Math. Z. 248, 477–493 (2004) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-004-0665-9

Quillen Grassmannians as non-modular homotopy fixed points*

Natàlia Castellana

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain (e-mail: `natalia@mat.uab.es`)

Received: 10 January 2002; in final form: 31 October 2003 /
Published online: 9 July 2004 –

Abstract. The aim of this paper is to compute the homotopy fixed points of the homotopy action described by unstable Adams operations on the classifying spaces of unitary groups. The same technique can be applied to compute homotopy fixed points of the action of certain automorphisms on p-compact groups called generalized Grassmannians. We use the description of Quillen Grassmannians to describe the set of homotopy representations of elementary abelian p-groups into them.

Mathematics Subject Classification (2000): 55R35, 55R40, 20D20

1 Introduction

The concept of a p-compact group was introduced by Dwyer and Wilkerson [13] in 1994. These objects are the homotopy generalization of the notion of a Lie group. A p-compact group is a triple (X, BX, e) where BX is a p-complete pointed connected space, X is a space such that $H^*(X; \mathbb{F}_p)$ is finite, and $e : \Omega BX \to X$ is a homotopy equivalence.

The first examples of p-compact groups are the p-completions (in the sense of Bousfield-Kan [6]) of compact connected Lie groups and their classifying spaces. The basic Leitmotiv in their study is the interpretation of properties of compact Lie groups in purely homotopical terms ([12]) and their generalization to p-compact groups. For example, they admit a maximal torus and a Weyl group which is a finite p-adic pseudoreflection group.

One of the motivations for this work is the attempt of describing a p-compact group analogue to the following statement for Lie groups, which can be found as an exercise in [4, Exercise §5.4]. Let G be an almost simple compact Lie group with

* The author is supported by CIRIT grant 1995FI-02105PG and partially supported by MCYT grant BFM 2001-2035.

an action of an automorphism Φ and H the subgroup of elements of G fixed by Φ. Denote by H_0 the component of H containing the unit. Under certain restrictions on Φ (Φ preserves an 'épinglage', see [4, Exercise §5.4]), the following hold:

1. If G is of type A_{2l} for $l \geq 1$ and Φ of order 2, then H_0 is of type B_l.
2. If G is of type A_{2l-1} for $l \geq 2$ and Φ of order 2, then H_0 is of type C_l.
3. If G is of type D_l for $l \geq 4$ and Φ of order 2, then H_0 is of type B_{l-1}.

Roughly speaking, this statement tells us about what type of classical compact Lie group you obtain when you consider the fixed points of a certain action on a classical compact Lie group.

The idea of looking at homotopy fixed points of the action of Adams operations on unitary groups goes back to Quillen [23] in his work on the K-theory of fields of finite characteristic. He analyzes the homotopy fixed points of BU by the action of stable Adams operations, and he shows that this homotopy fixed points have the same integral homology as the classifying space of the infinite general linear groups for certain finite fields. Using these techniques, some p-compact groups in the family of generalized Grassmannians were first described in this article [23] as the p-completion of classifying spaces of general linear groups over certain fields.

The generalized Grassmannians form a family of p-compact groups denoted by $(X(q, r, n), BX(q, r, n), e)$, where $r|q|p-1$, whose Weyl groups are the p-adic reflection groups $G(q, r, n)$ in the family $2a$ of the Clark-Ewing list [8]. In particular, some classical compact Lie groups ($SO(n)$ and $Sp(n)$) give rise to generalized Grassmannians when completed at p. A systematic description of the whole family has been given recently by Notbohm [20].

Given a p-compact group X, a self map $\psi^\xi : BX \to BX$ is called an unstable Adams operation of degree $\xi \in \hat{\mathbb{Z}}_p$ if ψ^ξ extends (up to homotopy) the selfmap of the classifying space of the maximal torus induced by $\rho : (\hat{\mathbb{Z}}_p)^n \to (\hat{\mathbb{Z}}_p)^n$ where ρ is the homomorphism given by multiplication by ξ. When X is a generalized Grassmannian, the existence of unstable Adams operations is proved in [20, Sec. 7]. In particular, if $\xi \in (\hat{\mathbb{Z}}_p)^*$ is a primitive qth root of unity, there is an action of $\mathbb{Z}/q\mathbb{Z}$ on BX up to homotopy. When X is a generalized Grassmannian, the existence of unstable Adams operations is proved in [20, Sec. 7].

The main results of this paper are the following ones. From now on, p is an odd prime, $q|p-1$, $r|q$, and $q > 2$.

Theorem 1.1. *If $\mathbb{Z}/q\mathbb{Z} \cong \langle\psi\rangle$ acts on $BU(n)^\wedge_p$ via unstable Adams operations of order q, then*

$$(BU(n)^\wedge_p)^{h\mathbb{Z}/q\mathbb{Z}} \simeq BX(q, 1, [\tfrac{n}{q}])$$

where $[m]$ denotes the greatest integer not greater than m.

Theorem 1.2. *If $\mathbb{Z}/q\mathbb{Z} \cong \langle\phi\rangle$ acts on $BX(q, r, n+1)$ via an automorphism ϕ which lifts to the maximal torus as* $\mathrm{id}^n \times \psi^\xi$ *where ξ is a primitive qth root of unity, then*

$$BX(q, r, n+1)^{h\mathbb{Z}/q\mathbb{Z}} \simeq BX(q, 1, n).$$

Recall that a space X is of finite $\hat{\mathbb{Z}}_p$-type if it is p-complete and the homotopy groups $\pi_*(X)$ are of finite type over $\hat{\mathbb{Z}}_p$.

Corollary 1.3. *If Z is a space of finite $\hat{\mathbb{Z}}_p$-type,*

$$[Z, BU(n)\hat{_p}]^{\mathbb{Z}/q\mathbb{Z}} = [Z, BX(q, 1, [\tfrac{n}{q}])],$$
$$[Z, BX(q, r, n+1)]^{\mathbb{Z}/q\mathbb{Z}} = [Z, BX(q, 1, n)],$$

where the action of $\mathbb{Z}/q\mathbb{Z}$ on $BU(n)\hat{_p}$ and $BX(q, r, n+1)$ is defined in Theorems 1.1 and 1.2.

