HOMOLOGY DECOMPOSITIONS FOR p-COMPACT GROUPS

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Abstract. We construct a homotopy theoretic setup for homology decompositions of classifying spaces of p-compact groups. This setup is then used to show that the existence of the Dwyer-Wilkerson centralizer decomposition with respect to the family of elementary abelian p-subgroups of a p-compact group X is equivalent to the existence of a subgroup decomposition for X with respect to any family of subgroups which contains the radical subgroups of X. The resulting subgroup decomposition we obtain generalizes the subgroup decomposition with respect to radical subgroups for a compact Lie group constructed by Jackowski, McClure and Oliver.

Homology decompositions are among the most useful tools in the study of the homotopy theory of classifying spaces. Roughly speaking, a homology decomposition for a space X, with respect to some homology theory h_*, is a recipe for gluing together spaces, desirably of a simpler homotopy type, such that the resulting space maps into X by a map which induces an h_*-isomorphism.

When constructing a homology decomposition for a classifying space of a group G, it is natural to do so using classifying spaces of subgroups of G. For compact Lie groups two types of mod-p homology decompositions are known: the centralizer decomposition with respect to elementary abelian p-subgroups, due to Jackowski and McClure [JM], and the subgroup decomposition with respect to certain families of p-toral subgroups, due to Jackowski, McClure and Oliver [JMO].

A *p-compact group* is an $\mathbb{F}_p$-finite loop space X (i.e., a loop space whose mod-p homology is of finite type and vanishes above a certain degree), whose classifying space BX is p-complete in the sense of [BK]. These objects, defined by Dwyer and Wilkerson [DW1], and extensively studied by them and others, are a far reaching homotopy theoretic generalization of compact Lie groups and their classifying spaces. Dwyer and Wilkerson also introduced in [DW2] a centralizer decomposition with respect to elementary abelian p-subgroups for p-compact groups, which generalizes the corresponding decomposition for compact Lie groups. The aim of this paper is to construct a subgroup decomposition for p-compact groups, analogous to the subgroup decomposition for compact Lie groups introduced by Jackowski, McClure and Oliver in [JMO]. We will in fact show that in the right setup, the Dwyer-Wilkerson theorem about existence of a centralizer decomposition for p-compact groups, with respect to their elementary abelian p-subgroups, is equivalent to the existence of subgroup decompositions with respect to certain other families of subgroups. More detail will be given shortly.

We start by explaining some of the concepts involved. A subgroup of a p-compact group X is a pair (Y, α) where Y is a p-compact group and $\alpha : BY \longrightarrow BX$ is a *monomorphism*, namely, a pointed map whose homotopy fibre is $\mathbb{F}_p$-finite. The phrase

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"(Y,α) is a subgroup of X" will frequently be abbreviated by $Y \leq_\alpha X$. Recall that a p-compact torus is a topological group of type $K(A,1)$, where A is isomorphic to a finite product of copies of the p-adic integers. A p-compact toral group is a group containing a p-compact torus as a normal subgroup with p-power index. Every p-compact group admits a distinguished family of p-compact toral subgroups (S,ι), which are maximal in the sense that if (P,β) is any other p-compact toral subgroup of X, then there exists a map $f\colon BP \longrightarrow BS$, such that $\iota\circ f \simeq \beta$. Any such subgroup will be called a Sylow subgroup of X (see Definition A.9 and the following discussion).

For any p-compact group X, we consider two categories: the orbit category $\mathcal{O}(X)$ and the fusion category $\mathcal{F}(X)$. The objects in both categories are given by all subgroups (Y,α) of X. A morphism $(Y,\alpha) \longrightarrow (Y',\alpha')$ in $\mathcal{O}(X)$ is a homotopy class of a map $h\colon BY \longrightarrow BY'$ such that $\alpha'\circ h \simeq \alpha$, whereas in $\mathcal{F}(X)$ such a morphism is a pointed homotopy class of a homomorphism $f\colon BY \longrightarrow BY'$ such that $\alpha'\circ f$ is freely homotopic to α.

For any p-compact group X, we consider certain full subcategories of $\mathcal{O}(X)$ and $\mathcal{F}(X)$, where the objects are restricted to particular collections of subgroups, defined by certain properties:

- A subgroup $Y \leq_\alpha X$ is said to be *quasicentric* if the homotopy fibre of the natural map
$$\alpha_{\#}\colon \mathrm{Map}(BY,BY)_{id} \longrightarrow \mathrm{Map}(BY,BX)_\alpha$$
is weakly homotopically discrete and *centric* if it is weakly contractible.
- A p-compact toral subgroup $Y \leq_\alpha X$ of a p-compact group X is said to be *radical* if it is quasicentric and if $\mathrm{Aut}_{\mathcal{O}(X)}(Y,\alpha)$ is finite and contains no normal non-trivial p-subgroup (i.e., it is finite and p-reduced).

Every centric subgroup is obviously quasicentric, and we will show later that every radical subgroup is centric (Lemma 4.1). For a p-compact group X, we denote by $\mathcal{O}_p^c(X)$ and $\mathcal{O}_p^r(X)$, the full subcategories of $\mathcal{O}(X)$ whose objects are the centric and radical p-compact toral subgroups, respectively. Similar notation will be used for the fusion category. These categories are not generally small, but have small (in fact at most countable) skeletal subcategories (see Proposition 1.7), so defining limits and colimits over them makes sense. Let Top denote the category of spaces, and hoTop its homotopy category. We are now ready to state our main theorem.

Theorem A. *For any p-compact group X there exists a functor*
$$\Phi\colon \mathcal{O}_p^r(X) \longrightarrow \mathsf{Top},$$
such that

(1) *For each object (P,α) in $\mathcal{O}_p^r(X)$, $\Phi(P,\alpha)\simeq BP$ and*
(2) *there is a natural map*
$$\operatorname*{hocolim}_{\mathcal{O}_p^r(X)} \Phi \longrightarrow BX,$$
which induces a mod-p homology equivalence.

More detail about the method used in the proof will be given in the following section.

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1. Approximations and Decompositions

In this section the strategy for proving Theorem A is described in detail and an auxiliary result, which might be of independent interest, is stated and shown to imply the theorem.

Let $\pi\colon \mathsf{Top} \longrightarrow \mathsf{hoTop}$ denote the obvious projection functor. For any p-compact group X, there are functors

$$\phi\colon \mathcal{O}(X) \longrightarrow \mathsf{hoTop} \quad \text{and} \quad \psi\colon \mathcal{F}(X)^{op} \longrightarrow \mathsf{hoTop},$$

defined as follows. The functor ϕ sends a subgroup (Y,α) to BY and any morphism to the respective homotopy class. The functor ψ takes a subgroup (Y,α) to the mapping space $\mathrm{Map}(BY,BX)_\alpha$ and a morphism to the homotopy class of the map induced by any representative. For a subgroup (Y,α) of X, we denote $\mathrm{Map}(BY,BX)_\alpha$ by $B\mathcal{C}_X(Y,\alpha)$ or $B\mathcal{C}_X(Y)$ for short, if no ambiguity can arise. The associated loop space $\mathcal{C}_X(Y)$ is called the *centralizer* of (Y,α) in X. Dwyer and Wilkerson showed in [DW1, Prop. 5.1, 5.2] that if (P,α) is p-compact toral subgroup of X, then $\mathcal{C}_X(P)$ is a p-compact group and that the evaluation map $B\mathcal{C}_X(P) \xrightarrow{ev} BX$ is a monomorphism. (Thus the pair $(\mathcal{C}_X(P), ev)$ is a subgroup of X.)

If $\mathcal{C}$ is a collection of p-compact toral subgroups of a p-compact group X, then we denote by $\mathcal{O}_\mathcal{C}(X)$ and $\mathcal{F}_\mathcal{C}(X)$ the full subcategories of $\mathcal{O}(X)$ and $\mathcal{F}(X)$, whose objects are the subgroups in $\mathcal{C}$. We denote by

$$\phi_\mathcal{C}\colon \mathcal{O}_\mathcal{C}(X) \longrightarrow \mathsf{hoTop} \quad \text{and} \quad \psi_\mathcal{C}\colon \mathcal{F}_\mathcal{C}(X)^{op} \longrightarrow \mathsf{hoTop},$$

the restriction of ϕ and ψ to the respective full subcategories. Given such a collection $\mathcal{C}$, one obtains two enlarged collections: the collection $\mathcal{C}_1$ obtained by adding the subgroup $(X, 1_{BX})$ to $\mathcal{C}$, and the collection $\mathcal{C}_0$ obtained by adding the trivial subgroup $(\{1\}, *)$, where $*\colon B\{1\} \longrightarrow BX$ is the inclusion of the base point. Let $\iota_1\colon \mathcal{O}_\mathcal{C}(X) \longrightarrow \mathcal{O}_{\mathcal{C}_1}(X)$ and $\iota_0\colon \mathcal{F}_\mathcal{C}(X) \longrightarrow \mathcal{F}_{\mathcal{C}_0}(X)$ be the respective inclusion functors.

The following definitions are introduced in [DK].

Definition 1.1. *If $\theta\colon \mathcal{D} \longrightarrow \mathsf{hoTop}$, then a realization of θ is a pair (Θ,γ), where $\Theta\colon \mathcal{D} \longrightarrow \mathsf{Top}$ is a functor and $\pi\circ\Theta \xrightarrow{\gamma} \theta$ a natural isomorphism of functors. Two realizations (Θ,γ) and (Θ',γ') are weakly equivalent if there exists a natural transformation $\epsilon\colon \Theta \longrightarrow \Theta'$, which is a weak equivalence on each object $d \in \mathcal{D}$ and such that $\gamma'\circ\pi(\epsilon) = \gamma$.*

With this terminology we can now define what we mean by subgroup and centralizer approximations.

Definition 1.2. *Let $\mathcal{C}$ be a collection of subgroups of a p-compact group X. A subgroup approximation for X with respect to $\mathcal{C}$ is a realization $(\Phi_{\mathcal{C}_1},\gamma_1)$ of $\phi_{\mathcal{C}_1}$. Similarly, a centralizer approximation for X with respect to $\mathcal{C}$ is a realization $(\Psi_{\mathcal{C}_0},\delta_0)$ of $\psi_{\mathcal{C}_0}$.*

The following lemma provides an alternative, somewhat more intuitive, definition of subgroup and centralizer approximations. For a category $\mathcal{C}$, and a space Y, let $1_Y\colon \mathcal{C} \longrightarrow \mathsf{Top}$ denote the constant functor, taking each object to Y, and each morphism to the identity map.

Lemma 1.3. *Let $\mathcal{C}$ be a collection of subgroups of a p-compact group X. Then*

(i) *X has a subgroup approximation with respect to $\mathcal{C}$ if and only if there exists a realization $(\Phi_{\mathcal{C}}, \gamma)$ of $\phi_{\mathcal{C}}$, and a natural transformation $\eta\colon \Phi_{\mathcal{C}} \longrightarrow 1_{BX}$.*

(ii) *X has a centralizer approximation with respect to $\mathcal{C}$ if and only if there exists a realization $(\Psi_{\mathcal{C}}, \delta)$ of $\psi_{\mathcal{C}}$, and a natural transformation $\zeta\colon \Psi_{\mathcal{C}} \longrightarrow 1_{BX}$.*

Proof. We prove only (i), as the proof of (ii) is totally analogous. Assume X admits a subgroup approximation $(\Phi_{\mathcal{C}_1}, \gamma_1)$ with respect to $\mathcal{C}$ in the sense of Definition 1.2. Since $(X, 1_{BX})$ is a terminal object in $\mathcal{O}_{\mathcal{C}_1}(X)$, the obvious map

$$\operatorname*{hocolim}_{\mathcal{O}_{\mathcal{C}_1}(X)} \Phi_{\mathcal{C}_1} \longrightarrow \Phi_{\mathcal{C}_1}(X, 1_{BX})$$

is a homotopy equivalence. Let $\rho_1\colon \pi \circ \Phi_{\mathcal{C}_1} \longrightarrow \phi_{\mathcal{C}_1}$ be a natural isomorphism. Then ρ_1 determines a homotopy class of a homotopy equivalence $\Phi_{\mathcal{C}_1}(X, 1_{BX}) \xrightarrow{\simeq} BX$. Fix a representative ι_X for this equivalence.

Let $\Phi_{\mathcal{C}}$ denote the composite

$$\mathcal{O}_{\mathcal{C}}(X) \xrightarrow{inc} \mathcal{O}_{\mathcal{C}_1}(X) \xrightarrow{\Phi_{\mathcal{C}_1}} \mathsf{Top},$$

and let γ denote the restriction of γ_1 to $\Phi_{\mathcal{C}}$. Then $(\Phi_{\mathcal{C}}, \gamma)$ is clearly a realization of the homotopy functor $\phi_{\mathcal{C}}$ on $\mathcal{O}_{\mathcal{C}}(X)$. Let η denote the natural transformation defined by taking an object (P, α) of $\mathcal{O}_{\mathcal{C}}(X)$ to the composite

$$\Phi_{\mathcal{C}}(P, \alpha) = \Phi_{\mathcal{C}_1}(P, \alpha) \xrightarrow{\Phi_{\mathcal{C}_1}(\alpha)} \Phi_{\mathcal{C}_1}(X, 1_{BX}) \xrightarrow{\iota_X} BX.$$

Conversely, let $(\Phi_{\mathcal{C}}, \gamma)$ be a realization of the homotopy functor $\phi_{\mathcal{C}}$ restricted to $\mathcal{O}_{\mathcal{C}}(X)$, and let $\eta\colon \Phi_{\mathcal{C}} \longrightarrow 1_{BX}$ be a natural transformation. We must extend $\Phi_{\mathcal{C}}$ to $\mathcal{O}_{\mathcal{C}_1}(X)$, and define the appropriate natural transformation γ_1. Define $\Phi_{\mathcal{C}_1}(X, 1_{BX}) = BX$. If $(P, \alpha) \xrightarrow{[f]} (X, 1_{BX})$ is any morphism in $\mathcal{O}_{\mathcal{C}_1}(X)$, then f and α have the same homotopy class. Define $\Phi_{\mathcal{C}_1}([f]) = \eta(P, \alpha)\colon \Phi_{\mathcal{C}}(P, \alpha) \longrightarrow BX$. Define $\gamma_1\colon \pi \circ \Phi_{\mathcal{C}_1} \longrightarrow \phi_{\mathcal{C}_1}$ to coincide with γ on objects different from $(X, 1_{BX})$, and let $\gamma_1(X, 1_{BX})$ be the identity map. Then $(\Phi_{\mathcal{C}_1}, \gamma_1)$ is a subgroup approximation for X with respect to $\mathcal{C}$ in the sense of Definition 1.2, and the proof is complete. □

Any subgroup approximation $(\Phi_{\mathcal{C}_1}, \gamma_1)$ for X with respect to $\mathcal{C}$ gives rise to a map

$$\operatorname*{hocolim}_{\mathcal{O}_{\mathcal{C}}(X)} \Phi_{\mathcal{C}} \xrightarrow{inc_*} \operatorname*{hocolim}_{\mathcal{O}_{\mathcal{C}_1}(X)} \Phi_{\mathcal{C}_1} \xrightarrow{\simeq} \Phi_{\mathcal{C}_1}(X, 1_{BX}) \simeq BX,$$

where $\Phi_{\mathcal{C}}$ is the functor $\Phi_{\mathcal{C}_1}$ pre-composed with the inclusion. By abuse of notation we denote this map by η (since with the interpretation suggested by Lemma 1.3, this map would be induced by the natural transformation η). Similarly, a centralizer approximation $(\Psi_{\mathcal{C}_0}, \delta_0)$ gives rise to a map ζ given by the composite

$$\operatorname*{hocolim}_{\mathcal{F}_{\mathcal{C}}(X)^{op}} \Psi_{\mathcal{C}} \longrightarrow \operatorname*{hocolim}_{\mathcal{F}_{\mathcal{C}_0}(X)^{op}} \Psi_{\mathcal{C}_0} \xrightarrow{\simeq} \Psi_{\mathcal{C}_0}(\{1\}, *) \simeq BX,$$

where $\Psi_{\mathcal{C}}$ is the functor $\Psi_{\mathcal{C}_0}$ pre-composed with the inclusion.

Generally the maps η and ζ are not guaranteed to have any good properties, which brings us to the following

Definition 1.4. *We say that a subgroup approximation $(\Phi_{\mathcal{C}_1}, \gamma_1)$ (resp. centralizer approximation $(\Psi_{\mathcal{C}_0}, \delta_0)$) is a subgroup (*resp. *centralizer) decomposition if the map η (resp. ζ) above induces a mod-p homology equivalence.*

A collection $\mathcal{C}$ of subgroups of a p-compact group X is called subgroup-ample if there exists a subgroup decomposition $(\Phi_{\mathcal{C}_1}, \gamma_1)$ for X with respect to $\mathcal{C}$. Similarly $\mathcal{C}$ is said *to be centralizer-ample if there exists a centralizer decomposition* $(\Psi_{\mathcal{C}_0}, \delta_0)$ *for* X *with respect to* $\mathcal{C}$*.*

The claim of our main theorem thus amounts to saying that any p-compact group admits a subgroup decomposition with respect to the collection of its radical subgroups, or equivalently that the collection of the radical subgroups of a p-compact group X is subgroup-ample. Using this terminology, the Dwyer-Wilkerson theorem on homology decompositions for p-compact groups can be stated as claiming that for any p-compact group X, the collection of all its elementary abelian subgroups is centralizer-ample. The term "ample" is borrowed from [D], although there it refers only to a collection and not to the attempted approximation. It is possible to show that if $\mathcal{C}$ is an arbitrary collection of subgroups of X, then $\mathcal{C}$ is centralizer-ample if and only if it is subgroup-ample. This would justify using the phrase "an ample collection" in the sense Dwyer does in [D], but we shall not discuss this terminology any further in this paper.

Proposition 1.5. *For any p-compact group X the following hold.*

(i) *If $\mathcal{C}$ is a collection of centric subgroups of X, then there exists a subgroup approximation $(\Phi_{\mathcal{C}_1}, \gamma_1)$ for X with respect to $\mathcal{C}$, which is unique up to a weak equivalence.*
(ii) *If $\mathcal{A}$ is a collection of finite abelian p-subgroups of X, there exists a centralizer approximation $(\Psi_{\mathcal{A}_0}, \delta_0)$ for X with respect to $\mathcal{A}$, which is unique up to a weak equivalence.*

Proof. We first recall some terminology from [DK] that will be used in the proof. For a small category $\mathcal{D}$, a functor $\theta\colon \mathcal{D} \longrightarrow \mathsf{hoTop}$ is said to define a centric diagram over $\mathcal{D}$ if for every morphism $c \xrightarrow{f} d$ in $\mathcal{D}$, $\theta(f)$ is a the homotopy class of a centric map, namely, if for any representative f' for $\theta(f)$, the induced map

$$f'_{\#}\colon \operatorname{Map}(\theta(c), \theta(c))_{id} \longrightarrow \operatorname{Map}(\theta(c), \theta(d))_{\theta(f)}$$

is a weak equivalence.

If θ defines a centric diagram over $\mathcal{D}$, one has a sequence of functors $\theta_i\colon \mathcal{D}^{op} \longrightarrow \mathcal{A}b$ given by

$$\theta_i = \pi_i(\operatorname{Map}(\theta(d), \theta(d))_{id}),$$

and by [DK, Theorem 1.1], if the groups $\varprojlim^j \theta_i$ vanish for all i and j, then there exists a realization Θ of θ which is unique up to weak equivalence.

Lemma 3.1 below implies that if $\mathcal{C}$ is a centric collection (i.e., a collection all of whose objects are centric) of p-compact toral subgroups of a p-compact group X, then $\phi_{\mathcal{C}}\colon \mathcal{O}_{\mathcal{C}}(X) \longrightarrow \mathsf{hoTop}$ defines a centric diagram. It is also immediate that the extended diagram defined by $\phi_{\mathcal{C}_1}$ is centric. The category $\mathcal{O}_{\mathcal{C}_1}(X)$ has a terminal object, and hence the higher limits of any contravariant functor from it to the category of abelian groups vanish. Thus by the Dwyer-Kan theorem stated above, a realization $(\Phi_{\mathcal{C}_1}, \gamma_1)$ of $\phi_{\mathcal{C}_1}$ exists and is unique up to weak equivalence.

