elements in $\Gamma^p_1(G)$ and $g \in G$ there exists an element $h \in \Gamma^p_1(G)$ such that $gg_i g^{-1} = hg_i h^{-1}$ for all $1 \leq i \leq n$.

This actually turns $(BG)^\wedge_p$ into a simple space, compare with [25, Lemma 1.3]. If one requires that G has virtually finite mod p cohomology, one obtains new examples of groups G such that $\Omega\big((BG)^\wedge_p\big)$ is $B\mathbb{Z}/p$-local.

Example 1.7. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, as defined by Broto, Levi, and Oliver in [8, Definition 1.8] and consider its classifying space $|\mathcal{L}|^\wedge_p$. We know from [8, Theorem 5.8] that $H^*(|\mathcal{L}|^\wedge_p; \mathbb{F}_p)$ is noetherian (it can be computed in fact by stable elements, just like the cohomology of an ordinary finite group). Therefore, the module of indecomposable elements is finite, and hence $\Omega(|\mathcal{L}|^\wedge_p)$ is $B\mathbb{Z}/p$-local by Lemma 1.1.

2. The Dwyer-Wilkerson theorem

We recall in this section the theorem of Dwyer and Wilkerson about homotopy groups of 2-connected spaces with locally finite module of indecomposable elements (their statement is about CW-complexes, but it holds under the more general

assumptions of [12, Theorem 1.2]). We explain then how it can be efficiently applied to understand certain spaces which are not 2-connected by considering their 2-connected cover.

Theorem 2.1. *[12, Theorem 1.3] Let X be a 2-connected space such that the mod p cohomology $H^*(X; \mathbb{F}_p)$ is of finite type. Assume that $\tilde{H}^*(X; \mathbb{F}_p) \neq 0$ and that the module of indecomposable elements $QH^*(X; \mathbb{F}_p)$ is locally finite. Then there exist infinitely many integers k such that $\pi_k X$ contains p-torsion.* ☐

The following elementary lemma (compare with [16, Lemma 1.4.4]) is the key to understand which are the spaces that make it impossible to relax the connectivity assumption in the theorem.

Lemma 2.2. *Let $X = K(A, 2)$ be a p-complete space such that $H^*(X; \mathbb{F}_p)$ is of finite type. Assume that ΩX is $B\mathbb{Z}/p$-local. Then A is isomorphic to a finite direct sum of copies of $\mathbb{Z}_p^\wedge$.*

Proof. Obviously A must be p-torsion free since $\Omega X \simeq K(A, 1)$ is assumed to be $B\mathbb{Z}/p$-local. Since $H^*(X; \mathbb{F}_p)$ is of finite type, we infer that $H_1(A; \mathbb{F}_p)$, which is isomorphic to $H_2(X; \mathbb{F}_p)$, is finite. Hence, by [4, Lemma 7.5], A is an abelian p-torsion free Ext-p-complete group of finite type, i.e. A is isomorphic to a finite direct sum of copies of $\mathbb{Z}_p^\wedge$ (use Harrison's classification [5, VI.4.5] or Bousfield's comment on p-adically polycyclic groups in [4, p. 347]). ☐

Theorem 2.3. *Let X be a p-complete space such that $H^*(X; \mathbb{F}_p)$ is of finite type. Assume that the module of indecomposable elements $QH^*(X; \mathbb{F}_p)$ is locally finite. Then one of the following properties is satisfied:*

(1) X is aspherical,
(2) $X\langle 1 \rangle$ is equivalent to a finite product of copies of $K(\mathbb{Z}_p^\wedge, 2)$,
(3) X has infinitely many homotopy groups with p-torsion.

In the last case the space ΩX has infinitely many non-trivial k-invariants.

Proof. Let $Y = X\langle 1 \rangle$ be the universal cover of X. Let us assume that X is not aspherical, and consider the 2-connected cover $Y\langle 2 \rangle$ of Y, which can be seen as the total space in a fibration

$$K(\pi_2 Y, 1) \to Y\langle 2 \rangle \to Y.$$

We know that the loop space ΩY is $B\mathbb{Z}/p$-local by [12, Theorem 3.2] and $\Omega K(\pi_2 Y, 1)$ is homotopically discrete. Therefore, if we loop once this fibration, we see that $\Omega(Y\langle 2 \rangle)$ must be $B\mathbb{Z}/p$-local as well. This means precisely that the module of indecomposable elements $QH^*(Y\langle 2 \rangle; \mathbb{F}_p)$ is locally finite. From Theorem 2.1 we infer that $Y\langle 2 \rangle$ (completed at p) is either contractible or has infinitely many homotopy groups with p-torsion, i.e. Y itself has infinitely many homotopy groups with p-torsion unless its p-completion is an Eilenberg-Mac Lane space of type $K(A, 2)$. In this case we infer from Lemma 2.2 that A is isomorphic to a finite direct sum of copies of $\mathbb{Z}_p^\wedge$.