Corollary 1.4. *There exists a mod p homotopy decomposition of H-spaces:*

$$U(n)\hat{_p} \simeq X(q, 1, [\tfrac{n}{q}]) \times U(n)\hat{_p}/X(q, q, [\tfrac{n}{q}]),$$
$$X(q, r, n+1) \simeq X(q, 1, n) \times (S^{2rn-1})\hat{_p}.$$

The next corollary deals with the infinite unitary classifying space $BU\hat{_p}$. In [7] there is a description of monomorphisms between generalized Grassmannians of type $BX(q, 1, n) \to BX(q, 1, n+1)$ which allow to define a direct system and the corresponding homotopy colimit $BX(q)$. The above results related to homotopy fixed points can be proved for $BX(q)$ so that we recover Quillen's splitting of $BU\hat{_p}$.

Corollary 1.5. *Let $q|p-1$, there is a weak homotopy equivalence*

$$\Theta : BX(q) \to (BU\hat{_p})^{h\mathbb{Z}/q\mathbb{Z}}.$$

Finally, the description of Quillen Grassmannians in Theorem 1.1 allows us to describe the set of homotopy representations of elementary abelian p-groups into $X(q, 1, n)$.

In Section 2 we include some preliminaries on the main properties and construction of generalized Grassmannians for completeness. Section 3 contains a discussion of the method used to prove the main theorems. The proof of the main theorems (Theorems 1.1 and 1.2) are given in Section 4. Section 5 is devoted to the action of stable Adams operations (Corollary 1.5). Section 6 contains an analysis of the representations of elementary abelian p-groups into generalized Grassmannians.

2 The family of generalized Grassmannians

This section is devoted to the description of a family of p-compact groups, namely the generalized Grassmannians. The following definitions are due to Dwyer and Wilkerson [13].

A morphism $f : X \to Y$ between p-compact groups is a pointed map $Bf : BX \to BY$. We say that f is a monomorphism if the homotopy fiber of Bf (usually denoted by $Y/f(X)$ or simply by Y/X) is $\mathbb{F}_p$-finite.

A p-compact torus is a triple of the form $(T\hat{_p}, BT\hat{_p}, e)$ where T is a torus, that is, $BT\hat{_p} \simeq K(\hat{\mathbb{Z}}_p, 2)^n$. Dwyer and Wilkerson ([13]) proved that every p-compact group has a maximal torus $i : T \to X$ in the sense that the induced homomorphism $T \to \Omega\,\mathrm{Map}(BT, BX)_{Bi}$ is an equivalence to the identity component.

They define the Weyl space of a p-compact group. We can assume that the inclusion of the maximal torus $Bi : BT \to BX$ is a fibration. The Weyl space $\mathcal{W}_X$ is the space of selfmaps f of BT such that $Bi \circ f = Bi$. It is a topological monoid. The Weyl group W_X is the group of components of the Weyl space, $W_X = \pi_0(\mathcal{W}_X)$. In the case X is connected, this group W_X can be easily described as the set of all homotopy classes of selfmaps $\omega : BT \to BT$ such that $Bi \circ \omega \simeq Bi$. Dwyer and Wilkerson [13] proved that W_X is a finite group and that the action of W_X on $H^2(BT; \hat{\mathbb{Z}}_p) \otimes \mathbb{Q}$ induces a faithful representation of W_X in $GL_n(\mathbb{Q}\hat{_p})$ as a pseudoreflection group.

The classification of p-compact groups [2] for odd primes states that p-compact groups are in bijective correspondence with finite pseudoreflection groups over $\hat{\mathbb{Z}}_p$. These groups have been classified by Notbohm [21] (using [11], and the corresponding classifications of [8] over $\mathbb{Q}\hat{_p}$, and [24] over $\mathbb{C}$).

We briefly describe the groups $G(q, r, n)$. For any $q > 1$ such that $q | p - 1$, let $\mu_q \subset \mathbb{C}^*$ be the group of qth roots of unity. For any $r | q | p - 1$, define

$$A(q, r, n) = \{(z_1, \dots, z_n) \in \mu_q^n \,|z_1 \cdots z_n \in \mu_{q/r}\}.$$

The finite group $G(q, r, n)$ is a split extension of Σ_n by $A(q, r, n)$

$$G(q, r, n) \cong A(q, r, n) \rtimes \Sigma_n$$

with Σ_n acting on $A(q, r, n)$ by permuting the factors. The group $G(q, r, n)$ can be identified with a subgroup of $GL_n(\hat{\mathbb{Z}}_p)$ when we fix an isomorphism $\mu_q \cong \mathbb{Z}/q$. GL(n, R) for any commutative group of qth roots of unity. The diagonal matrices and matrices.

Generalized Grassmannians are p-compact groups (X, BX, e) whose Weyl groups are the ones of type $G(q, r, n)$ in the family $2a$ ([8]). Quillen Grassmannians are those with Weyl group $G(q, 1, n)$ and they were first described in [23]. For each q and n Quillen describes a field k such that $H^*(BGL_{qn}(k); \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \dots, x_n]^{G(q,1,n)}$. The p-completion of these spaces are p-compact groups with Weyl group $G(q, 1, n)$.

$G(2, 2, n), G(2, 1, n), G(3, 3, 2), G(4, 4, 2)$ and $G(6, 6, 2)$ are the Weyl groups of the Lie groups $SO(2n)$, $SO(2n + 1)$, $SU(3)$, $Sp(2)$ and G_2. Observe that these Weyl groups are conjugate to subgroups of $GL_n(\mathbb{Z})$. In fact, they are the only ones in this family that are realizable over $\mathbb{Q}\hat{_2}$.

It is worth pointing out that $G(1, 1, n) = \Sigma_n$ but these groups of permutations are not irreducible acting on $(\hat{\mathbb{Z}}_p)^n$ in the natural way by permutations of the elements of a basis. However this is the Weyl group of $U(n)$. The groups in the families 1, $2b$ and 3 are also uniquely realizable as Weyl groups of p-compact groups.

We shall sketch the construction of generalized Grassmannians $BX(q, r, n)$ by Notbohm [20] which covers the realization of all groups in family 2a. Let G be a finite group and let $\mathcal{H}$ be a family of subgroups of G closed under conjugation. The orbit category $\mathcal{O}_{\mathcal{H}}(G)$ associated to $\mathcal{H}$ is the category whose objects are the orbits G/H for each $H \in \mathcal{H}$ and whose morphisms are the G-maps between the orbits, $\mathrm{Hom}_G(G/H, G/K)$.

We consider the family $\mathcal{H}$ of subgroups $H \leq G(q,r,n)$ conjugate to some subgroup $\Sigma(\Pi) := \Sigma_{n_1} \times \cdots \times \Sigma_{n_s} \leq \Sigma_n \leq G(q,r,n)$ for $\Pi = (n_1, \ldots, n_s)$ where $n_1 + \cdots + n_s = n$ and $n_i = p^{j_i}$ for some $j_i \in \mathbb{N} \cup \{0\}$.