Lemma 3.1 again, in conjunction with the fact that the centralizer in X of a p-compact toral subgroup is itself a p-compact group [DW1, Proposition 5.1], implies that if $\mathcal{A}$ is a collection of finite abelian p-subgroup of X, then the diagram defined by $\psi_{\mathcal{A}}\colon \mathcal{F}_{\mathcal{A}}(X)^{op} \longrightarrow \mathsf{hoTop}$ is centric [DW2, Lemma 11.15]. An identical argument now shows that there exists a realization $(\Psi_{\mathcal{A}_0}, \delta_0)$ of $\psi_{\mathcal{A}_0}$, which is unique up to weak equivalence. □

Remark 1.6. *In particular, notice that the uniqueness part of Proposition 1.5 implies that if a collection is subgroup ample, then any subgroup approximation is a decomposition. A similar comment applies to centralizer approximations.*

The main statement of this paper is not in fact an independent claim that the collection of radical subgroups in a p-compact group is subgroup-ample, but rather that this statement is equivalent to the claim that the collection of all elementary abelian subgroups is centralizer-ample. More precisely one has:

Theorem B. *For any p-compact group X, the following statements are equivalent:*

a) *The collection of all p-compact toral centric subgroups of X is subgroup ample.*
b) *The collection of all radical subgroups of X is subgroup ample.*

Furthermore, the following statements are equivalent:

(i) *For every p-compact group X the collection of all its non-trivial elementary abelian subgroups is centralizer ample.*
(ii) *For every p-compact group X the collection of all its centric p-compact toral subgroups is subgroup ample.*

Statement (i) of Theorem B is a theorem of Dwyer and Wilkerson [DW2, Theorem 8.1]. Thus, Theorem B implies Theorem A at once. Notice the difference between the two sets of equivalent conditions in the theorem: in the first set the conditions are stated for a given p-compact group, whereas in the second they are stated for all p-compact groups. The reason for this difference is the different methods we employ in proving the two sets of equivalences.

We end this section with a general statement which implies at once that all the categories considered in this paper have small skeletal subcategories. We say that two subgroups (Y, α) and (Y', α') of X are "*conjugate*", if they are isomorphic as objects in $\mathcal{O}(X)$. Using this terminology, it follows at once that all Sylow subgroups of a p-compact group X are conjugate (see Definition A.9). We also talk occasionally about"*conjugacy classes*" of subgroups of X, by which we mean simply isomorphism classes of objects in $\mathcal{O}(X)$. This proposition uses the existence of discrete approximations for p-compact toral groups, established in [DW1].

Proposition 1.7. *For any p-compact group X, conjugacy classes of p-compact toral subgroups of X form a countable set.*

Proof. Let $S \leq_\iota X$ be a Sylow subgroup, and fix a discrete approximation $\check{S}$ of S. Let (P, α) be an arbitrary p-compact toral subgroup of X, and let $\check{P}$ be a discrete approximation of P. Then, by the defining property of a Sylow subgroup, there is a map $f\colon BP \longrightarrow BS$, such that $\iota \circ f \simeq \alpha$. By Lemma A.22, f is homotopic to a map induced by a homomorphism (in fact, a monomorphism) $\check{f}\colon \check{P} \longrightarrow \check{S}$. Thus conjugacy classes of p-compact toral subgroups of X are in 1–1 correspondence with conjugacy classes in X of subgroups of the form $(Q, \check{\iota}_Q)$, where Q is a subgroup of $\check{S}$ in the ordinary sense and $\check{\iota}_Q\colon BQ^\wedge_p \longrightarrow BX$ is the p-completion of the inclusion followed by ι. But $\check{S}$ is a countable group, and the Proposition follows. □

The rest of the paper is organized as follows. Section 2 contains a discussion on the notion of the normalizer and Weyl spaces for subgroups of a p-compact group, as well as the Weyl group of a subgroup. The key properties of centric and radical subgroups are proven in Sections 3 and 4. Section 5 is a study of the orbit category of radical subgroups. A slightly stronger form of the equivalence between a) and b) in Theorem B is shown in Section 6 (Proposition 6.1). The equivalence of (i) and (ii) in the theorem is contained in Section 7 (Proposition 7.4, again in a slightly stronger form). Background material needed along the paper is collected in Appendix A. In Appendix B we show that our decomposition theorem is indeed a generalization of the Jackowski-McClure-Oliver decomposition theorem.

2. The Normalizer and Weyl spaces of a subgroup

In [DW3] Dwyer and Wilkerson construct a normalizer space $\mathcal{N}_X(Y) = \mathcal{N}_X(Y,\alpha)$ and a Weyl space $\mathcal{W}_X(Y) = \mathcal{W}_X(Y,\alpha)$ for a subgroup (Y,α) of a p-compact group X. In this section we give an alternative construction of these spaces and study some of their properties, which will be useful in analyzing centric and radical collections of subgroups.

For a subgroup $Y \leq_\alpha X$, we defined

$$B\mathcal{C}_X(Y) = B\mathcal{C}_X(Y,\alpha) \stackrel{\text{def}}{=} \mathrm{Map}(BY,BX)_\alpha.$$

We denote by $\mathcal{C}_X(Y) = \Omega B\mathcal{C}_X(Y)$ the space of Moore loops in $B\mathcal{C}_X(Y)$ based at α. When Y is a p-compact toral group, $B\mathcal{C}_X(Y)$ is the classifying space of a p-compact group [DW1, Propositions 5.1 and 6.1]. Let $\mathcal{P}(B\mathcal{C}_X(Y))$ denote the space of Moore paths in $B\mathcal{C}_X(Y)$ based at α, namely, paths $[0,r] \xrightarrow{\omega} B\mathcal{C}_X(Y)$, $r \geq 0$ with $\omega(0) = \alpha$. Let

$$ev\colon \mathcal{P}(B\mathcal{C}_X(Y)) \longrightarrow B\mathcal{C}_X(Y)$$

denote the evaluation map at the end point of a path. Then ev is a fibration, and the fibre over the base point α is $\mathcal{C}_X(Y)$.

Let $\mathrm{Map}(BY,BY)_{\{\alpha\}}$ denote the components of the mapping space which are mapped to the component of α under the map

$$\alpha_\#\colon\ \mathrm{Map}(BY,BY) \longrightarrow \mathrm{Map}(BY,BX).$$

Let $\mathrm{Map}_*(BY,BY)_{\{\alpha_*\}}$ denote the corresponding components of the pointed mapping space. Let $\mathcal{W}_X(Y,\alpha)$ and $\mathcal{N}_X(Y,\alpha)$ denote the pull-back spaces of ev along $\alpha_\#$ and $\alpha_\# \circ j$ respectively, as shown in the following commutative diagram

$$\begin{array}{ccccc} \mathcal{N}_X(Y,\alpha) & \xrightarrow{\quad\pi\quad} & \mathcal{W}_X(Y,\alpha) & \longrightarrow & \mathcal{P}(B\mathcal{C}_X(Y)) \\ {\scriptstyle\nu}\downarrow & & {\scriptstyle\theta}\downarrow & & {\scriptstyle ev}\downarrow \\ \mathrm{Map}_*(BY,BY)_{\{\alpha_*\}} & \xrightarrow{\quad i\quad} & \mathrm{Map}(BY,BY)_{\{\alpha\}} & \xrightarrow{\quad\alpha_\#\quad} & B\mathcal{C}_X(Y) \end{array}, \tag{1}$$

where the map i is the inclusion. Since both squares are pull-back squares by construction, the homotopy fibres of i and π are both homotopy equivalent to Y.

Definition 2.1. *The spaces $\mathcal{N}_X(Y) = \mathcal{N}_X(Y,\alpha)$ and $\mathcal{W}_X(Y) = \mathcal{W}_X(Y,\alpha)$ are called the* normalizer space *and* Weyl space *of (Y,α) respectively.*

As one should expect, these spaces admit a multiplicative structure, as detailed in the following lemma.

Lemma 2.2. *For any subgroup (Y,α) of a p-compact group X, the normalizer and Weyl spaces of (Y,α) admit a natural structure of topological monoids. Furthermore, the maps π, θ and ν in diagram (1) above, as well as the fibre inclusion $\mathcal{C}_X(Y) \longrightarrow \mathcal{N}_X(Y)$, are all maps of topological monoids.*

Proof. By construction, a point in $\mathcal{W}_X(Y)$ is a pair (f,ω), where $BY \xrightarrow{f} BY$ is a map, $[0,r] \xrightarrow{\omega} B\mathcal{C}_X(Y)$ is a path starting at α, and $\omega(r) = \alpha\circ f$. Define a monoidal structure on $\mathcal{W}_X(Y)$ as follows. If (f,ω) and (f',ω') are two points in $\mathcal{W}_X(Y)$, define $(f,\omega)\cdot(f',\omega')$ by $(f\circ f', f'(\omega)*\omega')$, where $f'(\omega)$ is the image of ω under the self map of $\mathrm{Map}(BY,BX)_\alpha$ induced by f', and $*$ means the juxtaposition of the two paths. This is clearly an associative composition with a two-sided unit given by the pair $(1_Y,\kappa_\alpha)$, where κ_α means the constant path at α. The map θ is a map of topological monoids with respect to this operation on $\mathcal{W}_X(Y)$ and composition of maps in $\mathrm{Map}(BY,BY)_{\{\alpha\}}$. The monoidal structure on $\mathcal{N}_X(Y)$ is defined similarly and makes the maps π and ν into maps of topological monoids. Finally, $\mathcal{C}_X(Y)$ can be identified as the subspace of $\mathcal{N}_X(Y)$ given by the fibre of ν over the identity map in $\mathrm{Map}(BY,BY)_{\{\alpha*\}}$, and inclusion is obviously a multiplicative map. □

The sets of components of $\mathcal{W}_X(Y)$ and $\mathcal{N}_X(Y)$ obviously have the structure of unital associative monoids. Moreover, using Lemma A.2, it is easy to see that both topological monoids consist of self maps of BY which are homotopy monomorphisms, and thus automorphisms. Therefore, the set of components are in fact groups. We will only use $\pi_0(\mathcal{W}_X(Y))$ in this article.

Definition 2.3. *For any subgroup (Y,α) of X, define its Weyl group by*

$$W_X(Y) = W_X(Y,\alpha) \stackrel{\text{def}}{=} \pi_0(\mathcal{W}_X(Y,\alpha)).$$

Remark 2.4. The definition of the Weyl space given above coincides up to homotopy with the one in [DW3, Definition 4.1]. In the same paper (see the discussion following Remark 4.2) Dwyer and Wilkerson also identify the Weyl space $\mathcal{W}_X(Y,\alpha)$ with the homotopy fixed point set $(X/Y)^{hY}$ of the Y-action on the homotopy fibre X/Y of the map α (or rather on a space of the same homotopy type, which admits such an action). This alternative description of the Weyl space will be useful in a number of occasions throughout this paper.

The next two statements give a few useful features of the Normalizer and Weyl spaces. For a space K we denote by $*_K$ a choice of a base point in K, and by $\Omega(K,*_K)$ the Moore loops of K based at $*_K$.

Lemma 2.5. *For any subgroup $Y \leq_\alpha X$, there is a map of topological monoids $\omega_\alpha\colon \mathcal{N}_X(Y) \longrightarrow \Omega(BX,\alpha(*_{BY}))$. Furthermore, the map $\eta_\alpha \stackrel{\text{def}}{=} B\omega_\alpha$ is an extension of both α, and the evaluation map $B\mathcal{C}_X(Y) \longrightarrow BX$.*

Proof. Let $(f,\omega)\in\mathcal{N}_X(Y)$ be a point. Thus f is a pointed self map of BY and ω is a Moore path in $B\mathcal{C}_X(Y)$, starting at α and ending at $\alpha\circ f$. Following ω with the evaluation map at the base point $B\mathcal{C}_X(Y) \xrightarrow{ev} BX$ gives a Moore loop in BX based at $\alpha(*_{BY})$. Define

$$\omega_\alpha\colon \mathcal{N}_X(Y) \longrightarrow \Omega(BX,\alpha(*_{BY}))$$

to be the map thus defined. One routinely verifies that ω_α is a map of topological monoids. It is also clear by construction that ω_α extends the loops on the evaluation map $\mathcal{C}_X(Y) \longrightarrow \Omega(BX,\alpha(*_{BY}))$.

Next, notice that Y is homotopy equivalent as a loop space to the normalizer space $\mathcal{N}_Y(Y) \stackrel{\text{def}}{=} \mathcal{N}_Y(Y,1_Y)$ and that α induces a loop map

$$\mathcal{N}_Y(Y) \longrightarrow \mathcal{N}_X(Y),$$

which can be identified up to homotopy with the fibre inclusion in the fibration of loop spaces and loop maps

$$Y \longrightarrow \mathcal{N}_X(Y) \longrightarrow \mathcal{W}_X(Y). \tag{2}$$

Furthermore, the following square commutes

$$\begin{array}{ccc} \mathcal{N}_Y(Y) & \longrightarrow & \mathcal{N}_X(Y) \\ \simeq \big\downarrow \omega_1 & & \big\downarrow \omega_\alpha \\ \Omega(BY, *_{BY}) & \xrightarrow{\Omega\alpha} & \Omega(BX, \alpha(*_{BY})). \end{array}$$

This shows that η_α extends α, and it clearly extends $ev\colon B\mathcal{C}_X(Y) \longrightarrow BX$, as claimed. □

Remark 2.6. Notice that the constructions of $\mathcal{C}_X(Y,\alpha)$, $\mathcal{N}_X(Y,\alpha)$ and $\mathcal{W}_X(Y,\alpha)$ do not actually require that X is a p-compact group, and that Y is a subgroup. Rather, these constructions can be carried out for any pair of spaces X and Y, and an arbitrary map $\alpha\colon X \longrightarrow Y$. Furthermore, Lemmas 2.2 and 2.5 both hold in this more general context.

Proposition 2.7. *Let X be a p-compact group and let $Y \leq_\alpha X$, be a p-compact toral subgroup. Then the following holds.*

(i) *The Weyl space $\mathcal{W}_X(Y)$ and normalizer space $\mathcal{N}_X(Y)$ are both $\mathbb{F}_p$-finite.*
(ii) *The map $\eta_\alpha\colon B\mathcal{N}_X(Y) \longrightarrow BX$ of Lemma 2.5 is a monomorphism (in the sense that its homotopy fibre is $\mathbb{F}_p$-finite).*
(iii) *If X is a p-compact toral group then $\mathcal{W}_X(Y)$ and $\mathcal{N}_X(Y)$ are also p-compact toral groups and $\mathcal{W}_X(Y)$ is non-contractible.*

Proof. By Corollary A.21, $\mathcal{W}_X(Y)$ is $\mathbb{F}_p$-finite, and $W_X(Y)$ is a finite group. The fibration

$$Y \longrightarrow \mathcal{N}_X(Y) \longrightarrow \mathcal{W}_X(Y) \tag{3}$$

then implies that $\pi_0(\mathcal{N}_X(Y))$ is finite. Since each space in this fibration is a loop space, the components of each are all homotopy equivalent and in particular each component of Y and $\mathcal{W}_X(Y)$ is $\mathbb{F}_p$-finite. Let $\mathcal{N}_X(Y)_0$ and $\mathcal{W}_X(Y)_0$ denote the identity components of the respective loop spaces. Then there is a fibration

$$Y_c \longrightarrow \mathcal{N}_X(Y)_0 \longrightarrow \mathcal{W}_X(Y)_0$$

where Y_c is the appropriate union of components of Y. A standard Serre spectral sequence argument, taking into account that this fibration is principal and hence orientable, shows that $\mathcal{N}_X(Y)_0$ is $\mathbb{F}_p$-finite. This completes the proof of Part (i).

In order to prove Part (ii) we will show that if $f\colon B\mathbb{Z}/p \to B\mathcal{N}_X(Y)$ is any map such that $\eta_\alpha \circ f$ is null-homotopic, then f is null-homotopic. By Proposition A.3, this is equivalent to the claim that the homotopy fibre of η_α is $\mathbb{F}_p$-finite.

Let BY' denote the homotopy pullback space of the system

$$B\mathbb{Z}/p \xrightarrow{\pi\circ f} B\mathcal{W}_X(Y) \xleftarrow{\pi} B\mathcal{N}_X(Y).$$

Then one has a fibration

$$BY \xrightarrow{\epsilon} BY' \xrightarrow{\gamma} B\mathbb{Z}/p$$

with a section σ. In particular Y' is a p-compact group.

Let $h\colon BY' \to B\mathcal{N}_X(Y)$ denote the map resulting from the pullback construction. Then $h \circ \sigma \simeq f$, and so $\eta_\alpha \circ h \circ \sigma \simeq \eta_\alpha \circ f$ is null-homotopic by assumption, and by

[MN, Theorem 2.17] it follows that $\eta_\alpha \circ h$ has a nontrivial kernel. Let K denote the kernel of $\eta_\alpha \circ h$, and let Y'' denote Y'/K (c.f., [DW1, §7]). Then one has a commutative diagram:

$$\begin{array}{ccccc} & & BY & & \\ & \swarrow^{\epsilon} & \downarrow & \searrow^{\alpha} & \\ BY' & \xrightarrow{h} & B\mathcal{N}_X(Y) & \xrightarrow{\eta_\alpha} & BX. \\ & \searrow^{\delta} & & \nearrow & \\ & & BY'' & & \end{array}$$

Since α is a monomorphism, so is the composite $\delta \circ \epsilon$ by Lemma A.2(i). Consider the following commutative diagram where the rows and columns are fibrations:

$$\begin{array}{ccccc} F & \longrightarrow & BK & \xrightarrow{\nu} & B\mathbb{Z}/p \\ \downarrow & & \downarrow & & \downarrow = \\ BY & \xrightarrow{\epsilon} & BY' & \xrightarrow{\gamma} & B\mathbb{Z}/p. \\ {\scriptstyle \delta\circ\epsilon}\downarrow & & {\scriptstyle \delta}\downarrow & & \downarrow \\ BY'' & \xrightarrow{=} & BY'' & \longrightarrow & * \end{array}$$

Since $\delta \circ \epsilon$ is a monomorphism, F is $\mathbb{F}_p$-finite, and hence ν is a monomorphism. Since BK is nontrivial, ν must be an equivalence, and so F is contractible. Thus $\delta \circ \epsilon$ is an isomorphism of p-compact groups. Consider the composite

$$BY' \xrightarrow[\simeq]{\gamma \top \delta} B\mathbb{Z}/p \times BY'' \xrightarrow[\simeq]{id \times g} B\mathbb{Z}/p \times BY,$$

where g is some homotopy inverse for $\delta \circ \epsilon$. The first map is a homotopy equivalence since for any choice ν' of a homotopy inverse for ν, the composite $\nu' \circ \gamma$ is a left homotopy inverse for the fibre inclusion of BK in BY'. The second map is obviously a homotopy equivalence, since g is. Let φ denote this composite and choose a homotopy inverse φ'. Then $\varphi \circ \sigma \colon B\mathbb{Z}/p \longrightarrow B\mathbb{Z}/p \times BY$ is equal to $\gamma \circ \sigma$ followed by inclusion to the first factor, and so $\varphi \circ \sigma$ is homotopic to the inclusion to the first factor.

Now, consider the sequence:

$$B\mathbb{Z}/p \xrightarrow{\varphi\circ\sigma} B\mathbb{Z}/p \times BY \xrightarrow{\varphi'} BY' \xrightarrow{h} B\mathcal{N}_X(Y) \xrightarrow{\eta_\alpha} BX.$$

Clearly, $\varphi' \circ (\varphi \circ \sigma) \simeq \sigma$, and so $h \circ \varphi' \circ (\varphi \circ \sigma) \simeq h \circ \sigma \simeq f$. Taking adjoints of the appropriate maps in the sequence, and using Lemma 2.5, one gets a diagram

$$\begin{array}{ccccc} B\mathbb{Z}/p & \xrightarrow{\mathrm{ad}(h\circ\varphi')} & \mathrm{Map}(BY, B\mathcal{N}_X(Y))_{inc} & \xrightarrow{ev} & B\mathcal{N}_X(Y) \\ & & {\scriptstyle \eta_{\alpha\#}}\downarrow & \nearrow & \downarrow{\scriptstyle \eta_\alpha} \\ & & \mathrm{Map}(BY, BX)_\alpha & \xrightarrow{ev} & BX \end{array},$$

which is clearly homotopy commutative, except for possibly at the top triangle. We proceed under the assumption that this triangle commutes as well, and will return to the justification after the argument is completed.

A straight forward diagram chase shows that $ev \circ \mathrm{ad}(h \circ \varphi') \simeq f$. Thus f factors through $B\mathcal{C}_X(Y) = \mathrm{Map}(BY, BX)_\alpha$. Since the evaluation map $B\mathcal{C}_X(Y) \longrightarrow BX$ is a monomorphism, and by assumption $\eta_\alpha \circ f \simeq ev \circ \eta_{\alpha\#} \circ \mathrm{ad}(h \circ \varphi')$ is null-homotopic, the map $\eta_{\alpha\#} \circ \mathrm{ad}(h \circ \varphi')$ factors through the homotopy fibre of the evaluation on $B\mathcal{C}_X(Y)$, which is $\mathbb{F}_p$-finite by [DW1, Proposition 5.2], and hence is null-homotopic by Miller's theorem on the Sullivan conjecture [Mi]. Thus f is null-homotopic, proving Part (ii), subject to showing that the composite

$$\mathrm{Map}(BY, B\mathcal{N}_X(Y))_{inc} \longrightarrow \mathrm{Map}(BY, BX)_\alpha \longrightarrow B\mathcal{N}_X(Y)$$

is homotopic to the evaluation map. To simplify notation let N denote $\mathcal{N}_X(Y)$. By Remark 2.6 we have a commutative diagram

$$\begin{array}{ccccc} B\mathcal{C}_N(Y) & \longrightarrow & B\mathcal{N}_N(Y) & \xrightarrow{\simeq} & BN \\ \downarrow & & \simeq\downarrow & & \downarrow \\ B\mathcal{C}_X(Y) & \longrightarrow & BN & \longrightarrow & BX, \end{array}$$

where each of the horizontal composites is the respective evaluation map. This completes the proof of the claim, and hence of Part (ii).