The statement about the k-invariants is a direct consequence of Proposition 1.3. Indeed if the loop space ΩX has only a finite number of non-trivial k-invariants, there exists an integer N such that the p-complete Eilenberg-Mac Lane space $K(\pi_n(\Omega X), n)$ is a retract of ΩX for any $n \geq N$. Therefore this Eilenberg-Mac Lane space is $B\mathbb{Z}/p$-local as well, which is only possible if $n \leq 2$. This implies that all higher homotopy groups are trivial and so X is aspherical by the first part of the theorem. $\quad\square$

Since an unstable algebra which is locally finite as a module over the Steenrod algebra obviously has also a locally finite module of indecomposable elements, the Dwyer-Wilkerson condition truly generalizes the previously accessible cases. It is in fact straightforward to obtain the Lannes-Schwartz theorem as a corollary.

Corollary 2.4. *Let X be a simply connected space such that $H^*(X; \mathbb{F}_p)$ is of finite type. Assume that $H^*(X; \mathbb{F}_p)$ is non-trivial and locally finite. Then there exists an infinite number of integers k such that $\pi_k X$ contains p-torsion.*

Proof. Since $X_p^\wedge$ is a $B\mathbb{Z}/p$-local space, so is its loop space $\Omega(X_p^\wedge)$. The previous proposition applies and we can conclude because the cohomology of $K(\mathbb{Z}_p^\wedge, 2)$ is not locally finite. $\quad\square$

In view of Theorem 2.3 a good understanding of the Dwyer-Wilkerson statement for arbitrary connected spaces goes through – compare with condition (2) – the study of 2-stage Postnikov pieces. In Lemma 2.2 we have identified the second homotopy group. As for the fundamental group we will assume that X is an $\mathbb{F}_p$-good space, so $X_p^\wedge$ is p-complete.

By [11, Proposition 3.4] such spaces include all virtually nilpotent spaces, that is, the action of the fundamental group on any homotopy group is virtually nilpotent. Bousfield characterizes the $H\mathbb{F}_p$-local spaces in [3, Theorem 5.5] in terms of their homotopy groups, which implies in particular that the n-connected covers and the n-th Postnikov sections of p-complete (and $\mathbb{F}_p$-good) spaces are p-complete.

In short, if X is a virtually nilpotent space, its p-completion $X_p^\wedge$ is an $H\mathbb{F}_p$-local space. Its second Postnikov section $Y = X_p^\wedge[2]$ is a p-complete space with only two homotopy groups, which can be seen as the total space of a fibration of the form

$$K(A, 2) \longrightarrow Y \longrightarrow K(G, 1)$$

where both $K(A, 2)$ and $K(G, 1)$ are p-complete spaces.

Lemma 2.5. *Let X be a virtually nilpotent space such that $H^*(X; \mathbb{F}_p)$ is of finite type. Then $\pi_1(X_p^\wedge)$ is a p-complete group isomorphic to $(\pi_1 X)_p^\wedge$. It is an extension of a finite p-group by a nilpotent p-complete one.*

Proof. The fundamental group G of $X_p^\wedge$ is isomorphic to that of $K(G, 1)_p^\wedge$ by the Whitehead type theorem [4, Proposition 4.1]. Since the fundamental group of X is a finitely generated virtually nilpotent group, it is in particular polycyclic-by-finite. We conclude by Bousfield $\mathbb{F}_p$-goodness result [4, Theorem 7.2] on polycyclic-by-finite spaces that $\pi_1(X_p^\wedge) \cong G_p^\wedge$.

It remains to describe this virtually nilpotent group. As in Example 1.4 we can find normal subgroups $N \leq G' \leq G$ such that N is nilpotent, finitely generated,

and torsion free, the quotient $Q = G'/N$ is p-perfect, and G/G' is a finite p-group. We will actually show that the inclusion $N \to G'$ induces an epimorphism $N_p^\wedge \to G_p'^\wedge$. By [4, Lemma 5.2] we only need to check that it induces an epimorphism on the first mod p homology group, i.e. the quotient by the first term of the mod p lower central series. Notice that the quotient $N/N \cap \Gamma_1^p G'$ is isomorphic to $G'/\Gamma_1^p G'$ because Q is p-perfect. Therefore the maximal quotient of N which is an elementary abelian group is at least as large as $H_1(G'; \mathbb{F}_p)$ and we are done. □

Summing up this result with Lemma 2.2 we can now describe quite accurately the p-complete 2-stage Postnikov pieces which have a $B\mathbb{Z}/p$-local loop space.