There is a functor $F : \mathcal{O}_{\mathcal{H}}(G(q,r,n)) \to \mathrm{hoTop}$ to the homotopy category of spaces defined by

$$F(G(q,r,n)/\Sigma(\Pi)) = BU(\Pi) := BU(n_1) \times \cdots \times BU(n_s).$$

The functor F is defined on morphisms by using unstable Adams operations and permutation of factors.

This functor lifts to the category of topological spaces Top in a way that the following holds.

Theorem 2.1. *[20] Fix any odd prime p, any $r|q|(p-1)$, and any $n > 1$. Let $G = G(q,r,n)$ and $\mathcal{O}_{\mathcal{H}}(G(q,r,n))$ as above. Then there exists a functor*

$$\Psi : \mathcal{O}_{\mathcal{H}}(G(q,r,n)) \longrightarrow \mathrm{Top}$$

such that:

1. *For any partition $\Pi = (n_1, \ldots, n_s)$ of n into a powers of p,*

$$\Psi(G/\Sigma(\Pi)) \simeq BU(\Pi)\hat{_p}.$$

2. *The composite $H^*(-; \mathbb{Z}\hat{_p}) \circ \Psi$ is isomorphic to the fixed point functor*

$$G(q,r,n)/H \longmapsto \mathbb{Z}\hat{_p}[x_1, \ldots, x_n]^H.$$

3. *If we set $BX(q,r,n) := (\mathrm{hocolim}_{\mathcal{O}_{\mathcal{H}}(G(q,r,n))} \Psi)\hat{_p}$ then*

$$H^*(BX(q,r,n); \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \ldots, x_n]^{G(q,r,n)}.$$

Moreover, Notbohm [20] describes the set $[BX(q,r,n), BX(q,r,n)]$ of homotopy classes of self maps of generalized Grassmannians. In particular he proves the existence of certain maps described in the following proposition.

Proposition 2.2. *[20]*

1. *For each $a \in A(q,1,n)/A(q,r,n)$ with representative $(a_1, \ldots, a_n)$, there exists a map*

$$\varphi^a : BX(q,r,n) \to BX(q,r,n)$$

such that $\varphi|_{BT^n} \simeq \psi^{a_1} \times \cdots \times \psi^{a_n}$, where ψ^{a_i} are the corresponding Adams operations associated to $a_i \in \mu_q$.

2. *For each $a \in (\hat{\mathbb{Z}}_p)^*$ there exists an unstable Adams operations*

$$\psi^a : BX(q,r,n) \to BX(q,r,n).$$

In [7], the author analyzes the existence of several morphisms between generalized Grassmannians which will be used in this paper.

Theorem 2.3. *[7] For $r|q|p-1$, $q>2$*

1. *There exists a monomorphism of p-compact groups*

$$c\colon X(q,r,n)\to U(nq)\hat{_p}$$

such that $\psi^a\circ Bc\simeq Bc$ where $a\in\mu_q$.

2. *There exists a monomorphism of p-compact groups Γ,*

$$\Gamma : X(q,1,n)\to X(q,r,n+1)$$

such that $\varphi^a\circ B\Gamma\simeq B\Gamma$ for any $a\in\mu_r$.

The homotopy type of the corresponding centralizers ([13]) described in the next proposition will be used in the proof of Theorems 1.1 and 1.2.

Proposition 2.4.

$$\mathrm{Map}(BX(q,r,n),BU(nq)\hat{_p})_{Bc}\simeq *$$
$$\mathrm{Map}(BX(q,1,n),BX(q,q,n+1))_{B\Gamma}\simeq (BS^1)\hat{_p}$$

Proof. The decomposition of $BX(q,r,n)$ as a homotopy colimit induces a description of the mapping space $\mathrm{Map}(BX(q,r,n),BU(nq)\hat{_p})_{Bc}$ as a homotopy inverse limit,

$$\operatorname*{holim}_{\mathcal{O}_{\mathcal{H}}(G(q,r,n))}\mathrm{Map}(\Psi(\Sigma(\Pi)),BU(nq)\hat{_p})_{Bc}$$

where $\Psi(\Sigma(\Pi))\simeq BU(\Pi)\hat{_p}$. The Bousfield-Kan spectral sequence [6] converging to the homotopy groups of a homotopy inverse limit has E^2-term

$$\varprojlim{}^i{}_{\mathcal{O}_{\mathcal{H}}(G(q,r,n))}\pi_j(\mathrm{Map}(\Psi(\Sigma(\Pi)),BU(nq)\hat{_p})_{Bc}).$$

These higher limits are zero ([7, Proof of Theorem E]). Therefore the spectral sequence collapses and

$$\pi_*(\mathrm{Map}(BX(q,r,n),BU(nq)\hat{_p})_{Bc})=0.$$

In the case of $\mathrm{Map}(BX(q,1,n),BX(q,q,n+1))_{B\Gamma}$,

$$\mathrm{Map}(BX(q,1,n),BX(q,q,n+1))_{B\Gamma}\simeq\operatorname*{holim}_{\mathcal{O}_{\mathcal{H}}(G(q,1,n))}\mathrm{Map}(\Psi(\Sigma(\Pi)),BX(q,q,n+1))_{Bc}$$

The E^2-term of the corresponding spectral sequence is

$$\varprojlim{}^i{}_{\mathcal{O}_{\mathcal{H}}(G(q,1,n))}\pi_j(\mathrm{Map}(\Psi(\Sigma(\Pi)),BX(q,q,n+1))_{B\Gamma}).$$

These higher limits are computed in [7, Proof of Theorem B]. It turns out that all vanish except for

$$\varprojlim{}^0{}_{\mathcal{O}_{\mathcal{H}}(G(q,1,n))}\pi_2(\mathrm{Map}(F(\Sigma(\Pi)),BX(q,q,n+1))_{B\Gamma})=\pi_2((BS^1)\hat{_p})=\hat{\mathbb{Z}}_p.$$

Therefore

$$\mathrm{Map}(BX(q,1,n),BX(q,q,n+1))_{B\Gamma}\simeq K(\hat{\mathbb{Z}}_p,2)\simeq(BS^1)\hat{_p}.$$

$\square$

3 Loop spaces and non-modular homotopy fixed points

Let X be a connected p-compact group with a G-action up to homotopy $G \to [BX, BX]$ where G is a finite group of order prime to p. The obstruction theory developed by Cooke [9] implies that we can always replace this homotopy action by a topological action and assume that BX is a G-space. Moreover, this lift to a topological action is unique in the sense that two lifts are homotopy equivalent by a G-equivariant map. This obstruction theory is a special case of the obstruction theory developed by Dwyer and Kan in [10] for a diagram indexed by the one-object category in which G is the group of morphisms.