It remains to prove Part (iii). If X is p-compact toral then $W_X(Y)$ is a finite p-group by Corollary A.21 and hence so is $\pi_0(\mathcal{N}_X(Y))$. Furthermore, since $\mathcal{W}_X(Y)$ is the homotopy fibre of the map

$$\mathrm{Map}(BY, BY)_{\{\alpha\}} \xrightarrow{\alpha_\#} B\mathcal{C}_X(Y),$$

and since each homotopy class in $\{\alpha\}$ is the class of a homotopy equivalence, $\mathcal{W}_X(Y)$ is homotopy equivalent as a space to a disjoint union of spaces of the form $\mathcal{C}_X(Y)/\mathcal{Z}(Y)$, where $\mathcal{Z}(Y)$ denotes the center of Y (see Definition A.15 and the following remarks). Under the hypotheses of Part (iii), $\mathcal{C}_X(Y)$ is a p-compact toral group. Hence $\mathcal{W}_X(Y)$ is p-compact toral and fibration (3) above implies that $\mathcal{N}_X(Y)$ is p-compact toral. Also, by Corollary A.21

$$\chi(\mathcal{W}_X(Y)) = \chi((X/Y)^{hY}) \equiv \chi(X/Y) \equiv 0 \mod p,$$

and so $\mathcal{W}_X(Y)$ is not contractible. □

3. Centric p-compact toral subgroups

We now specialize to centric collections of p-compact toral subgroups of a p-compact group X, i.e., collections all of whose objects are centric in X. We start by analyzing the automorphism group of a centric subgroup as an object in the orbit category $\mathcal{O}(X)$.

Lemma 3.1. *Let X be a p-compact group.*

(i) *If $Y \leq_\beta X$ is a centric subgroup, then $\mathcal{W}_X(Y)$ is homotopically discrete.*
(ii) *If $P \leq_\gamma Y \leq_\beta X$ and $P \leq_\alpha X$ is a p-compact toral centric subgroup, where α denote $\beta \circ \gamma$, then P is centric in Y and $(X/Y)^{hP}$ is homotopically discrete.*

Proof. The first statement follows directly from the definition. Thus it remains to prove the second.

Assume $P \leq_\gamma Y \leq_\beta X$, and P is p-compact toral and centric in X. Then the composite

$$\mathrm{Map}(BP, BP)_{id} \xrightarrow{\gamma_\#} \mathrm{Map}(BP, BY)_\gamma \xrightarrow{\beta_\#} \mathrm{Map}(BP, BX)_\alpha$$

is an equivalence, and since P is p-compact toral, all spaces in this sequence are classifying spaces of p-compact groups [DW1, Proposition 5.1]. Hence $\gamma_\#$ is a monomorphism and $\beta_\#$ an epimorphism by the first two statements of Lemma A.2. Since β

is a monomorphism by assumption, and the evaluation maps are monomorphisms by [DW1, Propositions 5.2 and 6.1], commutativity of the following diagram of fibrations

$$\begin{array}{ccccc} (X/Y)^{hP} & \xrightarrow{k} & \mathrm{Map}(BP,BY)_{\{\alpha\}} & \xrightarrow{\beta_\#} & \mathrm{Map}(BP,BX)_\alpha \\ \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} \\ X/Y & \xrightarrow{j} & BY & \xrightarrow{\beta} & BX, \end{array}$$

and Lemma A.2 again imply that $\beta_\#$ is a monomorphism on each component. (The subscript $\{\alpha\}$ in the diagram denotes the set of components mapped to the component of α under $\beta_\#$.) Hence Lemma A.2(iv) applies to show that $\beta_\#$ is an isomorphism on each component and hence $\gamma_\#$ is an isomorphism as well, proving that (P,γ) is centric in Y. In particular, since $\beta_\#$ is an equivalence on each component, this also shows that $(X/Y)^{hP}$ is homotopically discrete. □

Corollary 3.2. *Let (P,α) and (Y,β) be subgroups of a p-compact group X. Assume P is p-compact toral and centric in X. Then*

$$\mathrm{Mor}_{\mathcal{O}(X)}((P,\alpha),(Y,\beta)) = \pi_0((X/Y)^{hP}).$$

In particular $\mathrm{Aut}_{\mathcal{O}(X)}(P,\alpha) = W_X(P)$ and if X is p-compact toral and P is a proper subgroup of X (i.e., α is not an equivalence) then $W_X(P)$ is a non-trivial finite p-group.

Proof. The morphism set $\mathrm{Mor}_{\mathcal{O}(X)}((P,\alpha),(Y,\beta))$ is given by the set $\{\alpha\}$ of homotopy classes of maps $BP \xrightarrow{\zeta} BY$, such that $\beta\circ\zeta \simeq \alpha$. The homotopy fibre of the map

$$\beta_\#\colon\ \mathrm{Map}(BP,BY)_{\{\alpha\}} \longrightarrow \mathrm{Map}(BP,BX)_\alpha$$

is $(X/Y)^{hP}$ by [DW1, Lemma 10.4], and since P is centric in X, $(X/Y)^{hP}$ is homotopically discrete by Lemma 3.1(ii). In particular $\beta_\#$ is a homotopy equivalence on each component, and so the inclusion of its homotopy fibre

$$(X/Y)^{hP} \longrightarrow \mathrm{Map}(BP,BY)_{\{\alpha\}}$$

induces a bijection on components. The identification of automorphism group of (P,α) in $\mathcal{O}(X)$ as the respective Weyl group follows from the interpretation of the Weyl space as a homotopy fixed point space (see Remark 2.4) and Lemma 3.1(i). Finally, if X is p-compact toral, then $\mathcal{W}_X(P) \simeq (X/P)^{hP}$ is a non-trivial p-compact group by Proposition 2.7(iii), and since it is homotopically discrete, $W_X(P)$ must be a finite non-trivial p-group. □

The following lemma shows that the class of centric p-compact toral subgroups of a p-compact group is closed under supergroups.

Lemma 3.3. *Let $P \le_\gamma Q \le_\beta X$ be a pair of p-compact toral subgroups of a p-compact group X, and let α denote the composite $\beta\circ\gamma$. If (P,α) is centric in X, then so is (Q,β).*

Proof. Suppose first that P is a normal subgroup of Q of finite index (see Definition A.13). Thus there is a fibration

$$BP \xrightarrow{\gamma} BQ \longrightarrow B\pi,$$

where π is a finite p-group. Let $\widetilde{BP}$ denote the pull-back of the universal cover of $B\pi$ along the projection $BQ \longrightarrow B\pi$. Then $\widetilde{BP}$ is homotopy equivalent to BP and admits a free action of π with orbit space BQ. Let $\{\alpha\}$ denote the set of components of all maps $BQ \longrightarrow BX$ extending α. Then

$$\mathrm{Map}(BQ, BX)_{\{\alpha\}} \simeq \mathrm{Map}(\widetilde{BP}_{h\pi}, BX)_{\{\alpha\}} \simeq (\mathrm{Map}(\widetilde{BP}, BX)_\alpha)^{h\pi}$$

Similarly, let $\{\gamma\}$ denote the set of components of all self maps of BQ extending γ. Then

$$\mathrm{Map}(BQ, BQ)_{\{\gamma\}} \simeq (\mathrm{Map}(\widetilde{BP}, BQ)_\gamma)^{h\pi},$$

and there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(BQ, BQ)_{\{\gamma\}} & \xrightarrow{\simeq} & (\mathrm{Map}(\widetilde{BP}, BQ)_\gamma)^{h\pi} \\ {\scriptstyle \beta_\#}\big\downarrow & & \big\downarrow{\scriptstyle (\beta_\#)^{h\pi}} \\ \mathrm{Map}(BQ, BX)_{\{\alpha\}} & \xrightarrow{\simeq} & (\mathrm{Map}(\widetilde{BP}, BX)_\alpha)^{h\pi}. \end{array}$$

Since $P \leq_\alpha X$ is centric, $P \leq_\gamma Q$ is centric by Lemma 3.1, and so $\beta_\#^{h\pi}$ is induced by an equivariant map which is a homotopy equivalence, implying that it is itself a homotopy equivalence. Commutativity implies that $\beta_\#$ is a homotopy equivalence as well. Restricting to the component $\mathrm{Map}(BQ, BQ)_{id}$, this shows that $Q \leq_\beta X$ is centric in this case.

Assume now that $P \leq_\gamma Q$ is arbitrary. By Lemma A.23 there is a sequence

$$P = P_0 \leq P_1 \leq P_2 \leq \cdots \leq P_k \leq P_{k+1} \leq \cdots Q_0 \leq Q_1 \leq \cdots \leq Q_{n-1} \leq Q_n = Q$$

such that $BQ_0 = \mathrm{hocolim}_i\, BP_i$, and each group in the sequence is a normal subgroup of finite index in the following one. By induction P_k is centric in X for all k.

By [DW1, Propositions 5.2 and 6.18] there exists some sufficiently large N, such that

$$\mathrm{Map}(BP_k, BQ_0)_{\iota_k} \simeq \mathrm{Map}(BQ_0, BQ_0)_{id}$$

and

$$\mathrm{Map}(BP_k, BX)_{\alpha_k} \simeq \mathrm{Map}(BQ_0, BX)_{\beta_0}$$

for all $k \geq N$, where ι_k, α_k and β_0 denote the appropriate restrictions. But $P_k \leq_{\beta_k} X$ is centric by the argument given for the case of a normal subgroup of finite index, so $P_k \leq_{\iota_k} Q_0$ is centric by Lemma 3.1, and

$$\beta_\#\colon\ \mathrm{Map}(BP_k, BQ_0)_{\iota_k} \longrightarrow \mathrm{Map}(BP_k, BX)_{\alpha_k}$$

is an equivalence. Naturality now implies that

$$\beta_\#\colon\ \mathrm{Map}(BQ_0, BQ_0)_{id} \longrightarrow \mathrm{Map}(BQ_0, BX)_{\beta_0}$$

is an equivalence, and so $Q_0 \leq_\beta X$ is centric.

The statement for a normal subgroup of finite index applied inductively to the subgroups in the sequence

$$Q_0 \leq Q_1 \leq \cdots \leq Q_{n-1} \leq Q_n = Q$$

shows that $Q \leq_\beta X$ is centric. □

If $P \leq Y \leq X$, and P is centric in Y, then it is not generally the case that P is centric in X. The following lemma singles out a family of subgroups $Y \leq X$, which are a very useful exception to the rule.

Lemma 3.4. *Let $P \leq_\alpha X$ be a p-compact toral subgroup and let $E \leq_\gamma X$ be an elementary abelian p-subgroup. If $P \leq_\beta \mathcal{C}_X(E)$, then P is centric in $\mathcal{C}_X(E)$ if and only if it is centric in X.*

Proof. Denote $\mathcal{C}_X(E)$ by C for short and assume $P \leq_\beta C$. If P is centric in X, then it is centric in C by Lemma 3.1. Conversely, assume P is centric in C. Since E is central in C (see Definition A.15), for any p-compact toral subgroup $Q \leq C$,

$$E \leq \mathcal{Z}(C) = \mathcal{C}_C(C) \leq \mathcal{C}_C(Q).$$

In particular $E \leq \mathcal{C}_C(P) \simeq \mathcal{Z}(P)$ and $E \leq \mathsf{E}(P)$, where $\mathsf{E}(P)$ is the maximal central elementary abelian subgroup of P (see Lemma A.17). Hence

$$P \leq \mathcal{C}_X(\mathsf{E}(P)) \leq \mathcal{C}_X(E) = C,$$

and since P is centric in C, it is centric in $\mathcal{C}_X(\mathsf{E}(P))$ by Lemma 3.1. Write $\widehat{E} \stackrel{\text{def}}{=} \mathsf{E}(P)$ and $\widehat{C} \stackrel{\text{def}}{=} \mathcal{C}_X(\widehat{E})$ for short, and let

$$\widehat{\beta}\colon BP \longrightarrow B\widehat{C} \qquad \text{and} \qquad \widehat{\gamma}\colon B\widehat{E} \longrightarrow BX$$

be the obvious maps factoring β and γ respectively. Then

$$B\mathcal{C}_{\widehat{C}}(P) \stackrel{\text{def}}{=} \mathrm{Map}(BP, B\widehat{C})_{\widehat{\beta}} \stackrel{\text{def}}{=} \mathrm{Map}(BP, \mathrm{Map}(B\widehat{E}, BX)_{\widehat{\gamma}})_{\widehat{\beta}} \simeq$$
$$\mathrm{Map}(B\widehat{E}, \mathrm{Map}(BP, BX)_{\alpha})_{ad^2\widehat{\beta}} \stackrel{\text{def}}{=} \mathrm{Map}(B\widehat{E}, B\mathcal{C}_X(P))_{ad^2\widehat{\beta}} \xrightarrow[ev]{\simeq} B\mathcal{C}_X(P).$$

The first equivalence is in fact a homeomorphism given by double adjunction, and the second follows because $E(P) \leq \mathcal{Z}(P)$, which is central in $B\mathcal{C}_X(P)$. This composite of equivalences is easy to identify with the map induced by the evaluation map

$$B\mathcal{C}_{\widehat{C}}(P) \stackrel{\text{def}}{=} \mathrm{Map}(BP, \mathrm{Map}(B\widehat{E}, BX)_{\widehat{\gamma}})_{\widehat{\beta}} \xrightarrow{ev_\#} \mathrm{Map}(BP, BX)_\alpha \stackrel{\text{def}}{=} B\mathcal{C}_X(P).$$

Since P is centric in $\widehat{C}$, it now follows at once that it is centric in X. □

4. Radical subgroups

In this section we describe some basic properties of the collection of radical p-compact toral subgroups of a p-compact group X. Recall that a subgroup (Y, α) of a p-compact group is said to be radical if it is quasicentric and if $\mathrm{Aut}_{\mathcal{O}(X)}(Y, \alpha)$ is finite and p-reduced.

We start by observing that radical p-compact toral subgroups of a p-compact group are also centric. (This stands in contrast to the case of compact Lie group, where radical subgroups are not in general p-centric.)

Lemma 4.1. *Any radical p-compact toral subgroup P of a p-compact group X is centric in X.*

Proof. Let $P \leq_\alpha X$ be a radical p-compact toral subgroup. Then there is a homotopy fibre sequence

$$\mathcal{W}_X(P) \longrightarrow \mathrm{Map}(BP, BP)_{\{\alpha\}} \xrightarrow{\alpha_\#} B\mathcal{C}_X(P).$$

Notice that the total space and fibre in this fibration are topological monoids, and the fibre inclusion is a multiplicative map. Therefore, the associated homotopy long exact sequence gives an exact sequence of groups

$$\pi_1(B\mathcal{Z}(P)) \xrightarrow{\pi_1(\alpha_\#)} \pi_1(B\mathcal{C}_X(P)) \xrightarrow{\partial} W_X(P) \longrightarrow \pi_0(\mathrm{Map}(BP, BP)_{\{\alpha\}}) \longrightarrow 1.$$

The group $\pi_1(B\mathcal{C}_X(P))$ is a finite p-group, since $\mathcal{C}_X(P)$ is a p-compact group, and $\mathrm{Im}(\partial)$ is a normal subgroup of $W_X(P)$ isomorphic to $\mathrm{Coker}(\pi_1(\alpha_\#))$. Since (P, α) is radical, $\mathrm{Im}(\partial)$ has to be trivial, and hence $\pi_1(\alpha_\#)$ is an epimorphism. Moreover, since (P, α) is quasicentric in X, $\mathcal{W}_X(P)$ is homotopically discrete and so $\alpha_\#$ induces an isomorphism on all homotopy groups. Hence $\alpha_\#$ restricted to the identity component of its source is a homotopy equivalence, implying that (P, α) is centric in X. □

Next, we show that any Sylow subgroup of a p-compact group is radical.

Lemma 4.2. *Any Sylow subgroup $S \leq X$ of a p-compact group X is radical there.*

Proof. Let $S \leq_{\iota_S} X$ be a Sylow subgroup, and let $T \leq S$ be a maximal torus in S, and thus in X. Notice that T is in particular a normal subgroup of S. By [DW1, Proposition 8.10], $\mathcal{W}_S(T)$ and $\mathcal{W}_X(T)$ are homotopically discrete and $\mathbb{F}_p$-finite, so $W_S(T)$ and $W_X(T)$ are finite groups.

Consider the fibration

$$S/T \longrightarrow X/T \longrightarrow X/S$$

induced by the inclusions of subgroups $T \leq S \leq X$. Taking homotopy fixed point sets under the action of T and using the identification of Weyl spaces as homotopy fixed point sets (see Remark 2.4), one obtains a fibration

$$\mathcal{W}_S(T) \longrightarrow \mathcal{W}_X(T) \longrightarrow (X/S)^{hT}. \tag{4}$$

Notice that the fibre inclusion in this fibration is a homomorphism of topological monoids. Also, since $T \lhd S$, T acts trivially on S/T, and so there are equivalences of topological monoids $S/T \simeq (S/T)^{hT} = \mathcal{W}_S(T) \simeq W_S(T)$.

By Corollary A.21, $(X/S)^{hT}$ is $\mathbb{F}_p$-finite, and since $\mathcal{W}_S(T)$ and $\mathcal{W}_X(T)$ are homotopically discrete, the long exact homotopy sequence for the fibration (4) implies that $(X/S)^{hT}$ is aspherical. Since spaces of type $K(\pi, 1)$, with π a finite non-trivial p-group, are never $\mathbb{F}_p$-finite, and since $(X/S)^{hT}$ is p-complete, the fundamental group of any connected component of $(X/S)^{hT}$ is either infinite or trivial. Since the connecting map $\pi_1(X/S)^{hT} \xrightarrow{\partial} \pi_0(\mathcal{W}_S(T))$ must be a monomorphism, and since $\pi_0(\mathcal{W}_S(T))$ is finite, the fundamental group of each connected component of $(X/S)^{hT}$ must be trivial. This shows that $(X/S)^{hT}$ is homotopically discrete, and that the homomorphism $W_S(T) \longrightarrow W_X(T)$ is a monomorphism. By Corollary A.21, $\chi(X/S)^{hT} \equiv \chi(X/S)$ mod p, and $\chi(X/S)$ is prime to p, since S is a Sylow subgroup. Hence the image of $W_S(T)$ in $W_X(T)$ is a subgroup of index prime to p, (i.e., a Sylow p-subgroup).

Putting this together, one has

$$\mathcal{W}_X(S) \overset{\text{def}}{=} (X/S)^{hS} \simeq ((X/S)^{hT})^{hW_S(T)} \simeq (W_X(T)/W_S(T))^{hW_S(T)}.$$

The first equivalence follows from [DW1, Propositions 6.8, 6.9 and Lemma 10.5], and the second from the fact that the projection to the set of components is equivariant with respect to the action of $W_S(T)$. By definition, homotopy fixed points and ordinary fixed points of a discrete group action on a discrete set coincide. Thus the right hand side can be identified with $(W_X(T)/W_S(T))^{W_S(T)}$, which is clearly a finite set of order prime to p. This shows that $|W_X(S)|$ has order prime to p and the claim follows. □

The following lemma is a one-sided analogue of 3.4 for radical subgroups. Notice that if $P \leq Y \leq X$ and P is radical in X, it is not necessarily radical in Y.

Lemma 4.3. *Let $P \leq_\alpha X$ be a p-compact toral subgroup and let $E \overset{\text{def}}{=} \mathsf{E}(P) \leq_\epsilon X$ be a maximal central elementary abelian p-subgroup of P. Then $P \leq \mathcal{C}_X(E)$ and if P is radical in X, then it is radical in $\mathcal{C}_X(E)$.*

Proof. The first statement is obvious (see Remark A.19). Write $K \overset{\text{def}}{=} \mathcal{C}_X(E)$ for short. If P is radical in X, then it is centric there by Lemma 4.1, and hence $\mathcal{W}_X(P)$ is homotopically discrete by Lemma 3.1. By the same lemma, since $P \leq K$, P is also centric in K, and $\mathcal{W}_K(P)$ as well as $(X/K)^{hP}$ are homotopically discrete. The Weyl group $W_K(P)$ is finite by Corollary A.21, and it remains to show that it is p-reduced. To show that, we construct a homomorphism

$$W_X(P) \longrightarrow W_X(K),$$

with kernel $W_K(P)$. Having done that, the claim follows from the assumption that $W_X(P)$ is p-reduced and Lemma 4.4 below.