Proposition 2.6. *Let X be a virtually nilpotent space such that $H^*(X; \mathbb{F}_p)$ is of finite type and $\pi_n(X) = 0$ for any $n \geq 3$. Assume that $QH^*(X; \mathbb{F}_p)$ is locally finite. Then $\pi_1(X_p^\wedge)$ is isomorphic to $(\pi_1 X)_p^\wedge$, an extension of a finite p-group by a nilpotent p-complete group, and $\pi_2(X_p^\wedge)$ is isomorphic to a finite direct sum of copies of $\mathbb{Z}_p^\wedge$.* □

When the fundamental group is finite, it must be a p-group and we recover precisely the class of 2-stage Postnikov systems studied by Grodal in [13].

3. Universal covers of p-completed spaces

In this section we consider classifying spaces in four different families. We identify explicitly the universal covers of their p-completions, and we show they are p-completions of spaces inside the same family. We start by recalling the well-known case of finite groups, see [17], even though the next examples also contain all finite groups.

3.1. Finite groups

Let G be a finite group and $O^p(G)$ the maximal p-perfect subgroup of G. This is a normal subgroup and the quotient $P = G/O^p(G)$ is a p-group. Therefore the fibration $BO^p(G) \to BG \to BP$ is preserved by p-completion. Since $O^p(G)$ is p-perfect, $(BO^p(G))_p^\wedge$ is simply connected and thus is weakly equivalent to the universal cover of $(BG)_p^\wedge$.

Hence, for any classifying space of a finite group, the universal cover can be chosen, up to p-completion, to be another classifying space.

3.2. p-local finite groups

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group as in Example 1.7. We learn from [6, Theorem 4.4] that there exists a p-local finite group $(O^p(S), O^p(\mathcal{F}), O^p(\mathcal{L}))$ such that $|O^p(\mathcal{L})|_p^\wedge$ is the universal cover of $|\mathcal{L}|_p^\wedge$.

Again we see that the universal cover can be chosen, up to p-completion, inside the class of p-local finite groups.

3.3. Virtually nilpotent groups

For virtually nilpotent groups the idea to construct the universal cover of $(BG)^\wedge_p$ out of group theoretical information is already present in the work of Bastardas, [1, Section 5.3].

Let G be a finitely generated virtually nilpotent group. As in Example 1.4 we can find normal subgroups $N \leq G' \leq G$ such that N is nilpotent, finitely generated, and torsion free, the quotient $Q = G'/N$ is p-perfect, and G/G' is a finite p-group. Hence $\pi_n(BG)^\wedge_p \cong \pi_n(BG')^\wedge_p$ for all $n \geq 2$. As we only wish to identify the universal cover, we might as well assume from now on that G sits in an extension

$$N \longrightarrow G \longrightarrow Q$$

where Q is finite and p-perfect, and N is a nilpotent group, finitely generated, and torsion free. In fact we see from Lemma 2.5 that $\pi_1(BG)^\wedge_p \cong G^\wedge_p$ is a nilpotent group since Q is p-perfect.

The extension $N \to G \to Q$ gives rise to a fibration $BN \to BG \to BQ$, which can be fiberwise p-completed. Because $(BN)^\wedge_p$ is weakly equivalent to the classifying space of $N^\wedge_p$, the total space X of the new fibration is the classifying space of some group $\bar{G}$. Notice also that the map $BG \to B\bar{G}$ is a mod p equivalence.

Lemma 3.1. *Consider the extension $N^\wedge_p \to \bar{G} \to Q$. The p-completion homomorphism $\bar{G} \to \bar{G}^\wedge_p$ is then surjective.*

Proof. In the proof of Lemma 2.5 we showed that the inclusion $N \hookrightarrow G$ induces an epimorphism $N^\wedge_p \to G^\wedge_p$. Since $G^\wedge_p \cong \bar{G}^\wedge_p$, the morphism $\bar{G} \to \bar{G}^\wedge_p$ is onto as well. □

Let us define now K as the kernel of the completion morphism $\bar{G} \to \bar{G}^\wedge_p$ (the intersection of all the terms in the mod p lower central series). In Theorem 3.3 we identify the universal cover of $(BG)^\wedge_p$ as the p-completed classifying space of the group K.