Assume that we have a good candidate Y for X^{hG}, i.e. a connected p-compact group Y with a G-equivariant map j into $X(G$ acting trivially on $Y)$

$$BY \stackrel{Bj}{\to} BX.$$

The technique described in this section gives us conditions which assure that $BY \simeq BX^{hG}$ and is based on the analysis of the homotopy fiber of Bj.

If F is the homotopy fibre of Bj then all the maps in the fibration $F \to BY \stackrel{Bj}{\to} BX$ are G-equivariant. Thus, it induces a fibration when taking the homotopy fixed points:

$$F^{hG} \to BY^{hG} \xrightarrow{Bj^{hG}} BX^{hG}.$$

The Bousfield-Kan spectral sequence for the homotopy groups of BY^{hG} degenerates to $\pi_*(BX^{hG}) \cong \pi_*(BX)^G$. This is a consequence of $|G|$ being prime to p and $\pi_1(BX)$ being pro-p-groups (see [26]). In particular, BX^{hG} is connected. Since $|G|$ is prime to p and acts trivially on Y it follows that

$$BY^{hG} \simeq \mathrm{Map}(BG, BY) \simeq BY.$$

The aim is to describe conditions which assure the contractibility of F^{hG}. If $F^{hG} \simeq *$ then BY and BX^{hG} are homotopy equivalent.

In order to analyze the contractibility of F^{hG}, we will proceed by studying its free loop space. Notice that since a space Z is a retract of its free loop space $\Lambda(Z)$, the contractibility of $\Lambda(Z)$ implies the contractibility of Z.

Lemma 3.1. *Let F be a G-space and $L_g(F)$ defined by the following homotopy pullback diagram*

$$\begin{tikzcd} L_g(F) \arrow[r] \arrow[d] & F \arrow[d, "\Delta"] \\ F \arrow[r, "{(1,g)}"] & F \times F. \end{tikzcd} \tag{1}$$

If $g \in Z(G)$ then $L_g(F)^{hG} \simeq \Lambda(F^{hG})$.

Proof. If $g \in Z(G)$ the maps in the pullback diagram (1) are G-equivariant. Since taking homotopy fixed points is a homotopy inverse limit, it commutes with homotopy pullbacks [6, Example XI.4.3] and it induces another pullback diagram,

$$\begin{tikzcd}
(L_g(F))^{hG} \arrow[r] \arrow[d] & F^{hG} \arrow[d, "\Delta^{hG}"] \\
F^{hG} \arrow[r, "(1,g)^{hG}"] & (F \times F)^{hG}.
\end{tikzcd}$$

Notice that $(F \times F)^{hG} \simeq F^{hG} \times F^{hG}$. In fact, using the definition $F^{hG} = \mathrm{Map}_G(EG, F)$, one can see that

$$\mathrm{Map}_G(EG, F \times F) \simeq \mathrm{Map}_G(EG, F) \times \mathrm{Map}_G(EG, F)$$

with G acting diagonally on $F \times F$. Moreover, the induced map Δ^{hG} is the diagonal map on the homotopy fixed points:

$$\Delta^{hG} = \Delta : F^{hG} \to F^{hG} \times F^{hG}.$$

We claim that

$$(1, g)^{hG} \simeq \Delta : F^{hG} \to F^{hG} \times F^{hG}.$$

The map $(1, g)^{hG}$ is defined using composition

$$(1, g)^{hG} : \mathrm{Map}_G(EG, F) \to \mathrm{Map}_G(EG, F) \times \mathrm{Map}_G(EG, F)$$

by $(1, g)^{hG}(f) = (f, g \cdot f) = (f, f \circ g)$ where $g : EG \to EG$ is the left multiplication by $g \in G$ and f is a G-equivariant map from EG to F.

If $g \in Z(G)$ then the maps $g, \mathrm{id} : EG \to EG$ are G-equivariant, hence are G-homotopic by the universal property of EG ([15, Proposition 6.14]). It follows that $g \circ f \simeq_G f$ and therefore $(1, g)^{hG} \simeq \Delta$.

The proof finishes by remarking that

$$\Lambda(X) \simeq \mathrm{holim}(X \xrightarrow{\Delta} X \times X \xleftarrow{\Delta} X).$$

□

Remark 3.2. ([25]) Notice that $L_g(F)$ can be described by $\mathrm{Map}_{\mathbb{Z}/n\mathbb{Z}}(S^1, F)$ where n is the order of g and $\mathbb{Z}/n\mathbb{Z}$ acts on S^1 via $e^{\frac{2\pi i}{n}}$ (compare the diagonal map Δ with evaluation map at the boundary $ev(0) \times ev(1) : map(I, F) \to I \times I$). For every $g \in G$ we obtain space $L_g(F)$.

For a general group G not necessarily finite, if g is not of finite order, then one can easily check that $L_g(F) \simeq F^{h\mathbb{Z}}$.

If R is a unital and augmented algebra over a field k, we denote by $\mu : R \otimes R \to R$, $\eta : k \to R$ y $\epsilon : R \to k$ the morphisms corresponding to the multiplication, unit and augmentation respectively.

Proposition 3.3. *Let F be a p-complete G-space such that $|G|$ is prime to p. Assume that $R := H^*(F;\mathbb{F}_p)$ satisfies the following conditions:*

1. *There exists an algebra isomorphism $\kappa : R\otimes R \to R\otimes R$ such that $\epsilon\otimes\mathrm{id} = \mu\kappa$.*
2. *There exists $g \in Z(G)$ such that $\mu(\mathrm{id}\otimes g^*)\kappa(\mathrm{id}\otimes\eta) : R \to R$ is an isomorphism.*

Then $F^{hG} \simeq *$.

Proof. According to Lemma 3.1 it suffices to show that $L_g(F) \simeq *$. The Eilenberg-Moore spectral sequence associated to the pullback diagram (1) has E^2-term

$$E_2 \cong \mathrm{Tor}_{R\otimes R}(R, R)$$

and it converges to $H^*(L_g(F))$.