Notice first that the obvious group homomorphism $W_K(P) \to W_X(P)$ is a monomorphism. To see this, consider the fibration sequence

$$K/P \longrightarrow X/P \longrightarrow X/K.$$

Applying homotopy fixed points of P, we get a fibration of homotopically discrete spaces:

$$\mathcal{W}_K(P) \longrightarrow \mathcal{W}_X(P) \longrightarrow (X/K)^{hP}.$$

The claim follows by taking sets of components. We thus may identify $W_K(P)$ with its image in $W_X(P)$.

Each element $W_X(P)$ is the homotopy class of some self equivalence f of (P, α). If $[f] \in W_X(P)$, and f is any representative, then one has an induced self equivalence $f^\#$ of $(\mathcal{C}_X(P), ev)$. Since $B\mathcal{C}_X(P) \simeq B\mathcal{Z}(P)$, $f^\#$ induces a self equivalence of (E, j), where $j \colon BE \longrightarrow BX$ is the inclusion $BE \longrightarrow BP$ followed by α. By abuse of notation we denote the last self equivalence by $f^\#$ as well. This map in turn induces a self equivalence $\widehat{f}$ of (K, ev), whose restriction to BP is homotopic to f. Notice that this immediately implies that if $\widehat{f} \simeq 1_{BK}$, then $[f]$ is an element of $W_K(P)$. Conversely, if $[f] \in W_K(P)$, then for any representative f, the induced self map $\widehat{f}$ of (K, ev) is homotopic to 1_{BK}. Naturality of the construction implies at once that the map defined by $[f] \mapsto [\widehat{f}]$ is a group homomorphism $W_X(P) \longrightarrow W_X(K)$, and by the observations above its kernel is $W_K(P)$. □

Lemma 4.4. *Let $H \lhd G$ be a normal subgroup of the finite group G. If G is p-reduced, then so is H.*

Proof. Let $Q \leq H$ be the intersection of all Sylow p-subgroups of H. Then, $Q \lhd H$ and, since $H \lhd G$, it follows that $Q \lhd G$. Moreover, H is p-reduced if and only if Q is the trivial group. Thus, if H is not p-reduced then G is also not p-reduced, which proves the statement. □

The following lemma can be thought of as a homotopy version of [JMO, Lemma 1.5] for radical p-compact toral subgroups.

Lemma 4.5. *Let (P, α) and (Q, β) be p-compact toral subgroups of X. Assume that (P, α) is radical and let $(P, \alpha) \xrightarrow{\gamma} (Q, \beta)$ be a morphism in $\mathcal{O}(X)$. Consider the maps*

$$W_X(P) \xrightarrow{\gamma_\#} W_X(P, Q) \xleftarrow{\gamma^\#} W_X(Q),$$

where $W_X(P, Q) \stackrel{\text{def}}{=} \pi_0((X/Q)^{hP})$. Then γ is an isomorphism of p-compact groups if and only if there exists a monomorphism of groups $W_X(P) \xrightarrow{\delta} W_X(Q)$ such that $\gamma^\#\delta \simeq \gamma_\#$.

Proof. If γ is an isomorphism, then $\delta \stackrel{\text{def}}{=} (\gamma^\#)^{-1} \circ \gamma_\#$ gives the required group homomorphism. Thus it remains to prove the converse.

Since P is radical in X, it is also centric there by Lemma 4.1, and thus Q is centric in X by Lemma 3.3. By Lemma 3.1, P is centric in Q, and the spaces $(X/P)^{hP}$, $(X/Q)^{hQ}$, $(X/Q)^{hP}$ and $(Q/P)^{hP}$ are all homotopically discrete.

There is a short exact sequence of sets

$$W_Q(P) \xrightarrow{\beta_\#} W_X(P) \xrightarrow{\gamma_\#} W_X(P,Q)$$

obtained by taking components on the fibration of homotopically discrete spaces

$$(Q/P)^{hP} \xrightarrow{\beta_\#} (X/P)^{hP} \xrightarrow{\gamma_\#} (X/Q)^{hP}.$$

In particular $\beta_\#$ is a monomorphism and $W_X(P,Q)$ is isomorphic as a $W_X(P)$-set to $W_X(P)/\beta_\#(W_Q(P))$.

Assume γ is not an isomorphism of p-compact groups, and let $K \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\delta) \lhd W_X(P)$. Then the relation $\gamma^\# \circ \delta = \gamma_\#$ implies that the inclusion of K into $W_X(P)$ factors through $W_Q(P)$, which is a non-trivial finite p-group by Corollary 3.2. Hence K is itself a finite p-group. Since K is a normal subgroup of $W_X(P)$, and since P is assumed radical, K must be trivial, and therefore δ is injective.

Next we claim that $\gamma^\#$ is injective. Having shown that, it follows that $\gamma_\#$ is injective as well. Since $\gamma_\# \circ \beta$ is trivial, and since ι is injective, $W_Q(P)$ is the trivial group, and by Corollary 3.2 again this implies that γ is an equivalence. Indeed, the map $\gamma^\#$ is induced by the inclusion of homotopy fixed point sets $(X/Q)^{hQ} \longrightarrow (X/Q)^{hP}$. Without loss of generality, γ may be replaced by an inclusion of p-compact toral groups. By Lemma A.23, it suffices to prove the claim under the assumption that P is a normal subgroup of Q of finite index. But in that case $(X/Q)^{hQ} \simeq ((X/Q)^{hP})^{h(Q/P)}$, and since $(X/Q)^{hP}$ is homotopically discrete, $(X/Q)^{hQ}$ is homotopy equivalent to the fixed point set of the Q/P action on $\pi_0((X/Q)^{hP})$. It follows that $\gamma^\#$ is a monomorphism, as claimed. □

5. The orbit category of radical subgroups

In this section we study the orbit category of p-compact toral radical subgroups of a p-compact group X. In particular we study the behavior of the orbit category of radical subgroups and the respective subgroup approximations under extension of a p-compact group by a p-compact toral group. We also show that the orbit category of radical subgroups of a p-compact group has a finite skeletal subcategory. This last observation is crucial in carrying out inductive procedures later on.

Definition 5.1. *Let X and Z be p-compact groups. An extension of Z by X is a fibration*

$$BX \longrightarrow BY \longrightarrow BZ,$$

where both the projection and the fibre inclusion are homomorphisms (i.e. pointed maps).

Notice that an extension of p-compact groups automatically gives rise to a p-compact group. To see this, notice that the total space in a fibration defining an extension of p-compact groups is automatically p-complete, and its loop space is the total space in a fibration with $\mathbb{F}_p$-finite base and fibre, and is thus itself $\mathbb{F}_p$-finite by inspection of the associated Serre spectral sequence. Our first observation in this section is that extending a p-compact group by a p-compact toral group leaves the associated orbit category of radical subgroups unchanged up to equivalence.

From this section onwards it will be useful to consider subcategories of the orbit category (and later of the fusion category as well) of a p-compact group, whose objects are a particular family of subgroups of a fixed Sylow subgroup.

Definition 5.2. *Let X be a p-compact group with a Sylow subgroup $S \leq_{\iota_S} X$, and fix a discrete approximation $\check{S}$ of S. Let $\mathcal{O}_S(X)$ and $\mathcal{F}_S(X)$ denote the full subcategories of the orbit and fusion categories of X whose objects are of the form (P, ι_P), where $BP = B\check{P}^\wedge_p$ for some subgroup $\check{P} \leq \check{S}$, and ι_P is the p-completion of the composite $\check{P} \leq \check{S} \leq S \leq_{\iota_S} X$.*

Lemma 5.3. *For any p-compact group X and a Sylow subgroup $S \leq X$, the categories $\mathcal{O}_S(X)$ and $\mathcal{F}_S(X)$ of Definition 5.2 are equivalent to the full subcategories of $\mathcal{O}(X)$ and $\mathcal{F}(X)$ whose objects are all p-compact toral subgroups $P \leq X$.*

Proof. This is immediate from the defining property of a Sylow subgroup, and Lemma A.22. (See also Proposition 1.7.) □

Thus, given a p-compact group X, we shall fix, whenever necessary, a Sylow subgroup $S \leq_{\iota_S} X$ and a discrete approximation $\check{S}$ for S, and work with collections of subgroups of X of the form (P, ι_P) where P corresponds to a subgroup $\check{P} \leq \check{S}$, and ι_P is induced by the natural inclusion followed by ι_S. When this cannot lead to ambiguity, such subgroups will be denoted by P, Q, etc. (i.e., as opposed to (P, ι_P), (Q, ι_Q) etc.), but the symbol ι_P will always be understood to be in the background.

We are now ready to state and prove our main claim about extensions: an extension of a p-compact group by a p-compact toral group keeps invariant, up to an equivalence, the orbit category of p-compact toral radical subgroups. More precisely we have the following.

Proposition 5.4. *Let*

$$BK \xrightarrow{\iota} BX \xrightarrow{\pi} BY$$

be an extension of p-compact groups where K is p-compact toral. Then there is an equivalence of categories

$$\Psi\colon \mathcal{O}^r(Y) \longrightarrow \mathcal{O}^r(X).$$

Proof. Let (S, ι_S) be a Sylow subgroup for X, and let $\check{S} \leq S$ be a discrete approximation. Let $\check{K} \leq K$ denote the kernel of $\pi|_{B\check{S}}$. Then $\check{K}$ is a discrete approximation for K, and $\check{R} \overset{\text{def}}{=} \check{S}/\check{K}$ is a discrete approximation for a Sylow subgroup (R, ι_R) for Y. In particular one has an extension of discrete groups

$$1 \longrightarrow \check{K} \xrightarrow{\check{\iota}} \check{S} \xrightarrow{\check{\pi}} \check{R} \longrightarrow 1.$$

Applying the classifying space functor followed by p-completion, one has an extension of p-compact toral groups, which maps to the extension of Y by K given in the lemma. Notice that there is an obvious 1–1 correspondence between subgroups of $\check{R}$ and subgroups of $\check{S}$ containing $\check{K}$.

By Lemma 5.3, to prove the proposition, it suffices to show that there is an isomorphism of categories $\Psi_S\colon \mathcal{O}^r_R(Y) \longrightarrow \mathcal{O}^r_S(X)$, where $\mathcal{O}^r_S(X)$ is the full subcategory of $\mathcal{O}_S(X)$ whose objects are radical in X, and similarly $\mathcal{O}^r_R(Y)$. The functor Ψ_S we use is in fact defined on $\mathcal{O}_S(X)$, but will only be shown to be an isomorphism of categories when restricted to $\mathcal{O}^r_S(X)$.

Definition of Ψ_S: For an object P of $\mathcal{O}_R(Y)$ corresponding to a subgroup $\check{P} \leq \check{R}$, define $\Psi_S(P)$ to be the p-compact toral subgroup P' of S given by taking the inverse image of $\check{P}$ in $\check{S}$ under $\check{\pi}$, and applying the classifying space functor followed by p-completion to it. Notice that the square

$$\begin{array}{ccc} BP' & \xrightarrow{\iota_{P'}} & BX \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ BP & \xrightarrow{\iota_P} & BY \end{array}$$

is a homotopy pullback. Unless ambiguity may arise, we will denote $\Psi_S(P)$, $\Psi_S(Q)$ by P', Q' etc., for short.

We next define Ψ_S on morphisms. A morphism $(P, \iota_P) \xrightarrow{[f]} (Q, \iota_Q)$ in $\mathcal{O}_R(Y)$ is the homotopy class of a map $f\colon BP \longrightarrow BQ$, such that $\iota_Q \circ f \simeq \iota_P$. Thus given such a morphism, one has an induced map, well defined up to homotopy

$$\Psi_S([f])\colon \Psi_S(P) = BP' \longrightarrow BQ' = \Psi_S(Q)$$

such that $[\iota_{Q'}] \circ \Psi_S([f]) = [\iota_{P'}]$. This completes the definition of Ψ_S as a functor on $\mathcal{O}_R(Y)$.

Ψ_S is bijective on morphism sets: Let $P, Q \in \mathcal{O}_R(Y)$ be any two objects, and let $P', Q' \leq X$ denote $\Psi_S(P)$ and $\Psi_S(Q)$ respectively, as before. Then one has the following sequence of homotopy equivalences

$$(X/Q')^{hP'} \simeq (Y/Q)^{hP'} \simeq ((Y/Q)^{hK})^{hP} \simeq (Y/Q)^{hP}. \tag{5}$$

The first equivalence holds since $X/Q' \simeq Y/Q$. The second follows from [DW1, Lemma 10.5]. For the last equivalence, notice that K acts trivially on Y/Q, implying that $(Y/Q)^{hK} = \mathrm{Map}(BK, Y/Q)$. Furthermore, Y/Q is p-complete and $\mathbb{F}_p$-finite, and so the evaluation map

$$\mathrm{Map}(BK, Y/Q) \xrightarrow[ev]{\simeq} Y/Q$$

is an equivalence, by the Sullivan conjecture for p-compact groups [DW2, Theorem 9.3]. Taking components now implies that Ψ_S is a bijection on morphism sets:

$$\mathrm{Mor}_{\mathcal{O}(X)}(P', Q') \stackrel{\text{def}}{=} \pi_0((X/Q')^{hP'}) \cong \pi_0((Y/Q)^{hP}) \stackrel{\text{def}}{=} \mathrm{Mor}_{\mathcal{O}(Y)}(P, Q).$$

Ψ_S is injective on objects: Since objects in $\mathcal{O}_R(Y)$ are in 1–1 correspondence via Ψ_S with objects of $\mathcal{O}_S(X)$ which contain K, it follows that Ψ_S, and hence its restriction $\mathcal{O}^r_R(Y)$, is injective on objects.

Restriction to $\mathcal{O}^r_R(Y)$: Set $P = Q$ in Equation (5) above. Taking components, one obtains a group isomorphism $W_X(P') \simeq W_Y(P)$. Hence, $P' \leq_{\iota_{P'}} X$ is radical if and only if $P \leq_{\iota_P} Y$ is radical. This shows that Ψ_S restricted to $\mathcal{O}^r_R(Y)$ takes values in $\mathcal{O}^r_S(X)$.

Ψ_S is surjective on objects: To show that Ψ_S is surjective on objects, it suffices to show that any subgroup $\check{P}' \leq \check{S}$ such that $(P', \iota_{P'})$ is radical in X must contain $\check{K}$, and thus is in the image of Ψ_S. This part of the proof is the only place where radicality is used.

Let $P' \leq S$ be such a subgroup. Let $\check{Q} \stackrel{\text{def}}{=} \langle \check{P}', \check{K} \rangle \leq \check{S}$, and let $\check{K}_{P'} \stackrel{\text{def}}{=} \check{K} \cap \check{P}'$. Then $\check{Q}$ and $\check{K}_{P'}$ are discrete approximations for the subgroups (Q, ι_Q) and $(K_{P'}, \iota_{K_{P'}})$ of X, and if we let $P \leq Y$ denote the subgroup whose discrete approximation is given by $\check{P}'/\check{K}_{P'}$, then by definition $P' = \Psi_S(P)$.

We proceed by showing that the inclusion $P' \leq_j Q$ is in fact an isomorphism. Since this inclusion is induced by the corresponding inclusion of discrete approximations, the claim follows, proving that Ψ_S is surjective on objects.

Using the identification of Weyl spaces with the appropriate homotopy orbit spaces (see Remark 2.4), and Lemma 4.5, it suffices to show that $(X/Q)^{hQ} \xrightarrow{j^\#} (X/Q)^{hP'}$ is an equivalence, and that the map $\delta\colon W_X(P') \longrightarrow W_X(Q)$, induced by the composite

$$\mathcal{W}_X(P') \simeq (X/P')^{hP'} \xrightarrow{j\#} (X/Q)^{hP'} \xrightarrow{(j\#)^{-1}} (X/Q)^{hQ} \simeq \mathcal{W}_X(Q),$$

is a group homomorphism.

By [DW1, Lemma 10.5],

$$(X/Q)^{hQ} \simeq ((X/Q)^{hK})^{hP} \qquad \text{and} \qquad (X/Q)^{hP'} \simeq ((X/Q)^{hK_{P'}})^{hP},$$

and so to prove that $j^\#$ is an equivalence, it is enough to prove that the restriction $r\colon BK_{P'} \longrightarrow BK$ induces a homotopy equivalence $r^\#\colon (X/Q)^{hK} \longrightarrow (X/Q)^{hK_{P'}}$. In fact we will show that both spaces are equivalent to X/Q.

By construction one has the following homotopy commutative diagram where the columns are fibrations.

$$\begin{array}{ccccc}
BK_{P'} & \longrightarrow & BK & = & BK \\
\downarrow & & \downarrow & & \downarrow \\
BP' & \longrightarrow & BQ & \longrightarrow & BX. \\
\downarrow & & \downarrow & & \downarrow \\
BP & = & BP & \longrightarrow & BY
\end{array} \tag{6}$$

In particular, it follows that $X/Q \simeq Y/P$, and that $(X/Q)^{hK} \simeq (Y/P)^{hK}$. Furthermore, one has the following commutative diagram, where the rows are fibrations.

$$\begin{array}{ccccc}
(Y/P)^{hK} & \longrightarrow & \mathrm{Map}(BK, BP)_{\{c\}} & \longrightarrow & \mathrm{Map}(BK, BY)_c \\
\downarrow & & {\scriptstyle ev}\downarrow & & {\scriptstyle ev}\downarrow \\
Y/P & \longrightarrow & BP & \longrightarrow & BY
\end{array}.$$

The space $\mathrm{Map}(BK, BP)_{\{c\}}$ is in fact connected. This follows at once from the Sullivan conjecture for p-compact groups [DW2, Theorem 9.3] and the fact that Y/P is $\mathbb{F}_p$-finite. The Sullivan conjecture also implies that the two right vertical arrows in this diagram are homotopy equivalences. Hence, so is the induced map on homotopy fibres. This shows that $(X/Q)^{hK} \simeq X/Q$. The same argument applies to show that $(X/Q)^{hK_{P'}} \simeq X/Q$.

Finally we must show that the map $\delta\colon W_X(P') \longrightarrow W_X(Q)$ defined above is a group homomorphism. Let $x, y \in W_X(P')$ be any two elements. Let $\widehat{x}$, $\widehat{y}$ denote a choice of self maps of BP', whose homotopy classes are x and y respectively. Let $\widehat{\delta(x)}$, $\widehat{\delta(y)}$ and $\widehat{\delta(xy)}$ denote a choice of self maps of BQ whose homotopy classes are $\delta(x)$, $\delta(y)$ and $\delta(xy)$ respectively. We must show that $\widehat{\delta(xy)} \simeq \widehat{\delta(x)} \circ \widehat{\delta(y)}$. By construction we have a homotopy commutative diagram

$$\begin{array}{ccccc} BP' & \xrightarrow{\widehat{y}} & BP' & \xrightarrow{\widehat{x}} & BP' \\ {\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow \\ BQ & \xrightarrow{\widehat{\delta(y)}} & BQ & \xrightarrow{\widehat{\delta(x)}} & BQ \end{array},$$

which remains homotopy commutative if the bottom composite is replaced by $\widehat{\delta(xy)}$. Since P' is radical in X, it is centric there by Lemma 4.1. Hence by Lemma A.4, j is an epimorphism in the categorical sense, i.e., $\widehat{\delta(x)} \circ \widehat{\delta(y)} \simeq \widehat{\delta(xy)}$. This completes the proof. □

Proposition 5.4 allows to compare subgroup ampleness of radical subgroups of a p-compact group Y with that of radical subgroups of any extension of Y be a p-compact toral group.

Proposition 5.5. *Let X be an extension of a p-compact group Y by a p-compact toral group K. Then, the collection of all radical subgroups of X is subgroup-ample if and only if the collection of all radical subgroups of Y is subgroup-ample.*

Proof. Let $S \leq X$ and $R \leq Y$ be Sylow subgroups, such that $K \lhd S$ and $S/K \cong R$, and let $BS \xrightarrow{\pi} BR$ denote the projection. It suffices to prove the claim for the collections of all subgroups of S and R which are radical in X and Y respectively.