Proposition 3.2. *There is a fibration $(BK)^\wedge_p \longrightarrow (B\bar{G})^\wedge_p \longrightarrow B(G^\wedge_p)$.*

Proof. Let us consider the pair of fibrations $B\bar{G} \to BQ$ and $B\bar{G} \to B(G^\wedge_p)$ with fibers $(BN)^\wedge_p$ and BK respectively. We deduce from Lemma 3.1 that the induced map on fundamental groups $\bar{G} \to G^\wedge_p \times Q$ is onto. Since the first fiber $(BN)^\wedge_p$ is nilpotent, we can apply the "nilpotent action lemma" [11, 5.1] of Dwyer, Farjoun, and Kan. Therefore, the nilpotent group $G^\wedge_p$ acts nilpotently on all homology groups $H_n(BK; \mathbb{F}_p)$. The "nilpotent fibration lemma" [11, Proposition 4.2(i)] tells us now that p-completion preserves the fibration $BK \to B\bar{G} \to B(G^\wedge_p)$ and we are done. □

Theorem 3.3. *Let G be a virtually nilpotent group which is an extension of a finite, p-perfect group by a nilpotent, finitely generated, and torsion free one. The space $(BK)^\wedge_p$ is the universal cover of $(BG)^\wedge_p$.*

Proof. One knows that the fundamental group of $(BG)_p^\wedge$ is $G_p^\wedge$ ([4, Theorem 7.2]). In view of the above proposition we only need to remark that $BG \to B\bar{G}$ is an $H\mathbb{F}_p$-equivalence (the p-completion $N \to N_p^\wedge$ is so for N is nilpotent). □

We note that K is virtually nilpotent as well, being a subgroup of a virtually nilpotent one. We can even say more, since $N_p^\wedge$ is the p-completion of a finitely generated torsion free nilpotent group: K is p-adically polycyclic-by-finite. In the situation where G is actually finite, K coincides with $O^p(G)$, which is consistent with the approach of Levi [17], compare with 3.1.

3.4. Quasi p-perfect groups

By definition, see Example 1.6, a group G is quasi p-perfect if the subgroup $\Gamma_1^p(G)$ is p-perfect. The fibration

$$B\Gamma_1^p(G) \longrightarrow BG \longrightarrow BH_1(G; \mathbb{F}_p)$$

is preserved under p-completion since $H_1(G; \mathbb{F}_p)$ is a p-group and BG is $\mathbb{F}_p$-good. Hence the universal cover of $(BG)_p^\wedge$ is $\left(B\Gamma_1^p(G)\right)_p^\wedge$.

4. Homotopy groups of p-completed classifying spaces

We obtain in this section in a single proof the results which were known before about p-completions of classifying spaces of finite groups (Levi) and virtually nilpotent groups (Bastardas). We prove along the same lines a new result for quasi p-perfect groups and p-local finite groups.

Theorem 4.1. *Let X be the classifying space of a member of the following four families:*

(1) finite groups,
(2) p-local finite groups,
(3) finitely generated virtually nilpotent groups,
(4) quasi p-perfect groups of finite virtual mod p cohomology.

Then the p-completion of X is either aspherical or it has infinitely many homotopy groups with p-torsion. In this case the space $\Omega(X)_p^\wedge$ has infinitely many non-trivial k-invariants.

Proof. In view of the previous section it remains to prove that none of the four families can contain a space whose p-completion is $K(\mathbb{Z}_p^\wedge, 2)$. The mod p cohomology of such a space is polynomial on a generator in dimension 2, hence concentrated in even dimensions. Therefore there are no Bsocksteins at all, which means by an elementary Bockstein spectral sequence argument that there is no p-torsion in the higher integral homology groups.

When G is a finite group, choose L to be the trivial subgroup. When G is a virtually nilpotent group, we can assume as in 3.3 that $G = K$ is p-perfect and so

$(BG)^\wedge_p$ is simply connected. Choose then L to be the nilpotent subgroup of finite index $K \cap N^\wedge_p$. This is a subgroup of $N^\wedge_p$, which has finite homological dimension, [9, VIII.2]. Finally, when G is a quasi p-perfect group of virtually finite mod p cohomology, we choose L to be some subgroup of finite index which has finite mod p cohomology. A standard transfer argument applied to the subgroup $L < G$ shows now that in all three cases the multiplication by the (finite) index is zero on all integral homology groups of G but a finite number.

In case (2) the mod p cohomology of a p-local finite group is contained as a retract in the cohomology of its Sylow p-subgroup as an unstable subalgebra over the Steenrod algebra [8, Proposition 5.5] (see also [22, Proposition 9.4]). The cohomology of $K(\mathbb{Z}^\wedge_p, 2)$ which has no (higher) Bocksteins cannot be a retract of the cohomology of a finite group. □

We point out that the assumptions made on the virtually nilpotent group could hardly be relaxed. For example, if one drops the finitely generated hypothesis, the result is obviously false, as shown by the example of the Prüfer group: $(B\mathbb{Z}/p^\infty)^\wedge_p \simeq K(\mathbb{Z}^\wedge_p, 2)$.

The p-completed classifying spaces of p-perfect groups of finite virtual cohomological dimension have been studied by Levi. He proves in [18, Theorem 1.4] that $\Omega(BG)^\wedge_p$ is a retract of some finite complex.

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