The assumptions on R imply that $\kappa(\mathrm{id}\otimes\eta)$ induces an isomorphism ([5, Corollary 6.5])

$$\mathrm{Tor}_{R\otimes R}(R, R) \cong \mathrm{Tor}_R(\mathbb{F}_p, R)$$

where R is an R-module via $\mu(\mathrm{id}\otimes g^*)\kappa(\mathrm{id}\otimes\eta)$.

Since $\mu(\mathrm{id}\otimes g^*)\kappa(\mathrm{id}\otimes\eta)$ is an isomorphism, R is a free R-module and $\mathrm{Tor}_R(\mathbb{F}_p, R) = 0$. Therefore $L_g(F) \simeq *$. □

Remark 3.4. If $R \cong \Lambda(a_1,\ldots,a_s)\otimes\mathbb{F}_p[b_1,\ldots,b_r]/(b_1^{p_1},\ldots b_r^{p_r})$ where $\deg(a_i)$ is odd, $deg(b_i)$ is even and p_i are powers of p, the algebra isomorphism $\kappa : R\otimes R \to R\otimes R$ given by

$$\begin{aligned}
\kappa(a_i\otimes 1) &= a_i\otimes 1 - 1\otimes a_i\\
\kappa(1\otimes a_i) &= 1\otimes a_i\\
\kappa(b_i\otimes 1) &= b_i\otimes 1 - 1\otimes b_i\\
\kappa(1\otimes b_i) &= 1\otimes b_i
\end{aligned}$$

satisfies conditions in Proposition 3.3 1. Moreover $\mu(\mathrm{id}\otimes g^*)\kappa(\mathrm{id}\otimes\eta) = \mathrm{id} - g^*$. These cases are considered in [5, Proof of Proposition 6.2].

Proposition 3.5. *Let X be a p-complete G-space where $|G|$ is prime to p. Let Y be a p-complete space and let $f : Y \to X$ such that for every $g\in G$, $g\circ f \simeq f$. Assume that $\pi_*(\mathrm{Map}(Y, X)_f)$ are pro-p-groups. If the homotopy fiber of f satisfies the conditions in Proposition 3.3, then $Y^{hG} \simeq X$.*

Proof. The map $f : Y \to X$ is equivariant up to homotopy (G acting trivially on Y). That means that there is an induced action of G on the corresponding component of the mapping space $\mathrm{Map}(Y, X)_f$. The homotopy fixed point space $(\mathrm{Map}(Y, X)_f)^{hG}$ is non-empty because $\pi_*(\mathrm{Map}(Y, X)_f)$ are pro-p-groups and $(|G|, p) = 1$. This fact follows from [26] since

$$\pi_0((\mathrm{Map}(Y, X)_f)^{hG}) = \pi_0((\mathrm{Map}(Y, X)_f))^G = *.$$

There exists a map $f' \in \mathrm{Map}(X, Y)^{hG} = \mathrm{Map}(X, Y^{hG})$. The composite $ev \circ f'$ is homotopic to f by construction where $ev : X^{hG} \to X$ is the evaluation map which is equivariant. Moreover, $ev \circ f'$ is equivariant since ev is equivariant.

We can assume that f is equivariant. Applying homotopy fixed points to the fibration sequence $F \to X \to Y$, we conclude that $f^{hG} : X \to Y^{hG}$ is an equivalence if $F^{hG} \simeq *$. Proposition 3.3 implies the result. $\square$

4 Non-modular homotopy fixed points on generalized Grassmannians

This section contains a proof of the main theorems, Theorem 1.1 and 1.2.

Let G be a compact Lie group. A p-toral subgroup $P \subset G$ is called p-stubborn if $N(P)/P$ is finite and contains no nontrivial normal p-subgroups. $\mathcal{R}_p(G)$ denotes the category whose objects are orbits G/P for each p-stubborn $P \subset G$ and $\mathrm{Mor}(G/P, G/P')$ is the set of all G-maps between the orbits. The main result concerning this category is that the following map

$$\operatorname*{hocolim}_{P \in \mathcal{R}_p(G)} EG/P \to BG$$

is an $\mathbb{F}_p$-homology equivalence ([16]) where $EG/P \simeq BP$.

Lemma 4.1.

$$\mathrm{Map}(BU(n), BU(n+m))_i \simeq_p \mathrm{Map}(BU(n), BU(n) \times BU(m))_{i_1}$$

where i is the standard inclusion and i_1 is the inclusion in the first factor.

Proof. Using the mod p decomposition of unitary groups via p-stubborn subgroups [16] we obtain a decomposition of the mapping space as a homotopy inverse limit,

$$\mathrm{Map}(BU(n), BU(n+m))_{[i]} \simeq_p \operatorname*{holim}_{\mathcal{R}_p(U(n))} \mathrm{Map}(EU(n)/P, BU(n+m)\hat{_p})_i$$

where $[i]$ is the set of components of maps such that its restriction to every $P \in \mathcal{R}_p(U(n))$ is homotopy equivalent to $i(P) : BP \to BU(n)$. By the Dwyer-Zabrodsky Theorem for p-compact toral groups ([22]),

$$\mathrm{Map}(EU(n)/P, BU(n+m)\hat{_p})_i \simeq BC_{U(n+m)}(i(P))\hat{_p}.$$

Let ξ be a primitive pth root of unity. Notice that $\langle \xi \,\mathrm{Id}_n \rangle \subset Z(U(n)) \subset P$ for every p-stubborn $P < U(n)$. We have the inclusion

$$C_{U(n+m)}(i(P)) \leq C_{U(n+m)}(i(\xi \,\mathrm{Id}_n)) = C_{U(n+m)}(\xi \,\mathrm{Id}_n \oplus Id_m) = U(n) \times U(m).$$

The last equality follows from the Schur lemma. Therefore,

$$C_{U(n+m)}(i(P)) = C_{U(n) \times U(m)}(i(P)).$$

We may now consider the inclusion map $U(n) \times U(k) \to U(n+k)$ which induces a commutative map between diagrams

$$\{\mathrm{Map}(EU(n)/P, BU(n) \times BU(k))_{Bi}\}_{\mathcal{R}_p(U(n))} \to \{\mathrm{Map}(EU(n)/P, BU(n+k))_{Bi}\}_{\mathcal{R}_p(U(n))},$$

where the spaces are mod p homotopy equivalent.

$$\begin{aligned}\mathrm{Map}(BP, B(U(n) \times U(k)))_{Bi} &\simeq BC_{U(n)\times U(k)}(i(P))\\ &\simeq BC_{U(n+k)}(i(P))\\ &\simeq \mathrm{Map}(BP, BU(n+k))_{Bi}.\end{aligned}$$

This map between diagrams induces a homotopy equivalence between the corresponding homotopy inverse limits up to p-completion:

$$\mathrm{Map}(BU(n), BU(n+k))_{Bi} \simeq \mathrm{Map}(BU(n), BU(n) \times BU(k))_{B(\mathrm{id})\times c}.$$

□

Corollary 4.2.

$$\mathrm{Map}(BU(n) \times BU(m), BU(n+m))_{\oplus} \simeq_p BZ(U(n)) \times BZ(U(m)).$$

From now on we fix $G := \mathbb{Z}/q\mathbb{Z}$ for simplicity.