By Lemma 4.1 and Proposition 1.5, there exists a subgroup approximation (Definition 1.2)

$$\bar{\phi}\colon \mathcal{O}^r_R(Y) \longrightarrow \mathsf{Top}.$$

Let

$$\phi\colon \mathcal{O}^r_R(Y) \longrightarrow \mathsf{Top}$$

be the functor which takes a subgroup $P \leq R$ to the pullback space in the diagram

$$\begin{array}{ccc} \phi(P) & \longrightarrow & \bar{\phi}(P) \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \bar{\eta}} \\ BX & \xrightarrow{\pi} & BY = 1_{BY}(P) \end{array}.$$

Here 1_{BY} is the constant functors on $\mathcal{O}^r_R(Y)$ with value BY, and $\bar{\eta}$ is the natural transformation associated to $\bar{\phi}$. Notice that since π is a fibration, ϕ is well defined, and comes equipped with an obvious natural transformation $\eta\colon \phi \longrightarrow 1_{BX}$. Furthermore, since ϕ is defined using the pullback construction, the conditions of Puppe's theorem [Dr, pp.179] are automatically satisfied, and the commutative square above gives rise to a commutative diagram of fibrations

$$\begin{array}{ccccc} BK & \longrightarrow & \operatorname{hocolim}_{\mathcal{O}^r_R(Y)} \phi & \longrightarrow & \operatorname{hocolim}_{\mathcal{O}^r_R(Y)} \bar{\phi} \\ = \downarrow & & |\eta| \downarrow & & |\bar{\eta}| \downarrow \\ BK & \longrightarrow & BX & \longrightarrow & BY. \end{array}$$

Since Y and X are p-compact groups, their fundamental groups are finite p-groups. Hence $|\bar{\eta}|$ (resp. $|\eta|$) is a mod-p equivalence if and only if its homotopy fibre is mod-p acyclic. Since the fibres of η and $\bar{\eta}$ are homotopy equivalent, it follows that $|\eta|$ is a mod-p equivalence if and only if $|\bar{\eta}|$ is a mod-p equivalence.

Finally, by Lemma 5.4 the categories $\mathcal{O}^r_S(X)$ and $\mathcal{O}^r_R(Y)$ are isomorphic, and the composite

$$\mathcal{O}^r_S(X) \xrightarrow{\Psi_S^{-1}} \mathcal{O}^r_R(Y) \xrightarrow{\phi} \mathsf{Top}$$

is obviously a subgroup approximation for X. The proposition follows at once. $\square$

We end this section by showing the for every p-compact group X the category $\mathcal{O}^r(X)$ is equivalent to a finite category.

Proposition 5.6. *For any p-compact group X, the orbit category $\mathcal{O}^r(X)$ has a finite skeletal subcategory.*

Proof. Since all radical subgroups in X are centric in X by Lemma 4.1, each morphism set $\mathrm{Mor}_{\mathcal{O}^r(X)}(P,Q)$ is given by the set of components of the respective homotopy orbit space $(X/Q)^{hP}$, which is homotopically discrete by Lemma 3.1, and finite by Corollary A.21. Hence, it suffices to show that $\mathcal{O}^r(X)$ has a finite number of isomorphism classes of objects.

If X is a p-compact toral group and $Q \leq X$ is a proper subgroup, then the Weyl group $W_X(Q)$ is always a nontrivial finite p-group by Corollary 3.2. Hence, the only radical subgroup of X is X itself. This proves the claim in this case.

Let X be an arbitrary p-compact group, which is not p-compact toral. Let Y be the centerfree quotient of X, which exists by Lemma A.16. Since $\mathcal{O}^r(X) \simeq \mathcal{O}^r(Y)$ by Proposition 5.4, we are reduced to showing the statement for centerfree p-compact groups.

We proceed by downward induction on the order of X (see Definition A.7). Thus assume the claim for all p-compact groups of order strictly less than that of X. By [DW2, Proposition 8.3], there exist only finitely many conjugacy classes of elementary abelian subgroups of X. If $P \leq X$ is a radical subgroup, then $P \leq \mathcal{C}_X(\mathsf{E}(P))$, where $\mathsf{E}(P)$ is the maximal central elementary abelian subgroup of P, and P is radical there by Lemma 4.3. Since X is centerfree $|\mathcal{C}_X(\mathsf{E}(P))| < |X|$, and by induction hypothesis $\mathcal{C}_X(\mathsf{E}(P))$ has only finitely many conjugacy classes of radical p-compact toral subgroups. Hence the conjugacy class of P can only be one of a finite list of conjugacy classes of p-compact toral subgroups of X, each of which is radical in $\mathcal{C}_X(E)$ for some elementary abelian p-subgroup $E \leq X$. This completes the proof. $\square$

6. Subgroup ampleness of centric and radical collections

Let $\mathcal{C}$ be a family of centric subgroups of a p-compact group X, which contains at least one representative from the conjugacy class of each radical subgroup of X. The objective of this section is to show that the collection $\mathcal{C}$ is subgroup-ample if and only if the collection of all radical subgroups of X is subgroup ample.

It is convenient to restrict attention to collections which are contained in a fixed Sylow subgroup. Thus, let X be a p-compact group with a Sylow subgroup $S \leq_{\iota_S} X$, and let $\mathcal{C}$ be a collection of subgroups of S all of which are centric in X. We assume also that $\mathcal{C}$ contains all subgroups of S, which are radical in X. From this point onwards, our discussion becomes quite categorical in nature. The required material is collected in the Appendix for the convenience of the reader.

Let

$$\tau_{\mathcal{C}}\colon \mathcal{O}^r_S(X) \longrightarrow \mathcal{O}_{\mathcal{C}}(X)$$

denote the inclusion functor. For each object $P \in \mathcal{C}$, we denote by $P{\downarrow}\tau_{\mathcal{C}}$ the undercategory for of P with respect to $\tau_{\mathcal{C}}$. Objects in the undercategory are pairs $(Q, [u])$, where $Q \leq_{\iota_Q} X$ is radical, and $P \xrightarrow{[u]} Q$ is a morphism in $\mathcal{O}_{\mathcal{C}}(X)$. A morphism

$$[g]\colon (Q, [u]) \longrightarrow (Q', [u'])$$

is determined by a morphism $Q \xrightarrow{[g]} Q'$ in $\mathcal{O}^r_S(X)$, such that $[g] \circ [u] = [u']$. The functor $\tau_{\mathcal{C}}$ is said to be right cofinal if $|P{\downarrow}\tau_{\mathcal{C}}|$ is contractible for each $P \in \mathcal{C}$. The claim that this is indeed the case (Proposition 6.2) is the key ingredient in the proof of the main statement of the section, which is a slightly more general form of the equivalence of statements (a) and (b) in Theorem B.

Proposition 6.1. *Let X be a p-compact group with a Sylow subgroup $S \leq_{\iota_S} X$, and let $\mathcal{C}$ be a collection of subgroups of S, all of which are centric in X, and such that $\mathcal{C}$ contains all subgroups of S, which are radical in X. Then the collection of all radical subgroups in X is subgroup ample if and only if $\mathcal{C}$ is subgroup ample.*

Proof. By Proposition 6.2 the inclusion functor $\tau_{\mathcal{C}}\colon \mathcal{O}^r_S(X) \longrightarrow \mathcal{O}_{\mathcal{C}}(X)$ is right cofinal. If

$$\phi\colon \mathcal{O}_{\mathcal{C}}(X) \longrightarrow \mathsf{Top}$$

is a subgroup approximation, then its restriction $\phi \circ \tau_{\mathcal{C}}$ to $\mathcal{O}^r_S(X)$ is a subgroup approximation, and one has a commutative square

$$\begin{array}{ccc}
\operatorname*{hocolim}_{\mathcal{O}^r_S(X)} \phi \circ \tau_{\mathcal{C}} & \longrightarrow & \operatorname*{hocolim}_{\mathcal{O}_{\mathcal{C}}(X)} \phi \\
\downarrow & & \downarrow \\
BX & = & BX,
\end{array}$$

which implies the claim by Proposition 1.5 and Remark 1.6. □

It remains to prove:

Proposition 6.2. *For any p-compact group X, the inclusion functor*

$$\mathcal{O}^r_S(X) \xrightarrow{\tau} \mathcal{O}^c_S(X)$$

is right cofinal.

Notice that this statement suffices for the purpose of proving Proposition 6.1, since if $\mathcal{C}$ is a collection of subgroups of S which are centric in X, and $\mathcal{C}$ contains all subgroups of S which are radical in X, then the inclusion functor $\tau_{\mathcal{C}}$ is certainly right cofinal if the inclusion into the orbit category $\mathcal{O}^c_S(X)$ is right cofinal. The proof of Proposition 6.2 will occupy the rest of the section.

The following technical lemma is our main tool in an inductive proof of Proposition 6.2.

Lemma 6.3. *Let X be a p-compact group with a Sylow subgroup $S \leq_{\iota_S} X$. Let $P \leq S$ be a subgroup which is centric in X, and let*

$$P = P_0 \leq P_1 \leq P_2 \leq \cdots \leq P_j \leq P_{j+1} \leq \cdots Q = \operatorname*{colim}_j P_j \leq S$$

is a sequence of subgroups, such that for each $j \geq 0$, P_j is a normal subgroup of finite index in P_{j+1}. Then there exists a positive integer j_0 such that for all $j \geq j_0$ the functor

$$Q{\downarrow}\tau \longrightarrow P_j{\downarrow}\tau,$$

induced by the inclusion $P_j \leq Q$, is an equivalence of categories. Here, as before, $\tau\colon \mathcal{O}^r_S(X) \longrightarrow \mathcal{O}^c_S(X)$ denotes the inclusion.

Proof. Let $R \leq S$ be a subgroup, which is radical in X. Then one has a sequence of maps between the homotopy fixed point spaces

$$(X/R)^{hQ} \longrightarrow \ldots \longrightarrow (X/R)^{hP_{j+1}} \longrightarrow (X/R)^{hP_j} \longrightarrow \ldots . \tag{7}$$

Since R is radical, it is also centric by Lemma 4.1, and so all homotopy fixed point sets in the sequence are either empty or homotopically discrete by Lemma 3.1(ii). Since $P_j \lhd P_{j+1}$ for all j, one has $(X/R)^{hP_{j+1}} = ((X/R)^{hP_j})^{h(P_{j+1}/P_j)}$ by [DW1, Propositions 6.8, 6.9 and Lemma 10.5]. Furthermore, since $(X/R)^{hP_j}$ is homotopically discrete, and since the projection from any G-space to the G-set of its connected components is G-equivariant,

$$((X/R)^{hP_j})^{h(P_{j+1}/P_j)} \simeq (\pi_0((X/R)^{hP_j}))^{h(P_{j+1}/P_j)} = (\pi_0((X/R)^{hP_j}))^{P_{j+1}/P_j},$$

where the equality follows from the fact that for discrete G-sets homotopy fixed points and fixed points coincide by definition. This shows that all maps in the sequence (7) above induce monomorphisms on sets of path components.

Since $\pi_0((X/R)^{hP})$ is finite by Corollary A.21, the induced sequence on sets of path components has to stabilize above some sufficiently large $j(R)$, depending only on the isomorphism class of (R, ι_R) in $\mathcal{O}^r_S(X)$. By Proposition 5.6, there are only finitely many isomorphism classes of objects in this category. Hence one can define j_0' to be the maximum of all $j(R)$, where R runs over a set of representatives of isomorphism classes of objects in $\mathcal{O}^r_S(X)$. Thus, for all $j \geq j_0'$ and all $R \in \mathcal{O}^r_S(X)$, one has equivalences

$$(X/R)^{hP_{j_0}} \simeq (X/R)^{hP_j} \simeq (X/R)^{hQ}.$$

By definition of the undercategory, for all $j \geq 0$

$$Obj((P_j{\downarrow}\tau) = \bigcup_{R\in\mathcal{O}^r_S(X)} \mathrm{Mor}_{\mathcal{O}^c_S(X)}(P_j, R) = \bigcup_{R\in\mathcal{O}^r_S(X)} \pi_0((X/R)^{hP_j}).$$

Thus the same argument as above shows that the sequence of functors

$$\cdots \longrightarrow P_{j+1}{\downarrow}\tau \longrightarrow P_j{\downarrow}\tau \longrightarrow \cdots$$

stabilizes on objects for all $j > j_0'$.

It remains to show that the sequence stabilizes on morphism sets. The morphism set in each of the categories $P_j{\downarrow}\tau$ and in $Q{\downarrow}\tau$, between objects $(R, [u])$ and $(R', [u'])$, is a subset of $\mathrm{Mor}_{\mathcal{O}^r_S(X)}(R, R')$, which is finite for all $R, R' \in \mathcal{O}^r_S(X)$ by Proposition 5.6, and its cardinality depends only on the isomorphism classes of R and R' in $\mathcal{O}^r_S(X)$. By Proposition 5.6 again, there are only finitely many isomorphism classes of objects in $\mathcal{O}^r_S(X)$, and so the sequence of functors above must stabilize for all $j > j_0''$ for some sufficiently large j_0''. Let $j_0 = \max\{j_0', j_0''\}$. Then for all $j > j_0$ the functor $Q{\downarrow}\tau \longrightarrow P_j{\downarrow}\tau$ is an equivalence of categories, as claimed. □

Our next aim is to identify the nerve of the undercategories $P{\downarrow}\tau$. To do this, it is more convenient to use the inclusion functor $\tau\colon \mathcal{O}^r(X) \longrightarrow \mathcal{O}^c(X)$, rather than the respective categories where all objects are subgroups of a fixed Sylow subgroup.

Let X be a p-compact group, and let $P \leq_\alpha X$ be a centric p-compact toral subgroup. Let $N = \mathcal{N}_X(P, \alpha)$ be the normalizer space and $\eta_\alpha\colon BN \longrightarrow BX$ be the extension of α constructed in Lemma 2.5. Let $W = W_X(P, \alpha) \overset{\text{def}}{=} \pi_0(\mathcal{W}_X(P, \alpha))$ denote the Weyl group. Then there is a diagram

$$\begin{array}{ccccc} BP & \xrightarrow{inc} & BN & \longrightarrow & BW \\ & {\scriptstyle\alpha}\searrow & & \swarrow{\scriptstyle\eta_\alpha} & \\ & & BX & & \end{array},$$

where the row is a fibration, up to homotopy, since $\mathcal{W}_X(P,\alpha)$ is homotopically discrete. For every p-subgroup $\pi \leq W$ define a p-compact toral subgroup (P_π, α_π) of X, where P_π is the p-compact toral group whose classifying space is given as the pull-back space of the system

$$BN \longrightarrow BW \xleftarrow{inc} B\pi,$$

and where α_π is the composition of η_α with the obvious map $BP_\pi \longrightarrow BN$. By Lemma 2.7(ii), (P_π, α_π) is indeed a subgroup of X, and it is centric in X by Lemma 3.3 since it contains P as a subgroup. Notice that the map $BP \xrightarrow{\gamma_\pi} BP_\pi$ is a morphism in $\mathcal{O}^c(X)$. If $\pi \leq \pi' \leq W$ are p-subgroups, then one has an induced map $\delta_{\pi,\pi'} \colon BP_\pi \longrightarrow BP_{\pi'}$ such that $\delta_{\pi,\pi'} \circ \alpha_\pi \simeq \alpha_{\pi'}$. In other words $\delta_{\pi,\pi'}$ represents a morphism in $\mathcal{O}^c(X)$. Let $\mathcal{S}_p(W)$ denote the poset of all nontrivial p-subgroups of W (regarded as a category). Then, by naturality of the construction above, for every p-compact toral subgroup (P,α) which is centric in X one gets a functor $\delta = \delta_{(P,\alpha)} \colon \mathcal{S}_p(W) \longrightarrow \mathcal{O}^c(X)$, where $\delta(\pi) = (P_\pi, \alpha_\pi)$, and if we let $\iota_{\pi,\pi'}$ denote the inclusion $\pi \leq \pi'$, then $\delta(\iota_{\pi,\pi'}) = [\delta_{\pi,\pi'}]$.

Next, consider the functor $\tau^{op}{\downarrow}- \colon \mathcal{O}^c(X)^{op} \longrightarrow \mathcal{C}at$, where $\mathcal{C}at$ denotes the category of small categories, given by taking an object (Q,β) to the category $\tau^{op}{\downarrow}(Q,\beta)$. Let

$$\Delta = \Delta_{(P,\alpha)} \colon \mathcal{S}_p(W)^{op} \longrightarrow \mathcal{C}at,$$

denote the composite functor $\tau^{op}{\downarrow}\delta^{op}$. Thus, for each $\pi \leq W$,

$$\Delta(\pi) \stackrel{\text{def}}{=} \tau^{op}{\downarrow}(P_\pi, \alpha_\pi),$$

and if $\pi \leq \pi'$ then

$$\Delta(\iota^{op}_{\pi,\pi'}) \stackrel{\text{def}}{=} \delta(\iota_{\pi,\pi'})^* \colon \tau^{op}{\downarrow}(P_{\pi'}, \alpha_{\pi'}) \longrightarrow \tau^{op}{\downarrow}(P_\pi, \alpha_\pi).$$

We are now ready to identify the nerve of the undercategories $(P,\alpha){\downarrow}\tau$ for arbitrary centric p-compact toral subgroups of X. Clearly, it suffices to consider the case where (P,α) is non-radical in X, since if it is radical, then the category has an initial object and hence its nerve is contractible.

Lemma 6.4. *If $P \leq_\alpha X$ is a centric non-radical p-compact toral subgroup, then*

$$|(P,\alpha){\downarrow}\tau| \simeq \operatorname*{hocolim}_{\mathcal{S}_p(W)^{op}} |\Delta| \ .$$

Proof. We will show in fact that the right hand side is homotopy equivalent to the nerve of the opposite category $((P,\alpha){\downarrow}\tau)^{op} = \tau^{op}{\downarrow}(P,\alpha)$. This suffices since any small category and its opposite have canonically isomorphic nerves. Objects in $\tau^{op}{\downarrow}(P,\alpha)$ are triples $(Q,\beta,[h]^{op})$ where $(Q,\beta) \in \mathcal{O}^r(X)^{op}$, and $[h]^{op} \colon (Q,\beta) \longrightarrow (P,\alpha)$ is a morphism in $\mathcal{O}^c(X)^{op}$, i.e., $[h]$ is a homotopy class of a map $h \colon BP \longrightarrow BQ$, such that $\alpha \simeq \beta \circ h$. A morphism $(Q,\beta,[h]^{op}) \xrightarrow{[f]^{op}} (Q',\beta',[h']^{op})$ is determined by a morphism $(Q',\beta') \xrightarrow{[f]} (Q,\beta)$ in $\mathcal{O}^r(X)$ such that $[f] \circ [h'] = [h]$.

On the other hand, consider the Grothendieck category $Gr(\mathcal{S}_p(W)^{op}, \Delta)$, whose geometric realization gives the homotopy colimit of Δ over $\mathcal{S}_p(W)^{op}$ (see Appendix A). Its objects are quadruples $(\pi, (Q,\beta,[h]^{op}))$ where $\pi \leq W$ is a p-subgroup, and $(Q,\beta,[h]^{op})$

is an object of $\Delta(\pi) = \tau^{op}\downarrow(P_\pi, \alpha_\pi)$. More specifically, (Q, β) is an object of $\mathcal{O}^r(X)^{op}$, and $[h]^{op}\colon (Q,\beta) \longrightarrow (P_\pi, \alpha_\pi)$ is a morphism in $\mathcal{O}^c(X)^{op}$. A morphism

$$(\pi, (Q, \beta, [h]^{op})) \longrightarrow (\pi', (Q', \beta', [h']^{op}))$$

is a pair $(\iota^{op}_{\pi',\pi}, [f]^{op})$, where $\iota_{\pi',\pi}$ is the inclusion $\pi' \leq \pi$ (if $\pi' \not\leq \pi$ then there is no morphism between these objects), and

$$[f]^{op}\colon \Delta(\iota^{op}_{\pi',\pi})(Q, \beta, [h]^{op}) \longrightarrow (Q', \beta', [h]^{op})$$

is a morphism in $\Delta(\pi') = \tau^{op}\downarrow(P_{\pi'}, \alpha_{\pi'})$, i.e., $[f]\colon (Q', \beta') \longrightarrow (Q, \beta)$ is a morphism in $\mathcal{O}^r(X)$, such that $[f] \circ [h'] = [h] \circ \delta(\iota_{\pi',\pi})$.

To prove the Lemma, we will produce functors

$$Gr(\mathcal{S}_p(W)^{op}, \Delta) \underset{R}{\overset{L}{\rightleftarrows}} \tau^{op}\downarrow(P, \alpha),$$

such that $L \circ R = Id$, and a natural transformation from $R \circ L$ to the identity functor.

Definition of L. Define $L\colon Gr(\mathcal{S}_p(W)^{op}, \Delta) \longrightarrow (P, \alpha)\downarrow\tau$ by

$$L(\pi, (Q, \beta, [h]^{op})) = (Q, \beta, [h_\pi]^{op}),$$

where $[h_\pi] = [h] \circ [\gamma_\pi]$, and $BP \xrightarrow{\gamma_\pi} BP_\pi$ is the obvious map. A morphism

$$(\iota^{op}_{\pi',\pi}, [f]^{op})\colon (\pi, (Q, \beta, [h]^{op})) \longrightarrow (\pi', (Q', \beta', [h']^{op}))$$

is sent by L to $[f]^{op}\colon (Q, \beta, [h_\pi]^{op}) \longrightarrow (Q', \beta', [h'_{\pi'}]^{op})$. An easy diagram chasing, using the definitions, shows that L is well defined, i.e., that $f \circ h'_{\pi'} \simeq h_\pi$.

Definition of R. Define, $R\colon \tau^{op}\downarrow(P, \alpha) \longrightarrow Gr(\mathcal{S}_p(W)^{op}, \Delta)$ as follows. Observe first that if $(Q, \beta, [h]^{op})$ be an object in $\tau^{op}\downarrow(P, \alpha)$ then there is a homotopy commutative diagram of fibrations

$$\begin{array}{ccccc}
\mathcal{W}_Q(P, h) & \longrightarrow & \mathrm{Map}(BP, BP)_{\{h\}} & \longrightarrow & \mathrm{Map}(BP, BQ)_h \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \beta_\#} \\
\mathcal{W}_X(P, \alpha) & \longrightarrow & \mathrm{Map}(BP, BP)_{\{\alpha\}} & \longrightarrow & \mathrm{Map}(BP, BX)_\alpha,
\end{array}$$

where $BP \xrightarrow{h} BQ$ is a choice of a representative for $[h]$. Taking components on Weyl spaces we get a subgroup

$$W_Q \overset{\text{def}}{=} \mathrm{Im}(W_Q(P, h) \overset{\beta_\#}{\rightarrowtail} W_X(P, \alpha)) \leq W.$$

Furthermore, W_Q is a p-subgroup by Proposition 2.7(i), and by construction there is an equivalence of p-compact toral groups $P_{W_Q} \xrightarrow{\simeq} \mathcal{N}_Q(P, h)$. Define

$$R(Q, \beta, [h]^{op}) = (W_Q, (Q, \beta, [\rho_h]^{op})),$$

where ρ_h denotes the composite $BP_{W_Q} \xrightarrow{\simeq} B\mathcal{N}_Q(P, h) \xrightarrow{\eta_h} BQ$, and η_h is the extension of h constructed in Lemma 2.5. This defines R on objects. Notice that R cannot be defined at all if (P, α) is radical in X, since in that case $(P, \alpha, [Id_P]^{op})$ is an object in the over category, and the corresponding W_P, as defined above, is the trivial subgroup, which is not an object of $\mathcal{S}_p(W)$.

Next, we define R on morphisms. Let $(Q, \beta, [h]^{op}) \xrightarrow{[f]^{op}} (Q', \beta', [h']^{op})$ be a morphism in $\tau^{op}\downarrow(P, \alpha)$. Thus $[f]$ is a homotopy class of a map $BQ' \xrightarrow{f} BQ$ such that $[\beta'] = [\beta] \circ [f]$ and $[f] \circ [h'] = [h]$. Fix a choice of a representing map h for each object

$(Q, \beta, [h]^{op})$ of $\tau^{op}{\downarrow}(P, \alpha)$. Then any choice of a representative f for $[f]$ gives rise to a homotopy commutative triangle

$$\begin{array}{ccc} \mathrm{Map}(BP, BQ')_{h'} & \xrightarrow{\quad f_\# \quad} & \mathrm{Map}(BP, BQ)_h \\ & {\scriptstyle \beta'_\#}\searrow \quad \swarrow{\scriptstyle \beta_\#} & \\ & \mathrm{Map}(BP, BX)_\alpha, & \end{array}$$

which in turn induces a commutative triangle of groups and monomorphisms

$$\begin{array}{ccc} W_{Q'}(P, h') & \xrightarrow{\ f_\# \ } & W_Q(P, h) \\ & {\scriptstyle \beta'_\#}\searrow \quad \swarrow{\scriptstyle \beta_\#} & \\ & W & \end{array}$$

Hence

$$W_{Q'} \overset{\text{def}}{=} \beta'_\#(W_{Q'}(P, h')) \leq \beta_\#(W_Q(P, h)) \overset{\text{def}}{=} W_Q,$$

and we let $W_{Q'} \xrightarrow{\omega_{Q',Q}} W_Q$ denote the inclusion. Furthermore, it is easy to check that f induces a homomorphism of p-compact groups

$$f_\# \colon \mathcal{N}_{Q'}(P, h') \longrightarrow \mathcal{N}_Q(P, h),$$

which is compatible with the projections to the respective Weyl groups, and so $f_\#$ coincides up to homotopy with $\delta(\omega_{Q',Q})$. It is also immediate that $f \circ \eta_{h'} \simeq \eta_h \circ f_\#$. Thus for a morphism $[f]^{op}$ in $\tau^{op}{\downarrow}(P, \alpha)$ as above, define

$$R([f]^{op}) \overset{\text{def}}{=} (\omega^{op}_{Q',Q}, [f]^{op}) \colon (W_Q, (Q, \beta, [\eta_h]^{op})) \longrightarrow (W_{Q'}, (Q', \beta', [\eta_{h'}]^{op})).$$

The discussion above implies that R is well defined.

The composites $L \circ R$ and $R \circ L$. We first show that $L \circ R = Id$. By definition for any object $(Q, \beta, [h]^{op})$ in $\tau^{op}{\downarrow}(P, \alpha)$

$$L \circ R(Q, \beta, [h]^{op}) = L(W_Q, (Q, \beta, [\rho_h]^{op})) = (Q, \beta, [\eta_h \circ \gamma_Q]^{op}),$$

where $BP \xrightarrow{\gamma_Q} B\mathcal{N}_Q(P, h)$ is the obvious map. Hence $[\eta_h \circ \gamma_Q] = [h]$ by Lemma 2.5, and $L \circ R$ is the identity functor on objects. If $(Q, \beta, [h]^{op}) \xrightarrow{[f]^{op}} (Q', \beta', [h']^{op})$ is a morphism in $\tau^{op}{\downarrow}(P, \alpha)$, then $L \circ R([f]^{op}) = [f]^{op}$, and so $L \circ R$ is in fact the identity functor on $\tau^{op}{\downarrow}(P, \alpha)$.

Next, we construct a natural transformation ζ from $R \circ L$ to the identity functor on the Grothendieck category. By definition

$$(R \circ L)(\pi, (Q, \beta, [h]^{op})) = R(Q, \beta, [h \circ \gamma_\pi]^{op}) = (W_Q, (Q, \beta, [\rho_{h\gamma_\pi}]^{op})),$$

where $\rho_{h\gamma_\pi}$ denotes the composite

$$BP_{W_Q} \xrightarrow{\ \simeq\ } B\mathcal{N}_Q(P, h \circ \gamma_\pi) \xrightarrow{\eta_{h\gamma_\pi}} BQ.$$

Notice that (P_π, h) is a subgroup of Q, and P_π contains P as a normal subgroup (with quotient π). Thus h induces a map

$$BP_\pi \simeq B\mathcal{N}_{P_\pi}(P, \gamma_\pi) \xrightarrow{h_\#} B\mathcal{N}_Q(P, h \circ \gamma_\pi) \simeq BP_{W_Q},$$

which implies that π is a subgroup of $W_Q \stackrel{\text{def}}{=} W_Q(P, h\circ\gamma_\pi)$, and if we let ι_{π,W_Q} denote the inclusion $\pi \le W_Q$, then $\delta(\iota_{\pi,W_Q}) \simeq h_\#$. Let ζ be the transformation taking the object $(\pi,(Q,\beta,[h]^{op}))$ to the morphism

$$(\iota^{op}_{\pi,W_Q},[1_{BQ}]^{op})\colon (W_Q,(Q,\beta,[\eta_{h\gamma_\pi}]^{op})) \longrightarrow (\pi,(Q,\beta,[h]^{op})).$$

Since the composite

$$BP_\pi \simeq B\mathcal{N}_{P_\pi}(P,\gamma_\pi) \xrightarrow{h_\#} B\mathcal{N}_Q(P,h\circ\gamma_\pi) \xrightarrow{\eta_{h\gamma_\pi}} BQ$$

is homotopic to h, ζ is well defined for each object $(\pi,(Q,\beta,[h]^{op}))$, and it remains to show naturality.

Let

$$(\iota^{op}_{\pi',\pi},[f]^{op})\colon (\pi,(Q,\beta,[h]^{op})) \longrightarrow (\pi',(Q',\beta',[h']^{op}))$$

be a morphism in the Grothendieck category. By definition

$$R\circ L(\iota^{op}_{\pi',\pi},[f]^{op}) = (\iota^{op}_{Q',Q},[f]^{op}).$$

Thus naturality of ζ amounts to showing that the diagram

$$\begin{array}{ccc}
(W_Q,(Q,\beta,[\eta_{h\gamma_\pi}]^{op})) & \xrightarrow{(\iota^{op}_{\pi,Q},[1_{BQ}]^{op})} & (\pi,(Q,\beta,[h]^{op})) \\
\downarrow{\scriptstyle (\iota^{op}_{Q',Q},[f]^{op})} & & \downarrow{\scriptstyle (\iota^{op}_{\pi',\pi},[f]^{op})} \\
(W_{Q'},(Q',\beta',[\eta_{h'\gamma_{\pi'}}]^{op})) & \xrightarrow[(\iota^{op}_{\pi',Q'},[1_{BQ'}]^{op})]{} & (\pi',(Q',\beta',[h']^{op}))
\end{array}$$

commutes. By definition of composition in the Grothendieck category

$$(\iota^{op}_{\pi',\pi},[f]^{op})\circ(\iota^{op}_{\pi,Q},[1_{BQ}]^{op}) \stackrel{\text{def}}{=} (\iota^{op}_{\pi',\pi}\circ\iota^{op}_{\pi,Q},[f]\circ\Delta(\iota^{op}_{\pi',\pi})([1_{BQ}]^{op})) = (\iota^{op}_{\pi',Q},[f]^{op}),$$

while

$$(\iota^{op}_{\pi',Q'},[1_{BQ'}]^{op})\circ(\iota^{op}_{Q',Q},[f]^{op}) \stackrel{\text{def}}{=} (\iota^{op}_{\pi',Q'}\circ\iota^{op}_{Q',Q},[1_{BQ'}]^{op}\circ\Delta(\iota^{op}_{\pi',Q'})([f]^{op})) = (\iota^{op}_{\pi',Q},[f]^{op}).$$

This shows that the diagram commutes, and hence completes the proof. □

We are now ready to conclude the section with a proof of Proposition 6.2, and thus complete the proof of Proposition 6.1.

Proof of Proposition 6.2. We have to show that, for all centric p-compact toral subgroups $P \le_\alpha X$ the undercategories $(P,\alpha)\downarrow\tau$ are contractible. We do this by a descending induction on the order of objects in $\mathcal{O}^c(X)$.

For every p-compact toral radical subgroup (P,α) of X, the undercategory $(P,\alpha)\downarrow\tau$ has an initial object, and is therefore contractible. Thus, the claim holds for all radical subgroups of X, and in particular for any Sylow subgroup (which is radical by Lemma 4.2).

Let (P,α) be an object in $\mathcal{C}$ which is not radical. Then by [Q], the nerve of the poset $\mathcal{S}_p(W_X(P))$ is contractible, since $W_X(P)$ contains a nontrivial normal p-subgroup. Assume first that the claim holds for any centric p-compact toral subgroup $Q \le_\beta X$ of the same dimension as that of P, and such that $|\pi_0(Q)| > |\pi_0(P)|$. In particular we may assume that it holds for the p-compact toral group P_π corresponding to a p-subgroup $\pi \le W_X(P)$. Thus for each such subgroup, $|(P_\pi,\alpha_\pi)\downarrow\tau| \simeq |((P_\pi,\alpha_\pi)\downarrow\tau)^{op}| = |\tau^{op}\downarrow(P_\pi,\alpha_\pi)| = |\Delta(\pi)|$ is contractible. Hence, by Lemma 6.4, $|(P,\alpha)\downarrow\tau| \simeq |\mathcal{S}_p(W_X(P))|$, which is contractible.

Next, assume that the claim holds for all subgroups whose dimension is strictly larger than that of P, and that it does not hold for (P, α). By Lemma 6.4

$$|(P, \alpha)\downarrow\tau| \simeq \operatorname*{hocolim}_{\mathcal{S}_p(W_X(P))^{op}} |\Delta|.$$

By assumption, the left hand side is not contractible, and, since $|\mathcal{S}_p(W_X(P))|$ is contractible, there must exist a centric p-compact toral subgroup

$$(P_1, \alpha_1) \stackrel{\text{def}}{=} (P_\pi, \alpha_\pi)$$

for some non-trivial p-subgroup $\pi \leq W_X(P)$, such that $|(P_1, \alpha_1)\downarrow\tau|$ is not contractible. Moreover, $P \leq_{\gamma_\pi} P_1$ is a proper subgroup, since π is nontrivial. Repeating this argument produces a chain of infinite length

$$(P, \alpha) \lneqq (P_1, \alpha_1) \lneqq \cdots \lneqq (P_n, \alpha_n) \lneqq \cdots$$

of centric p-compact toral subgroups. Furthermore, since all P_n have the same cohomological dimension, and since all homomorphisms $P_n \longrightarrow P_{n+1}$ are proper monomorphisms, the order of $\pi_0(P_n)$ must increase strictly with n. Let $Q \stackrel{\text{def}}{=} \operatorname{colim}_n P_n$ and let (Q, β) be the resulting subgroup of X. Then, $\dim(Q) \gneqq \dim(P)$ and, by Lemma 6.3, $|(Q, \beta)\downarrow\tau| \simeq |(P_n, \alpha_n)\downarrow\tau|$ for all sufficiently large values of n . Therefore, the claim can not hold for (Q, β), which contradicts the induction hypothesis and completes the proof. □

7. Subgroup ampleness of centric subgroups and centralizer ampleness of elementary abelian subgroups

This section contains the proof of Theorem B. The core of the proof is a comparison result between the homotopy type of homotopy colimits over $\mathcal{O}^c(X)$ and $\mathcal{F}^e(X)^{op}$.

Let X be a p-compact group, and fix a Sylow subgroup $S \leq_{\iota_S} X$. We start by constructing a functor $\mathsf{E}\colon \mathcal{O}^c_S(X) \longrightarrow \mathcal{F}^e_S(X)^{op}$. Let $P \leq S$ be a subgroup which is centric in X. Define $\mathsf{E}(P)$ to be the maximal central elementary abelian subgroup of P (see Lemma A.17). Then $\mathsf{E}(P)$ is an elementary abelian subgroup of X via the inclusion to P followed by ι_P. The following lemma shows how to define E on morphisms, and will be useful throughout the section.

Lemma 7.1. *Let X be a p-compact group with a Sylow subgroup $S \leq_{\iota_S} X$. Let $P, Q \leq S$ be subgroups which are centric in X. Then for every morphism $[h]\colon P \longrightarrow Q$ in $\mathcal{O}^c_S(X)$, there is a unique homomorphism $\mathsf{E}[h]\colon \mathsf{E}(Q) \longrightarrow \mathsf{E}(P)$ such that the diagram*

$$\begin{array}{ccc} B\mathsf{E}(P) & \xrightarrow{inc} & BP \\ {\scriptstyle B\mathsf{E}[h]}\uparrow & & \downarrow{\scriptstyle h} \\ B\mathsf{E}(Q) & \xrightarrow{inc} & BQ \end{array}$$

commutes up to homotopy.

Proof. Fix a discrete approximation $\check{S}$ for S, and let $\check{P}, \check{Q} \leq \check{S}$ be the corresponding discrete approximations for P and Q respectively. Then the morphism $[h]$ can be represented by a homomorphism $\check{h}\colon \check{P} \longrightarrow \check{Q}$, unique up to conjugacy. Assume first that P and Q are abelian p-compact toral groups, not necessarily centric in X. Then, the maximal elementary abelian subgroups $\mathsf{E}(\check{P}) \leq \check{P}$ and $\mathsf{E}(\check{Q}) \leq \check{Q}$ coincide with $\mathsf{E}(P)$ and $\mathsf{E}(Q)$ respectively, and both are fully characteristic. Hence $\check{h}$ restricted to

$\mathsf{E}(\check{P})$ takes values in $\mathsf{E}(\check{Q})$. Applying classifying spaces and p-completion, one gets a homotopy commutative square

$$\begin{array}{ccc} B\mathsf{E}(P) & \xrightarrow{inc} & BP \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ B\mathsf{E}(Q) & \xrightarrow{inc} & BQ \end{array},$$

where $h' = B(\check{h}|_{\mathsf{E}(\check{P})})^\wedge_p$. Notice that in this case the homomorphism h' is uniquely determined by the homotopy class of h.

Let $P, Q \leq S$ be any subgroups which are centric in X, and let $[h]\colon P \longrightarrow Q$ be a morphism in $\mathcal{O}^c_S(X)$. Then one has the following homotopy commutative diagram

$$\begin{array}{cccc} B\mathcal{Z}(P) & \stackrel{\text{def}}{=} \mathrm{Map}(BP, BP)_{id} & \xrightarrow{ev} & BP \\ \uparrow & \simeq\downarrow{\scriptstyle h_\#} & & \downarrow{\scriptstyle h} \\ {\scriptstyle h_{\mathcal{Z}}}\Big| & \mathrm{Map}(BP, BQ)_h & \xrightarrow{ev} & BQ \\ \Big| & {\scriptstyle h^\#}\uparrow & & \| \\ B\mathcal{Z}(Q) & \stackrel{\text{def}}{=} \mathrm{Map}(BQ, BQ)_{id} & \xrightarrow{ev} & BQ \end{array} \tag{8}$$

where $h_{\mathcal{Z}} \stackrel{\text{def}}{=} (h_\#)^{-1} \circ h^\#$ for some choice of a homotopy inverse for $h_\#$. The lower right square commutes up to homotopy since h can be chosen to be a pointed map within its unpointed homotopy class. The two other squares commute automatically. Notice that the homotopy classes of $h_\#$ and $h^\#$ as unpointed maps depends only on the unpointed homotopy class of h, and not on its pointed homotopy class. Since all the spaces in the middle column of the diagram are classifying spaces of abelian p-compact toral groups, pointed and unpointed homotopy classes of maps coincide. Hence the homotopy class of $h_{\mathcal{Z}}$ depends only on the unpointed homotopy class of h, and so there exists a unique homomorphism $\mathcal{Z}(h)\colon \mathcal{Z}(Q) \longrightarrow \mathcal{Z}(P)$ such that $B\mathcal{Z}(h) \simeq h_{\mathcal{Z}}$. Applying the construction of the previous paragraph to $h_{\mathcal{Z}}$ completes the proof. □

Lemma 7.1, provides the definition of E on morphisms in $\mathcal{O}^c_S(X)$, and shows that E is well defined as a functor. The next proposition provides a useful identification of overcategories of E.

Proposition 7.2. *Let X be a p-compact group with a Sylow subgroup $S \leq_{\iota_S} X$. Let $E_0 \leq S$ be an elementary abelian subgroup and let $S_0 \leq \mathcal{C}_X(E_0)$ be a Sylow subgroup. Then there is an equivalence of categories*

$$\Theta_{E_0}\colon \mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0)) \longrightarrow \mathsf{E} \downarrow E_0.$$

Proof. Fix an elementary abelian subgroup $E_0 \leq S$ and a Sylow subgroup $S_0 \leq \mathcal{C}_X(E_0)$. Objects in the overcategory $\mathsf{E} \downarrow E_0$ are pairs $(P, [f])$, where $P \leq S$ is centric in X, and $[f]\colon E_0 \longrightarrow \mathsf{E}(P)$ is a morphism in $\mathcal{F}^e_S(X)$. A morphism $[h]\colon (P, [f]) \longrightarrow (P', [f'])$ is determined by a morphism $[h]\colon P \longrightarrow P'$ in $\mathcal{O}^c_S(X)$, such that $\mathsf{E}[h] \circ [f'] = [f]$ in $\mathcal{F}^e_S(X)$.

Construction of Θ_{E_0}. By the defining property of a Sylow subgroup, there is a map $f_0\colon BS_0 \longrightarrow BS$, such that the following square commutes up to homotopy

$$\begin{array}{ccc} BS_0 & \xrightarrow{f_0} & BS \\ {\scriptstyle \iota_{S_0}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_S} \\ B\mathcal{C}_X(E_0) & \underset{ev}{\longrightarrow} & BX. \end{array}$$

Thus for any subgroup $P \leq_{\iota_P} S_0$, the pair (P, κ_P), where $\kappa_P = \iota_S \circ f_0 \circ \iota_P$ is a p-compact toral subgroup of X. Moreover, $P \leq S_0$ is centric in $\mathcal{C}_X(E_0)$ if and only if (P, κ_P) is a centric subgroup of X by Lemma 3.4.