Proposition 4.3. *If $s < q$ then*

$$(BU(qn+s)\hat{_p})^{hG} \simeq (BU(qn)\hat{_p})^{hG}.$$

Proof. Consider the inclusion map $Bi : BU(qn)\hat{_p} \to BU(qn+s)\hat{_p}$. Notice that it is G-equivariant up to homotopy with respect to the action of the unstable Adams operations $G = \langle\psi\rangle$ on both spaces. In order to rigidify this homotopy action, we will deal with the Whitney sum map

$$BU(qn)\hat{_p} \times BU(s)\hat{_p} \xrightarrow{\oplus} BU(qn+s)\hat{_p}$$

which is also G-equivariant up to homotopy with G acting diagonally on $BU(qn)\hat{_p} \times BU(s)\hat{_p}$. It suffices to show that the Whitney sum map lifts to a G-equivariant map because its restriction to $BU(qn)\hat{_p}$ will gives us the desired G-equivariant map.

We can describe the above situation in terms of functors from a finite category $\mathbb{I}$. Let $\mathbb{I}$ be an index category with two objects $\{0, 1\}$ and morphisms groups $Mor_{\mathbb{I}}(0, 1) = G$, $End_{\mathbb{I}}(0) = G$ and $End_{\mathbb{I}}(1) = G$ where the composition law is given by right and left multiplication in G. Let $F_1 : \mathbb{I} \to \mathrm{HoTop}$ be the functor defined by $F_1(0) = BU(qn)\hat{_p} \times BU(s)\hat{_p}$, $F_1(1) = BU(qn+s)\hat{_p}$, $F_1(j) = \oplus$. It is a well-defined functor to the homotopy category of spaces. This $\mathbb{I}$-diagram is centric:

$$BZ(U(qn) \times U(s)) \simeq \mathrm{Map}(BU(qn) \times BU(s), BU(nq+s))_{\oplus}$$

by Corollary 4.2. Then the obstructions to lifting F_1 to the topological category of spaces lie in

$$\varprojlim{}^{i+2}_{\mathbb{I}}\pi_i(BZ(-))$$

for $i \geq 1$ ([27]). Since $|G|$ is prime to p, the long exact sequence of higher limits over categories with two objects [19, Example 13.7.4] shows the vanishing of these higher limits hence there exists a lifting to the topological category. Two such lifts are unique up to a natural transformation since the obstructions for the uniqueness also vanish. We can assume that $\psi \circ \oplus = \oplus \circ \psi$.

We may now consider the fibration

$$F^G \to (BU(qn)\hat{_p})^{hG} \to (BU(qn+s)\hat{_p})^{hG}$$

where F is the homotopy fibre of Bi. We will show that $F^{h\mathbb{Z}/q\mathbb{Z}} \simeq *$ by checking conditions in Proposition 3.3.

The Eilenberg-Moore spectral sequence is a spectral sequence of commutative algebras $\{E_r, d_r\}$ converging to $H^*(F; k)$ where F is the fiber with $E_2 = \mathrm{Tor}_{H^*(BU(nq+s);\mathbb{F}_p)}(H^*(BU(nq); \mathbb{F}_p), \mathbb{F}_p)$.

The use of this spectral sequence for computing the cohomology of homogeneous spaces has been discussed in [3]. It converges at the E_2-page and the E_2-term can be computed using the description in [3, Lemma 4.11]. Roughly speaking, it is an exterior algebra on the desuspension of the kernel of Bi^* tensor the cokernel of Bi^*,

$$H^*(F) \cong \Lambda(e_1, \dots, e_s)$$

where $deg(e_i) = 2(n+i) - 1$. Moreover, the naturality of the Eilenberg-Moore spectral sequence shows that the induced map in cohomology $\mathrm{id} - \psi^{2n-1} = (1-\xi^n)\cdot : H^{2n-1}(F) \to H^{2n-1}(F)$ is an isomorphism, where ξ is a qth root of unity modulo p. $\square$

Corollary 4.4. *If $n < q$ and $\mathbb{Z}/q\mathbb{Z} = \langle\psi\rangle$ is generated by an unstable Adams operation of order $q | p-1$ then*

$$(BU(n)\hat{_p})^{h\mathbb{Z}/q\mathbb{Z}} \simeq *.$$

Proof of Theorem 1.1. The complexification map c ([7], Theorem 2.3)

$$Bc : BX(q,1,n) \to BU(qn)\hat{_p}$$

is a G-equivariant map up to homotopy, where G acts trivially on $BX(q,q,n)$.

By Proposition 2.4, the mapping space

$$\mathrm{Map}(BX(q,1,n), BU(nq)\hat{_p})_{Bc} \simeq *$$

is p-complete hence c can be lifted to a map which is G-equivariant on the nose in a unique way (up to G-equivariant homotopy). The homotopy fiber of Bc satisfies conditions in Proposition 3.3 ([7, Proof of Theorem E]). Note that $\mathrm{id} - \psi^{2m-1} =$

$(1-\xi^m)\cdot$ is an isomorphism by the naturality of the Eilenberg-Moore spectral sequence. Therefore Bc induces a homotopy equivalence

$$BX(q,1,n) \simeq (BU(nq)\hat{_p})^{hG}.$$

Finally

$$BU(nq+s)^{hG} \simeq BU(nq)^{hG} \simeq BX(q,1,n)$$

where the first equivalence follows from Proposition 4.3. □

Proof of Theorem 1.2. The monomorphism Γ ([7], Theorem 2.3)

$$B\Gamma : BX(q,1,n) \to BX(q,q,n+1)\hat{_p}$$

is a G-equivariant map up to homotopy, where G acts trivially on $BX(q,1,n)$.