Let $P \leq \mathcal{C}_X(E_0)$ be a centric subgroup. Then, one has obvious maps

$$BE_0 \longrightarrow B\mathcal{Z}(\mathcal{C}_X(E_0)) \xrightarrow{\iota_P^{\#}} B\mathcal{C}_{\mathcal{C}_X(E_0)}(P) \simeq B\mathcal{Z}(P),$$

whose composite defines E_0 as a central subgroup of P. Hence, by maximality of $\mathsf{E}(P)$, the above composite factors up to homotopy through a map $j_P\colon BE_0 \longrightarrow B\mathsf{E}(P)$. Define

$$\Theta_{E_0}(P) = (P, [j_P]).$$

Notice that the pointed homotopy class of j_P is uniquely defined (since all the maps involved are monomorphisms, and all groups are abelian), and so Θ_{E_0} is well defined on objects.

Let $[h]\colon P \longrightarrow P'$ be a morphism in $\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))$. Then one has a diagram in which the external square commutes up to homotopy as do all triangles except for possibly the right hand side one.

$$\begin{array}{ccccc} \mathrm{Map}(BP', B\mathcal{C}_X(E_0))_{\iota_{P'}} & & \xleftarrow{\quad inc \quad} & & B\mathsf{E}(P') \\ & \nwarrow & & \nearrow{\scriptstyle j_{P'}} & \\ {\scriptstyle h^{\#}}\big\downarrow & & BE_0 & & \big\downarrow{\scriptstyle \mathsf{E}[h]} \\ & \swarrow & & \searrow{\scriptstyle j_P} & \\ \mathrm{Map}(BP, B\mathcal{C}_X(E_0))_{\iota_P} & & \underset{inc}{\xleftarrow{\qquad\quad}} & & B\mathsf{E}(P) \end{array}.$$

The left triangle commutes by definition of morphisms in $\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))$ and naturality, the two middle triangles commute by the argument in the previous paragraph, and the external rectangle by Lemma 7.1 and naturality. Since all maps are monomorphisms of abelian p-compact groups, it follows that $\mathsf{E}[h] \circ j_{P'} \simeq j_P$. Hence $[h]$ induces a morphism $[h]\colon (P, [j_P]) \longrightarrow (P', [j_{P'}])$ in $\mathsf{E} \downarrow E_0$, which is defined to be the image of $[h]$ under Θ_{E_0}. We proceed by showing that Θ_{E_0} is an equivalence of categories.

Θ_{E_0} induces an isomorphism on morphism sets. The map

$$\mathrm{Mor}_{\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))}(P, P') \longrightarrow \mathrm{Mor}_{\mathsf{E}\downarrow E_0}(\Theta_{E_0}(P), \Theta_{E_0}(P'))$$

induced by Θ_{E_0} is injective by definition. On the other hand, if $[h]\colon (P, [j_P]) \to (P', [j_{P'}])$ is a morphism in $\mathrm{Mor}_{\mathsf{E}\downarrow E_0}(\Theta_{E_0}(P), \Theta_{E_0}(P'))$, then $\mathsf{E}[h] \circ [j_{P'}] = [j_P]$, and by choosing representatives and applying $B\mathcal{C}_X(-)$ we get a homotopy commutative diagram

$$
\begin{array}{ccc}
BP & \xrightarrow{\quad h \quad} & BP' \\
\big\downarrow & & \big\downarrow \\
B\mathcal{C}_X(\mathsf{E}(P)) & \xrightarrow{\mathsf{E}[h]^{\#}} & B\mathcal{C}_X(\mathsf{E}(P')) \\
 & {\searrow}^{j_P^{\#}} \quad {\swarrow}_{j_{P'}^{\#}} & \\
 & B\mathcal{C}_X(E_0) &
\end{array}
$$

where the triangle homotopy commutes. Thus to prove surjectivity it suffices to show that the square at the top of the diagram commutes as well.

It is easy to check that the following diagram is strictly commutative:

$$
\begin{array}{ccc}
BP & \xrightarrow{\quad i \quad} & \mathrm{Map}(\mathrm{Map}(BP, BP)_{id}, BX)_{ev\circ\iota_P} = B\mathcal{C}_X(\mathcal{Z}(P)) \\
\Big\| & & \simeq\Big\uparrow (h_{\#})^{\#} \\
BP & \xrightarrow{\quad j \quad} & \mathrm{Map}(\mathrm{Map}(BP, BP')_h, BX)_{ev\circ\iota_{P'}} = B\mathcal{C}_X(\mathcal{C}_{P'}(P)), \\
h\Big\downarrow & & \Big\downarrow (h^{\#})^{\#} \\
BP' & \xrightarrow{\quad k \quad} & \mathrm{Map}(\mathrm{Map}(BP', BP')_h, BX)_{ev\circ\iota_{P'}} = B\mathcal{C}_X(\mathcal{Z}(P'))
\end{array}
$$

where $i(b)(f) \overset{\text{def}}{=} \iota_P(f(b))$, and j and k are defined similarly (compare with Diagram 8 in the proof of Lemma 7.1). The map $((h_{\#})^{\#})^{-1} \circ (h^{\#})^{\#}$ is, up to homotopy, the restriction of $\mathsf{E}[h]^{\#}$, and the maps i and k factor the inclusions $BP \to B\mathcal{C}_X(\mathsf{E}(P))$ and $BP' \to B\mathcal{C}_X(\mathsf{E}(P'))$. This completes the proof of surjectivity.

Θ_{E_0} is injective on isomorphism classes of objects. Let $P, P' \leq S_0$ be two subgroups which are centric in $\mathcal{C}_X(E_0)$, and assume

$$(P, [j_P]) = \Theta_{E_0}(P) \cong \Theta_{E_0}(P') = (P', [j_{P'}]).$$

Then, there is an isomorphism $P \xrightarrow{[h]} P'$ in $\mathcal{O}^c(X)$, such that $\mathsf{E}[h] \circ [j_{P'}] = [j_P]$ in $\mathcal{F}_S^e(X)$. Picking representatives $\mathsf{E}(h)$, j_P and $j_{P'}$ for the respective maps between classifying spaces, applying $\mathrm{Map}(-, BP)$, and taking the appropriate components (using Lemma 7.1), one obtains a homotopy commutative diagram

$$
\begin{array}{ccl}
BP \simeq \mathrm{Map}(B\mathsf{E}(P), BP)_{inc} & & \\
\mathsf{E}(h)^{\#}\Big\downarrow\simeq & {\searrow}^{j_P^{\#}} & \\
\mathrm{Map}(B\mathsf{E}(P'), BP)_{inc\circ\mathsf{E}(h)} & & \mathrm{Map}(BE_0, BP)_{inc\circ[f]} \xrightarrow{\iota_{P\#}} \mathrm{Map}(BE_0, BX)_{\iota_{E_0}}. \\
h_{\#}\Big\downarrow\simeq & {\nearrow}_{j_{P'}^{\#}} & \\
BP' \simeq \mathrm{Map}(B\mathsf{E}(P'), BP')_{inc} & &
\end{array}
$$

This shows that P and P' are isomorphic as objects of $\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))$, and hence that Θ is injective on isomorphism classes of objects.

Θ is surjective on isomorphism classes. Let $(P, [f])$ be an object of $\mathsf{E}{\downarrow}E_0$. Then $E_0 \xrightarrow{[f]} \mathsf{E}(P)$ is a morphism in $\mathcal{F}_S^e(X)$, and the composite

$$BP \xrightarrow{\ i\ } B\mathcal{C}_X(\mathsf{E}(P)) \xrightarrow{f^{\#}} B\mathcal{C}_X(E_0)$$

factors through BS_0, by the defining property of a Sylow subgroup. Let $\iota_f\colon BP \to B\mathcal{C}_X(E_0)$ denote the composite $f^{\#} \circ i$. In particular, there exists a subgroup $P' \leq S_0$, such that $(P', \iota_{P'})$ and (P, ι_f) are isomorphic objects of $\mathcal{O}(\mathcal{C}_X(E_0))$ (the full orbit category), and P' is centric in $\mathcal{C}_X(E_0)$. We claim that $\Theta_{E_0}(P') \overset{\text{def}}{=} (P', [j_{P'}])$ is isomorphic to $(P, [f])$ in $\mathsf{E}{\downarrow}E_0$. Showing that amount to the claim that Θ_{E_0} is surjective on isomorphism classes of objects.

The map ι_f factors through an isomorphism $P \longrightarrow P'$ of objects in $\mathcal{O}^{\mathrm{c}}_S(X)$, which we also denote by ι_f. To show that this morphism is in fact an isomorphism in $\mathsf{E}{\downarrow}E_0$, it suffices to show that $\mathsf{E}(\iota_f) \circ [j_{P'}] = [f]$. But, by Lemma 7.1 on has a diagram of abelian p-compact groups, where the square and the external circle commute:

$$\begin{array}{ccccc} & & \mathsf{E}(P) & \xrightarrow{inc} & \mathcal{Z}(P) \\ & \overset{[f]}{\nearrow} & \cong\uparrow \mathsf{E}(\iota_f) & & \cong\uparrow [f]^{\#}. \\ E_0 & & & & \\ & \underset{[j_{P'}]}{\searrow} & & & \\ & & \mathsf{E}(P') & \xrightarrow{inc} & \mathcal{Z}(P') \end{array}$$

Since all maps are monomorphisms, the triangle commutes as well. This shows that Θ_{E_0} is surjective, and hence bijective on isomorphism classes of objects. Thus Θ_{E_0} is an equivalence of categories as claimed. □

The next proposition is a key ingredient in our analysis, as it sets the ground for an inductive proof of the equivalence of statements (i) and (ii) in Theorem B.

Proposition 7.3. *Let X be a p-compact group with a Sylow subgroup $S \leq_{\iota_S} X$. Assume that for every elementary abelian p-subgroup $E_0 \leq S$, and any Sylow subgroup S_0 of $\mathcal{C}_X(E_0)$, the collection of all subgroups of S_0 which are centric in $\mathcal{C}_X(E_0)$ is subgroup-ample for $\mathcal{C}_X(E_0)$. Let*

$$\Phi\colon \mathcal{O}^{\mathrm{c}}_S(X) \longrightarrow \mathsf{Top}$$

be a subgroup approximation functor, and let $\mathsf{E}\colon \mathcal{O}^{\mathrm{c}}_S(X) \longrightarrow \mathcal{F}^{\mathrm{e}}_S(X)^{op}$ be the functor constructed above. Let Φ_{E_0} denote the functor on $\mathsf{E}{\downarrow}E_0$, which takes an object $(P, [f])$ to the mapping space $\mathrm{Map}(BE_0, \Phi(P))_f$. Define a functor $\Psi\colon \mathcal{F}^{\mathrm{e}}_S(X)^{op} \longrightarrow \mathsf{Top}$ by

$$\Psi(E_0) \overset{\text{def}}{=} \operatorname*{hocolim}_{\mathsf{E}\downarrow E_0} \Phi_{E_0}.$$

Then Ψ is a centralizer approximation functor for X. Furthermore, the functor Ψ is naturally homotopy equivalent to the left homotopy Kan extension $L_{\mathsf{E}}(\Phi)$.

Proof. We first observe that Ψ is well defined. The definition on objects is clear, and we only need to check the action of Ψ on morphisms. Let $E_0 \xrightarrow{[h]} E_1$ be a morphism in $\mathcal{F}^{\mathrm{e}}_S(X)^{op}$. Then $[h]$ induces an obvious functor $\mathsf{E}{\downarrow}E_0 \xrightarrow{[h]^{\#}} \mathsf{E}{\downarrow}E_1$, which in turn induces a map $\operatorname{hocolim}_{\mathsf{E}\downarrow E_0} \Phi_{E_1} \circ [h]^{\#} \longrightarrow \operatorname{hocolim}_{\mathsf{E}\downarrow E_1} \Phi_{E_1}$. Furthermore, $[h]$ induces a natural transformation $\gamma_h\colon \Phi_{E_0} \to \Phi_{E_1} \circ [h]^{\#}$, taking an object $(P, [f])$ to the map

$$\mathrm{Map}(BE_0, \Phi(P))_f \xrightarrow{h^{\#}} \mathrm{Map}(BE_1, \Phi(P))_{f\circ h}.$$

The composite

$$\Psi(E_0) = \operatorname*{hocolim}_{\mathsf{E}\downarrow E_0} \Phi_{E_0} \xrightarrow{\gamma_{h*}} \operatorname*{hocolim}_{\mathsf{E}\downarrow E_0} \Phi_{E_0} \circ [h]^{\#} \longrightarrow \operatorname*{hocolim}_{\mathsf{E}\downarrow E_1} \Phi_{E_1} = \Psi(E_1)$$

defines the action of Ψ on morphisms.

Since Φ is a subgroup approximation for X, it comes equipped with a natural transformation $\Phi \to 1_{BX}$, which induces natural transformations $\Phi_{E_0} \to 1_{B\mathcal{C}_X(E_0)}$ and $\Phi_{E_0} \to 1_{BX}$ (the second by composing the first with the evaluation). Thus, to finish the proof that Ψ is a centralizer approximation, it remains to show that Ψ is a realization of the homotopy functor on $\mathcal{F}^e_S(X)^{op}$ taking an elementary abelian subgroup to the classifying space of its centralizer in X.

Let $E_0 \leq S$ be an elementary abelian subgroup, and let $S_0 \leq \mathcal{C}_X(E_0)$ be a Sylow subgroup. By Proposition 7.2, the functor $\Theta_{E_0} \colon \mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0)) \xrightarrow[\simeq]{} \mathsf{E}{\downarrow}E_0$ is an equivalence of categories, and hence induces a homotopy equivalence

$$\operatorname*{hocolim}_{\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))} \Phi_{E_0} \circ \Theta_{E_0} \xrightarrow{\simeq} \operatorname*{hocolim}_{\mathsf{E}{\downarrow}E_0} \Phi_{E_0} = \Psi(E_0).$$

By the definitions,

$$\Phi_{E_0} \circ \Theta_{E_0}(P) = \Phi_{E_0}(P, [j_P]) = \mathrm{Map}(BE_0, \Phi(P))_{j_P} \simeq BP$$

for any object P in $\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))$. The last equivalence follows since $j_P \colon BE_0 \to \Phi(P)$ is a central map. It follows easily from the definitions that the composite functor $\Phi_{E_0} \circ \Theta_{E_0}$ sends any morphism in $\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_{E_0}))$ to a map in the same homotopy class. Hence $\Phi_{E_0} \circ \Theta_{E_0}$ followed by the projection to the homotopy category is naturally isomorphic to the homotopy functor on $\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))$ sending a subgroup $P \leq S_0$ to its classifying space. The natural transformation $\Phi \to 1_{BX}$ induces, as above, a natural transformation $\Phi_{E_0} \circ \Theta_{E_0} \longrightarrow 1_{B\mathcal{C}_X(E_0)}$, and so by Lemma 1.3, $\Phi_{E_0} \circ \Theta_{E_0}$ is a subgroup approximation functor for $\mathcal{C}_X(E_0)$.

By assumption, the collection of all subgroups of S_0 which are centric in $\mathcal{C}_X(E_0)$ is subgroup-ample. Hence, by Proposition 1.5 any subgroup approximation for $\mathcal{C}_X(E_0)$ with respect to this collection is a decomposition. This shows that $\Psi(E_0) \simeq B\mathcal{C}_X(E_0)$, and what remains to be shown is that this equivalence is natural with respect to morphisms in $\mathcal{F}^e_S(X)^{op}$.

Any morphism $E_0 \xrightarrow{\alpha} E_1$ in $\mathcal{F}^e_S(X)^{op}$ induces a map $\mathcal{C}_X(E_0) \xrightarrow{\alpha^\#} \mathcal{C}_X(E_1)$. Let $S_1 \leq \mathcal{C}_X(E_1)$ be a Sylow subgroup, and let $S_0 \xrightarrow{\alpha'} S_1$ be a map factoring the composite $\alpha^\# \circ \iota_{S_0}$ up to homotopy. If $P \leq S_0$ is any p-compact toral subgroup, then P is centric in $\mathcal{C}_X(E_0)$ if and only if it is centric in X, or equivalently if and only if its image in S_1 under α' is centric in $\mathcal{C}_X(E_1)$, both by Lemma 3.4. Thus one has an induced functor $\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0)) \xrightarrow{\alpha_*} \mathcal{O}^c_{S_1}(\mathcal{C}_X(E_1))$, and it is not hard to verify that the square

$$\begin{array}{ccc} \mathcal{O}_{S_0}(\mathcal{C}_X(E_0)) & \xrightarrow{\alpha_*} & \mathcal{O}_{S_1}(\mathcal{C}_X(E_1)) \\ {\scriptstyle \Theta_{E_0}}\downarrow & & \downarrow{\scriptstyle \Theta_{E_1}} \\ \mathsf{E}{\downarrow}E_0 & \xrightarrow{\mathsf{E}{\downarrow}\alpha} & \mathsf{E}{\downarrow}E_1 \end{array}$$

commutes, where the vertical maps are equivalences of categories, by Proposition 7.2. Thus α induces a map

$$\alpha_* \colon \operatorname*{hocolim}_{\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))} \Phi_{E_0} \circ \Theta_{E_0} \longrightarrow \operatorname*{hocolim}_{\mathcal{O}^c_{S_1}(\mathcal{C}_X(E_1))} \Phi_{E_1} \circ \Theta_{E_1},$$

such that the diagram

$$\begin{array}{ccccc}
B\mathcal{C}_X(E_0) & \xleftarrow{\simeq} & \operatorname*{hocolim}_{\mathcal{O}^c_{S_0}(\mathcal{C}_X(E_0))} \Phi_{E_0} \circ \Theta_{E_0} & \xrightarrow{\simeq} & \operatorname*{hocolim}_{\mathsf{E}\downarrow E_0} \Phi_{E_0} = \Psi(E_0) \\
\downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \Psi(\alpha)} \\
B\mathcal{C}_X(E_1) & \xleftarrow{\simeq} & \operatorname*{hocolim}_{\mathcal{O}^c_{S_1}(\mathcal{C}_X(E_1))} \Phi_{E_1} \circ \Theta_{E_1} & \xrightarrow{\simeq} & \operatorname*{hocolim}_{\mathsf{E}\downarrow E_1} \Phi_{E_1} = \Psi(E_1)
\end{array}$$

commutes up to homotopy. This shows that Ψ followed by the projection to the homotopy category is naturally isomorphic to the centralizer homotopy functor on $\mathcal{F}^e_S(X)^{op}$, and finishes the proof that Ψ is a centralizer approximation for X with respect to the collection of all elementary abelian subgroups in S.

It remains to show that Ψ is naturally homotopy equivalent to the left homotopy Kan extension $L_{\mathsf{E}}(\Phi)\colon \mathcal{F}^e_S(X)^{op} \longrightarrow \mathsf{Top}$ of Φ along E. By definition, for each object $E_0 \leq S$ of $\mathcal{F}^e_S(X)^{op}$,

$$L_{\mathsf{E}}(\Phi)(E_0) \overset{\text{def}}{=} \operatorname*{hocolim}_{\mathsf{E}\downarrow E_0} \widehat{\Phi}.$$

Where $\widehat{\Phi}$ is the composite of Φ with the obvious functor $\mathsf{E}\downarrow E_0 \to \mathcal{O}^c_S(X)$. For each object $(P,[f])$ of $\mathsf{E}\downarrow E_0$, the evaluation map

$$\Phi_{E_0}(P,[f]) = \mathrm{Map}(BE_0, \Phi(P))_f \to \Phi(P) = \widehat{\Phi}(P,[f])$$

is a homotopy equivalence, since E_0 is central in P. Hence one has an induced natural transformation $\Psi \longrightarrow L_{\mathsf{E}}(\Phi)$ which is a homotopy equivalence on every object. □

We are now ready to prove the equivalence of statements (i) and (ii) in Theorem B. The following proposition claims the equivalence of two more general statements, which implies the equivalence claimed in the theorem. The proposition makes use of the concept of the order $|X|$ of a p-compact group X, i.e., the pair (d_X, o_X), where d_X is the cohomological dimension of X, and $o_X = |\pi_0(X)|$. (See Definition A.7)

Proposition 7.4. *Fix an ordered pair of nonnegative integers (d,o). Then the following statements are equivalent.*

- (i) *For any p-compact group X with a Sylow subgroup S, such that $|X| \leq (d,o)$, the collection of all subgroups of S which are centric in X is subgroup ample.*
- (ii) *For any p-compact group X with a Sylow subgroup S, such that $|X| \leq (d,o)$, the collection of all elementary abelian subgroups $E \leq S$ is centralizer ample.*

Proof. (i)⇒(ii): Fix a p-compact group X of order $|X| \leq (d,o)$ with a Sylow subgroup $S \leq X$. For any elementary abelian subgroup $E \leq S$, $|\mathcal{C}_X(E)| \leq (d,o)$. Hence, assumption (i) applied to $\mathcal{C}_X(E)$ for any elementary abelian $E \leq S$ is that for any Sylow subgroup $S' \leq \mathcal{C}_X(E)$, the collection of all subgroups $Q \leq S'$ which are centric in $\mathcal{C}_X(E)$ is subgroup ample. By Lemma 7.3, if Φ is a subgroup approximation functor for X with respect to the collection of all $P \leq S$ which are centric in X, then the left Kan extension $L_{\mathsf{E}}(\Phi)$ is a centralizer approximation functor for X with respect to all elementary abelian subgroups $E \leq S$. Moreover, by the universal property of the left homotopy Kan extension,

$$\operatorname*{hocolim}_{\mathcal{O}^c_S(X)} \Phi \simeq \operatorname*{hocolim}_{\mathcal{F}^e_S(X)^{op}} L_{\mathsf{E}}(\Phi).$$

Hence if Φ is a subgroup decomposition of X, then $L_{\mathsf{E}}(\Phi)$ is a centralizer decomposition. In particular, assuming (i) for X, every subgroup approximation for X with respect to the collection of all subgroups $P \leq S$, which are centric in X, is a subgroup decomposition. Hence (ii) holds for X.