By Proposition 2.4, the homotopy groups $\pi_*(\mathrm{Map}(BX(q,1,n), BX(q,q,n+1))_{B\Gamma})$ are pro-p-groups. The homotopy fiber of $B\Gamma$ satisfies conditions in Proposition 3.3 (see [7, Proposition 3.1]). Note that $\mathrm{id} - \psi^{2m-1} = (1-a^m)\cdot$ is an isomorphism by the naturality of the Eilenberg-Moore spectral sequence. Therefore $B\Gamma$ induces a homotopy equivalence

$$BX(q,1,n) \simeq BX(q,q,n+1)^{hG}.$$

□

Proof of Corollary 1.3. This result is an immediate consequence of the formula $map(X, Y^{hG}) = map(X,Y)^{hG}$ and the fact that $\pi_*(X^{hG}) = \pi_*(X)^G$ when $\pi_*(X)$ is a pro-p-group and p does not divide $|G|$ (see [26]). Notice that $BX(q,r,n+1)$ and $BU(n)\hat{_p}$ are p-complete, nilpotent and of finite $\hat{\mathbb{Z}}_p$-type. These properties assure that the homotopy groups $\pi_1(\mathrm{Map}(X, BU(n)\hat{_p})_f)$ and $\pi_1(\mathrm{Map}(X, BX(q,r,n+1)\hat{_p})_f)$ are pro-p-groups when X is of finite $\hat{\mathbb{Z}}_p$-type (see [26, Lemma 4]). □

Proof of Corollary 1.4. The inclusion of the homotopy fibre $U(n)\hat{_p}/X(q,1,[\frac{n}{q}])$ of Bc into $BX(q,1,[\frac{n}{q}])$ is homotopic to the constant map by Corollary 1.3. Then the fibration

$$X(q,1,[\tfrac{n}{q}]) \to U(n)\hat{_p} \to U(n)\hat{_p}/X(q,1,[\tfrac{n}{q}])$$

has a section which induces the corresponding mod p decomposition.

The same argument applies to prove that there exists a homotopy decomposition

$$X(q,r,n+1) \simeq_p X(q,1,n) \times X(q,r,n+1)/X(q,1,n).$$

Finally, the homogeneous space $X(q,r,n+1)/X(q,1,n)$ is homotopic to $(S^{2rn-1})\hat{_p}$ ([7, Corollary 3.3]). □

For classical compact Lie groups (that is, p odd and $q=2$) this splitting is the mod p homotopy decomposition given by Harris in [17].

5 Adams maps on stable exotic Grassmannians

The monomorphisms studied in [7] between generalized Grassmannians led to the definition of the infinite generalized Grassmannian

$$BX(q) := \operatorname*{hocolim}_{\mathbb{N}} BX(q, 1, n).$$

The complexification morphisms $Bc : BX(q, 1, n) \to BU(nq)\hat{_p}$ induce a morphism $Bc : BX(q) \to BU\hat{_p}$ between the corresponding homotopy direct limits. These monomorphisms Bc are G-equivariant therefore this map is homotopy equivariant with respect to the action of the unstable Adams' operation of order q on $BU\hat{_p}$. Thus, we have a factorization

$$Bc : BX(q) \to (BU\hat{_p})^{h\mathbb{Z}/q\mathbb{Z}}.$$

Proof of Corollary 1.5. On the one hand the homotopy groups $\pi_*(BU\hat{_p})$ are the direct limit $\operatorname{colim} \pi_*(BU(n)\hat{_p})$ induced by the inclusion maps between unitary groups. On the other hand, from Corollary 1.3 we know that $\pi_*(BX(q, 1, n)) = \pi_*(BU(nq)\hat{_p})^G$. Note that M^G is the inverse limit of M over the category with one object and G as the automorphism group. Because of the interchange property of higher limits [18, Theorem IX.2.1], Bc induces an isomorphism in the homotopy groups. □

Corollary 5.1.

$$\pi_i(BX(q)) = \begin{cases} \hat{\mathbb{Z}}_p & i \equiv 0 \mod 2q \\ 0 & otherwise \end{cases}$$

Proof. By the formula $\pi_i(BX(q)) \cong \pi_i(BU\hat{_p})^{\mathbb{Z}/q\mathbb{Z}}$, the task is now to describe the action of the Adams' operation on $\pi_{2i}(BU\hat{_p})$, that is on the p-adic K-theory of the spheres. From Adams' paper on vector fields on spheres [1], we know that this action corresponds to multiplication by ξ^i on $\pi_{2i}(BU\hat{_p})$ where ξ is a p-adic qth root of the unity. An easy computation finishes the proof. □

The referee pointed out that for $q = p - 1$, the inclusion $BX(q) \to BU\hat{_p}$ is the p-completion of the map induced on zero-spaces by the spectrum inclusion of the Adams summand $l \to ku$, where $\pi_*(l) = \mathbb{Z}_{(p)}[v_1]$.

6 Homotopy representations of elementary abelian p-groups

This section contains a description of the homotopy representations of elementary abelian p-groups into generalized Grassmannians.

The complex irreducible representations of $BV = B(\mathbb{Z}/p\mathbb{Z})^n$ are 1-dimensional and they can be described in the following way:

$$\Theta_{a_1,\dots,a_n} : (\mathbb{Z}/p\mathbb{Z})^n \to S^1,$$

where $\Theta_{a_1,\dots,a_n}(x_1, \dots, x_n) = exp(2\pi(a_1x_1 + \cdots + a_nx_n)/p)$ for all a_i with $0 \le a_i \le p - 1$. By the Dwyer-Zabrodsky theorem [14], there is a bijection

$[BV, BS^1] \cong \mathrm{IRep}(V)$ where IRep is the set of complex irreducible representations of V. Therefore, the unstable Adams operations of order $q|p-1$ act on the set IRep. This action can be described explicitly: if ξ is a qth root of unity, consider $0 \leq \bar{\xi} \leq p-1$ such that $\bar{\xi} = \xi$ modulo p. Then

$$\psi^{\xi} \cdot \Theta_{a_1,\dots,a_n} = \Theta_{\bar{\xi}a_1,\dots,\bar{\xi}a_n}.$$

This action extends to an action on the set $\mathrm{Rep}(V, U(n))$ by linearality.

Proposition 6.1. *Let V be an elementary abelian p-group. There is a bijection*

$$[BV, BX(q,1,n)] \cong \mathrm{Rep}(V, U(qn))^{\psi^{\xi}},$$

where ψ^{ξ} are unstable Adams operations of order q defined on complex representations.

Proof. By Corollary 1.3 there is a bijection

$$[BV, BX(q,1,n)] \to [BV, BU(qn)]^{\psi^{\xi}}.$$

Then proposition follows from the Dwyer-Zabrodsky theorem [14],

$$[BV, BU(qn)] \cong \mathrm{Rep}(V, U(qn)).$$

□

Remark 6.2. The action of $\mathbb{Z}/q\mathbb{Z}$ on the irreducible representations of V fixes the trivial representation and defines $\frac{p-1}{q}$ orbits, each of cardinality q. In particular, the set

$$\mathrm{Rep}(V, X(q,1,1)) := [BV, BX(q,1,1)] \cong [BV, BU(q)]^{\mathbb{Z}/q\mathbb{Z}}$$

consists of the trivial representation and the orbits of the irreducible non-trivial representations of V.

In general, we can state that any homotopy representation of V into $X(q,1,n)$ splits as a direct sum of irreducible homotopy representations
$\mathrm{IRep}(V) = [V, X(q,1,1)]$ where $X(q,1,1)$ is the Sullivan sphere $(S^{2q-1})\hat{_p}$.

Let Y and X be a p-compact groups. The set of representations $Rep(X, Y)$ is given by $[BX, BY]$, homotopy classes of free maps. We say that a homotopy representation $f \in \mathrm{Rep}(Y, X(q,q,n))$ splits as a direct sum of homotopy representations if f factors through a product of generalized Grassmannians. We say that a homotopy representation is irreducible if it does not split as direct sum of nontrivial homotopy representations.

It is known that complex representations of finite groups split as a direct sum of irreducible representations. The following result may be proved using the above discussion and it expresses the analogy with the situation for complex and real representations of finite groups.

Corollary 6.3. *Let V be an elementary abelian p-groups. Any homotopy representation $f \in \mathrm{Rep}(V, X(q,1,n))$ splits as a direct sum of irreducible homotopy representations. Moreover, the set of irreducible homotopy representations of V into $(S^{2q-1})\hat{_p}$ is $[BV, BU(q)]^{\mathbb{Z}/q\mathbb{Z}}$.*

Acknowledgements. The author is greatly indebted to Carlos Broto for several helpful comments concerning Section 3 and wishes to express her thanks to Jesper Møller for pointing out the existence of the above mentioned exercise in [4] and helpful suggestions to improve results in Section 4. We also thank the referees for their careful reading and many suggestions.

References

1. Adams, F.: Vector fields on spheres. Ann. of Math. **75**(3), 603–632 (1962)
2. Andersen, K., Grodal, J., Møller, J., Viruel, A.: The classification of p-compact groups for p odd. Preprint arXiv:math.AT/0302346
3. Baum, B.F.: On the cohomology of homogeneous spaces. Topology **7**, 15–38 (1968)
4. Bourbaki, N.: Éléments de mathématique. Groupes et algèbres de Lie. Chapitre 9, Masson, Paris, 1982
5. Broto, C., Møller, J.M.: Embeddings of DI_2 in F_4. Trans. Am. Math. Soc. **353**, 4461–3379 (2001)
6. Bousfield, A. K., Kan, D. M.: Homotopy limits, completions and localizations. Lecture Notes in Mathematics, Vol. **304**, Springer-Verlag, Berlin, 1972
7. Castellana, N.: On the p-compact groups corresponding to the reflection groups $G(q,r,n)$. To appear in Trans. Am. Math. Soc.
8. Clark, A., Ewing, J.: The realization of polynomial algebras as cohomology rings. Pacific J. of Math. **5**, 425–434 (1974)
9. Cooke, G.: Replacing homotopy actions by topological actions. Trans. Am. Math. Soc **237**, 391–406 (1978)
10. Dwyer, W. G., Kan, D. M.: Centric maps and the realization of diagrams in the homotopy category. Proc. Am. Math. Soc. **114**(2), 575–584 (1992)
11. Dwyer, W. G., Miller, H. R., Wilkerson, C. W.: Homotopical uniqueness of classifying spaces. Topology **31**(1), 29–45 (1992)
12. Dwyer, W. G., Wilkerson, C. W.: The elementary geometric structure of compact Lie groups. Bull. London Math. Soc. **30**, 337–364 (1998)
13. Dwyer, W. G., Wilkerson, C. W.: Homotopy fixed-point methods for Lie groups and finite loop spaces. Ann. Math. **139**(2), 395–442 (1994)
14. Dwyer, W.G., Zabrodsky, A.: Maps between classifying spaces. Proceedings of the 1986 Barcelona Conference on Algebraic Topology, Lecture Notes in Math. 1298, Springer Verlag, 1988
15. tom Dieck, T.: Transformation groups, de Gruyter Studies in Mathematics, 8. Walter de Gruyter & Co, Berlin-New York, 1987
16. Jackowski, S., McClure, J., Oliver, B.: Homotopy classification of self-maps of BG via G actions, I,II. Ann. of Math **135**, 183–270 (1992)
17. Harris, B.: On the homotopy groups of classical groups. Ann. of Math. **74**, 407–413 (1961)
18. Mac Lane, S.: Categories for the Working Mathematician. Graduate Text in Mathematics **5** (1971), Springer-Verlag
19. Møller, J.: N-determined p-compact groups. Fund. Math. **173**(3), 201–300 (2002)

20. Notbohm, D.: A topological realization of a family of pseudoreflection groups. Fund. Math. **155**, 1–31 (1998)
21. Notbohm, D.: p-adic lattices of pseudo reflection groups. Proceedings of the Barcelona Conference 1994, Algebraic Topology: New trends in localization and periodicity, ed.: C. Broto, C. Casacuberta and G. Mislin, pp. 337–352
22. Notbohm, D.: Maps between classifying spaces. Math. Z. **207**, 153–168 (1991)
23. Quillen, D.: On the cohomology and K-theory of the general linear groups over a finite field. Ann. of Math. **96**, 552–586 (1972)
24. Sheppard, G.C., Todd, J.A.: Finite unitary reflection groups. Canad. J. Math. **6**, 274–304 (1954)
25. Smith, L.: On the charateristic zero cohomology of the free loop space. Am. J. Math. **103**, 887–910 (1981)
26. Wojtkowiak, Z.: Maps from Bπ into X. Quart. J. Math. Oxford **39**, 117–127 (1988)
27. Wojtkowiak, Z.: On maps from holim F to Z. Algebraic Topology (Barcelona 1986). Lecture Notes in Math. **1298**, Springer, 227–236 (1987)