(ii)⇒(i): Notice first that for finite p-groups (i.e., p-compact groups of cohomological dimension 0), (i) and (ii) hold independently of each other (see for instance [D]).

Assume by induction that (ii)⇒(i) for all p-compact group Y, such that $|Y| < (d, o)$. Let X be a p-compact group of order (d, o), and assume (ii) holds for X. We must show that (i) hold for X as well.

Consider first the special case where X is assumed to be centerfree. With this assumption, $|\mathcal{C}_X(E)| < |X|$ for any nontrivial elementary abelian subgroup $E \leq S$. For a fixed elementary abelian subgroup $E \leq S$, assumption (ii), applied to $\mathcal{C}_X(E)$ is that for any Sylow subgroup $S' \leq \mathcal{C}_X(E)$ the collection of all nontrivial elementary abelian subgroups $F \leq S'$ is centralizer ample for $\mathcal{C}_X(E)$. By induction hypothesis, the collection of all subgroups $Q \leq S'$ which are centric in $\mathcal{C}_X(E)$ is subgroup ample. Let $\Phi\colon \mathcal{O}^c_S(X) \longrightarrow \mathsf{Top}$ be a subgroup approximation functor, which exists by Proposition 1.5, and let $L_{\mathsf{E}}(\Phi)$ denote the left Kan extension of Φ along E. By Proposition 7.3, $L_{\mathsf{E}}(\Phi)$ is a centralizer approximation functor for X with respect to the collection of all elementary abelian subgroups $F \leq S$. Assumption (ii) applied to X is that this collection is centralizer ample, and so there are homotopy equivalences:

$$\operatorname*{hocolim}_{\mathcal{O}^c_S(X)} \Phi \simeq \operatorname*{hocolim}_{\mathcal{F}^e_S(X)^{op}} L_{\mathsf{E}}(\Phi) \simeq BX.$$

The first by the universal property of the left homotopy Kan extension [HV], and the second by ampleness. This shows that the collection of all subgroups $P \leq S$ which are centric in X is subgroup ample and completes the proof in this case.

Let X be an arbitrary p-compact group of order (d, o). By Proposition 6.1, the collection of all subgroups $P \leq S$ which are centric in X is subgroup ample if and only if the collection of all subgroups of S which are radical in X is subgroup ample. By Proposition A.16, X is an extension of a centerfree p-compact group X' by a p-compact toral group K. Let S' be a Sylow subgroup for X'. By Proposition 5.5, it suffices to show that the collection of all subgroups $P' \leq S'$ which are radical in X' is subgroup ample, which is equivalent to the statement that the collection of all subgroups of S' which are centric in X' is subgroup ample, by Proposition 6.1. But $|X'| \leq |X|$ and X' is centerfree, in which case we have already proven the claim. This completes the proof in the general case. □

Appendix A. p-compact groups

A p-compact group is a triple (X, BX, e) such that X is $\mathbb{F}_p$-finite, BX is pointed and p-complete, and $e\colon X \longrightarrow \Omega BX$ is a homotopy equivalence. The space BX is called the classifying space of X.

Homomorphisms. A homomorphism $f\colon X \to Y$ of p-compact groups $f\colon X \to Y$ is a pointed map $Bf\colon BX \to BY$ between their classifying spaces. The homotopy fibre of Bf is denoted by $Y/f(X)$.

Definition A.1. *A homomorphism of p-compact groups $f\colon X \longrightarrow Y$ is said to be*

- *a monomorphism if $Y/f(X)$ is $\mathbb{F}_p$-finite,*
- *an epimorphism if $Y/f(X)$ is the classifying space of a p-compact group, and*
- *an isomorphism if Bf is an homotopy equivalence.*

A short exact sequence of p-compact groups is a sequence of homomorphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $BX \xrightarrow{Bf} BY \xrightarrow{Bg} BZ$ is a fibration, up to homotopy. Such a short exact sequence is also called an extension of Z by X.

The behavior of monomorphisms and epimorphisms between p-compact groups with respect to composition was considered in several papers (e.g. [DW1, DW3, MN]). The following Lemma gives a summary for future reference.

Lemma A.2. *Let $X \xrightarrow{g} Y \xrightarrow{f} Z$ be homomorphisms of p-compact groups.*

(i) *If $f \circ g$ is a monomorphism, then g is a monomorphism.*
(ii) *If g is a monomorphism and $f \circ g$ is an epimorphism, then f is an epimorphism.*
(iii) *If f is a monomorphism and an epimorphism then it is an isomorphism.*
(iv) *If $f\colon X \to X$ is a monomorphism then f is an isomorphism.*

Proof. (i) is implicit in [DW1, Theorem 7.3] and [MN, Theorem 2.17]. For (iii), see [DW1, Remark 3.3]. Finally (iv) is proved in [DW3, Proposition 4.3]. To prove (ii), consider the following diagram of fibrations

$$\begin{array}{ccccc} \Omega(Z/f(Y)) & \longrightarrow & Y/g(X) & \longrightarrow & BK \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & BX & \xrightarrow{=} & BX \\ \downarrow & & \downarrow g & & \downarrow f\circ g \\ Z/f(Y) & \longrightarrow & BY & \xrightarrow{f} & BZ \end{array}$$

where $Y/g(X)$ is $\mathbb{F}_p$ finite since g is a monomorphism and K is a p-compact group since the composite $f \circ g$ is an epimorphism. The space $Z/f(Y)$ is p-complete, being the homotopy fibre of a map between p-complete spaces, and $\Omega(Z/f(Y))$ is $\mathbb{F}_p$ finite by inspection of the Serre spectral sequence of the fibration $K \to \Omega(Z/f(Y)) \to Y/g(X)$. Hence, $Z/f(Y)$ is the classifying space of a p-compact group, which means that f is an epimorphism. □

A triple (Y, BY, e) is called an extended p-compact group if BY is the total space of a fibration $BZ \longrightarrow BY \longrightarrow BG$, where BZ is the classifying space of a p-compact group, G is a finite group, and $e\colon Y \longrightarrow \Omega BY$ is a homotopy equivalence. Obviously any p-compact group is extended, but the converse isn't true. The next lemma is well known for p-compact groups, but we actually use it in the following stronger form.

Proposition A.3. *Let Y be an extended p-compact group, and let X be any p-compact group. Given a pointed map $f\colon BY \longrightarrow BX$, the following conditions are equivalent:*

(i) *The homotopy fibre of f is $\mathbb{F}_p$-finite.*
(ii) *If $\alpha\colon B\mathbb{Z}/p \longrightarrow BY$ is such that $f \circ \alpha$ is null homotopic, then α is null homotopic.*

Proof. By definition, there is a fibration

$$BZ \longrightarrow BY \longrightarrow BG$$

where BZ is the classifying space of a p-compact group and G is a finite group. Let G_p be a fixed Sylow p-subgroup of G, and let BY_p be the pullback space of the system

$$BG_p \xrightarrow{inc} BG \longleftarrow BY.$$

Then $Y_p \overset{\text{def}}{=} \Omega BY_p$ is a p-compact group (in fact, and extension of the finite p-group G_p by the p-compact group Z).

If Y itself is a p-compact group, the proposition follows from [MN, Theorem 2.17]. Therefore, it suffices to prove that each of (i) and (ii) holds for f if and only if it holds for $f \circ \iota$, where $\iota\colon Y_p \longrightarrow Y$ is the map just constructed.

Consider the following diagram of fibrations

$$\begin{array}{ccccc} G/G_p & \longrightarrow & G/G_p & \longrightarrow & * \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ F' & \longrightarrow & BY_p & \xrightarrow{f\circ\iota} & BX \\ \big\downarrow & & \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle =} \\ F & \longrightarrow & BY & \xrightarrow{f} & BX \end{array}$$

Since G/G_p is and finite, F' is $\mathbb{F}_p$-finite if and only if F is $\mathbb{F}_p$-finite. Hence (i) holds for f if and only if it holds for $f \circ \iota$.

Next, we show that (ii) holds for f if and only if it holds for $f \circ \iota$. Notice first that if $\alpha\colon B\mathbb{Z}/p \longrightarrow BY_p$ is any map, the $\iota \circ \alpha$ is null homotopic if and only if α is null homotopic. One direction is clear, and for the other notice that if $\iota \circ \alpha$ is null homotopic, then α lifts to G/G_p which is discrete, and so α is null homotopic.

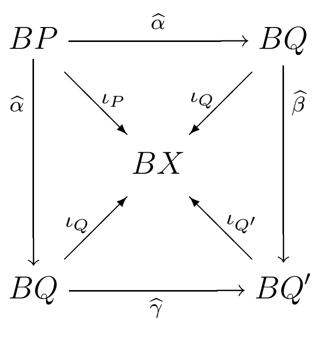
Assume (ii) is satisfied for f and let $\alpha\colon B\mathbb{Z}/p \to BY_p$ be a map such that $f \circ \iota \circ \alpha$ is null homotopic. Then $\iota \circ \alpha$ is null homotopic by (ii), and by the previous paragraph α is null homotopic.

Conversely, assume that (ii) is satisfied for $f \circ \iota$, and let $\alpha\colon B\mathbb{Z}/p \longrightarrow BY$ be a map such that $f \circ \alpha$ is null homotopic. The map ι is up to homotopy a finite covering of index prime to p, and so by the theory of covering spaces, α lifts to a map $B\mathbb{Z}/p \xrightarrow{\alpha'} Y_p$, such that $\iota \circ \alpha' \simeq \alpha$. Thus $f \circ \iota \circ \alpha' \simeq f \circ \alpha$ is null homotopic. But $f \circ \iota$ satisfies (ii), and so α' is null homotopic, and thus so is α. □

The following useful lemma (see the proofs of Lemma 4.5 and Proposition A.20) stand in contrast to the definition of epimorphisms of p-compact groups as formulated in [DW1].

Lemma A.4. *Let X be a p-compact group. Then all morphisms $\mathcal{O}^c(X)$ are epimorphisms in the categorical sense (i.e., if α, β, γ are morphisms in $\mathcal{O}^c(X)$, such that $\beta \circ \alpha = \gamma \circ \alpha$, then $\beta = \gamma$.*

Proof. Let $\alpha\colon P \longrightarrow Q$ and $\beta, \gamma\colon Q \longrightarrow Q'$ be morphisms in $\mathcal{O}^c(X)$ such that $\beta \circ \alpha = \gamma \circ \alpha$. Let $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\gamma}$ be maps between the appropriate classifying spaces, representing α, β and γ respectively. Then the diagram



commutes up to homotopy. We must show that $\widehat{\beta} \simeq \widehat{\gamma}$.

Assume first that $P \lhd Q$, i.e., that $\mathcal{N}_Q(P, \widehat{\alpha}) \cong Q$ as p-compact groups. Thus there is an extension of p-compact toral groups

$$BP \xrightarrow{\widehat{\alpha}} BQ \longrightarrow BW,$$

where $W = \mathcal{W}_Q(P, \widehat{\alpha})$. Let $\widetilde{BP} \overset{\text{def}}{=} EQ_{hP}$, where EQ is a free contractible Q space. Then $\widetilde{BP} \simeq BP$, and W acts on $\widetilde{BP}$ with homotopy orbit space $\widetilde{BP}_{hW} \simeq BQ$. Let $\{\beta\}$ denote the set of all homotopy classes of maps $BQ \to BQ'$, which can replace $\widehat{\beta}$ in the diagram above, leaving it homotopy commutative (in particular the homotopy class of $\widehat{\gamma}$ is there). Thus

$$\mathrm{Map}(BQ, BQ')_{\{\beta\}} \simeq \mathrm{Map}(\widetilde{BP}_{hW}, BQ')_{\{\beta\}} \simeq (\mathrm{Map}(\widetilde{BP}, BQ')_{\widehat{\beta}\widehat{\alpha}})^{hW},$$

and similarly, $\mathrm{Map}(BQ, BX)_{\iota_Q}$ is equivalent to a component of $(\mathrm{Map}(\widetilde{BP}, BX)_{\iota_P})^{hW}$. Furthermore, the map

$$\iota_{Q'\#}\colon \mathrm{Map}(\widetilde{BP}, BQ')_{\widehat{\beta}\widehat{\alpha}} \longrightarrow \mathrm{Map}(\widetilde{BP}, BX)_{\iota_P}$$

is equivariant with respect to the action of W, and a homotopy equivalence, since P is centric in X (and hence in Q', by Lemma 3.1). Thus the induced map

$$\mathrm{Map}(BQ, BQ')_{\{\beta\}} \simeq (\mathrm{Map}(\widetilde{BP}, BQ')_{\widehat{\beta}\widehat{\alpha}})^{hW} \xrightarrow{\iota_{Q'\#}} (\mathrm{Map}(\widetilde{BP}, BX)_{\iota_P})^{hW},$$

is a homotopy equivalence, and upon identifying the target space with a subspace of $\mathrm{Map}(BQ, BX)$, takes $\widehat{\beta}$ and $\widehat{\gamma}$ to $\iota_{Q'} \circ \widehat{\beta}$ and $\iota_{Q'} \circ \widehat{\gamma}$ respectively. But, in $\mathrm{Map}(BQ, BX)$ both maps are in the component of ι_Q by commutativity. Hence $\widehat{\beta} \simeq \widehat{\gamma}$, as claimed.

Assume now that P is not necessarily normal in Q. Fix discrete approximations $\check{P}$, $\check{Q}$ and $\check{Q}'$ for P, Q and Q' respectively. Then $\widehat{\alpha}$ is induced up to homotopy by a homomorphism $\check{P} \xrightarrow{\check{\alpha}} \check{Q}$ which is unique up to conjugacy, and without loss of generality we may assume that $\check{\alpha}$ is an inclusion, so $\check{P} \leq \check{Q}$. Define a sequence of p-discrete toral subgroups

$$\check{P} = \check{P}_0 \lhd \check{P}_1 \lhd \check{P}_2 \lhd \cdots \check{P}_{j-1} \lhd \check{P}_j \lhd \cdots \check{P}_\omega$$

by setting for each $j \geq 0$, $\check{P}_j \overset{\text{def}}{=} \mathcal{N}_{\check{Q}}(\check{P}_{j-1})$, and $\check{P}_\omega \overset{\text{def}}{=} \mathrm{colim}_j\, \check{P}_j$. Let $\iota_j\colon \check{P}_j \longrightarrow \check{Q}$ denote the inclusion, and let $\check{\beta}, \check{\gamma}\colon \check{Q} \longrightarrow \check{Q}'$ denote the homomorphisms, unique up to conjugacy, inducing $\widehat{\beta}$ and $\widehat{\gamma}$ respectively, up to homotopy. Set $\check{\beta}_j = \check{\beta} \circ \iota_j$ and $\check{\gamma}_j = \check{\gamma} \circ \iota_j$. By the argument for the case where $P \lhd Q$, and induction, for each $j \geq 0$, $\check{\beta}_j$ and $\check{\gamma}_j$ are conjugate in $\check{Q}'$. Thus the induced maps $\check{\beta}_\omega$ and $\check{\gamma}_\omega$ are conjugate in $\check{Q}'$, and so the maps $B\check{P}_\omega \longrightarrow B\check{Q}'$ are homotopic. Let P_ω be the subgroup of Q whose classifying space is $(B\check{P}_\omega)^\wedge_p$. Then, either the sequence constructed above is finite, in which case $P_\omega = Q$, and the proof is complete, or the dimension of P_ω is strictly larger that of P. Induction on the difference between the dimensions of P and Q now finishes the proof. □

Subgroups, Maximal Tori, and Sylow Subgroups. One of the most important concepts in this paper is that of a subgroup.

Definition A.5. *A subgroup of a p-compact group X, is a pair (Y, α), where Y is a p-compact group, and $\alpha\colon BY \longrightarrow BX$ is a monomorphism.*

A p-compact torus of rank n is a p-compact group T, such that

$$BT \simeq K((\mathbb{Z}_p^\wedge)^n, 2) \simeq (B(\mathbb{Z}/p^\infty\mathbb{Z})^n)_p^\wedge,$$

where $\mathbb{Z}/p^\infty$ is the direct limit of all cyclic groups $\mathbb{Z}/p^r$ under inclusion. The group $(\mathbb{Z}/p^\infty\mathbb{Z})^n$ is called a p-discrete torus or a discrete approximation for T (where the prime p is understood). A maximal torus of a p-compact group X is a subgroup (T_X, ι), where T_X is a p-compact torus, which is maximal in the sense that if (T, j) is any other subgroup with T a p-compact torus, then there exists a homomorphism $k\colon T \to T_X$ such that $j \simeq \iota \circ k$.

Theorem A.6. [DW1, Theorem. 8.13] *Any p-compact group admits a maximal torus unique up to conjugacy.*

The next useful concepts we introduce are those of the *order* of a p-compact group, and the index of a subgroup.

Definition A.7. *Let X be a p-compact group. Define the order of X to be the pair (d_X, o_X), where d_X is the mod-p cohomological dimension of X and o_X is the order of its group of components. The order of X is denoted by $|X|$. If Y is another p-compact group, then we say $|Y| \leq |X|$ if $d_Y \leq d_X$ or if $d_Y = d_X$ and $o_Y \leq o_X$. More generally, if $Y \leq X$ is a subgroup, define the index of Y in X, denoted as usual $|X\colon Y|$, to be the pair $(d_{X/Y}, o_{X/Y})$, where $d_{X/Y}$ is the mod-p cohomological dimension of X/Y, and $o_{X/Y}$ the order of the set of components of X/Y. Thus $|X| = |X\colon 1|$. We say that Y is a subgroup of X of finite index, or of index n, if $|X\colon Y| = (0, n)$ for some n (essentially finite).*

Similarly, for a compact Lie group G, define the order $|G|$ of G to be the pair (d_G, o_G), where d_G is the dimension of G, and $o_G \overset{\text{def}}{=} |\pi_0(G)|$. Lexicographical ordering, as above, endows the class of all compact Lie groups with a linear order.

The next lemma shows that the order behaves as one would expect under taking subgroups.

Lemma A.8. *Let X be a p-compact group and let $Y \leq_\alpha X$ be a subgroup. Then $|Y| \leq |X|$ with equality holding if and only if α is an isomorphism.*

Proof. By [DW1, Proposition 6.15], $d_X = d_Y + d_{X/Y}$. Hence $d_Y \leq d_X$ with equality if and only if $d_{X/Y} = 0$, i.e., if and only if X/Y is homotopically discrete. By [DW1, Remark. 6.16] this is the case if and only if α induces a homotopy equivalence between Y and a union of components of X, or equivalently if and only if $o_Y \leq o_X$. Hence $|Y| \leq |X|$ and equality holds if and only if α is a homotopy equivalence, i.e., an isomorphism of p-compact groups. □

A p-compact toral group is a p-compact group P, which is an extension of a finite p-group π by a p-compact torus. An important family of p-compact toral subgroups of any p-compact group is the collection of its maximal p-compact toral subgroups, which behaves in many ways like Sylow p-subgroups do in a finite group.

Definition A.9. *A Sylow subgroup of a p-compact group X is a p-compact toral subgroup $S \leq_\iota X$, which is maximal in the sense that every other p-compact toral subgroup $Y \leq_\alpha X$ factors through it. In other words, there exists a homomorphism $BY \xrightarrow{f} BS$, such that $\iota \circ f \simeq \alpha$.*

Notice that a Sylow subgroup of X is unique up to conjugacy. Notice also that since the prime p is fixed and since a p-compact toral group is in general not a p-group, we omit the prime p from the terminology and use "Sylow subgroup", rather than "Sylow p-subgroup".

By [DW2, Proposition 2.10] the p-normalizer of the maximal torus in a p-compact group X, $\mathcal{N}_p(T)$ is a subgroup such that $\chi(X/\mathcal{N}_p(T))$ is relatively prime to p, and by Proposition 2.14 in the same paper $\mathcal{N}_p(T)$ is a Sylow subgroup of X in the sense defined here. With the existence of at least one Sylow subgroup granted, the following lemma demonstrates the analogy of our concept with Sylow p-subgroups in the usual sense.

Lemma A.10. *Let X be a p-compact group, and let $P \leq_\alpha X$ be a p-compact toral subgroup. Then the following conditions are equivalent.*

(i) *(P, α) is a Sylow subgroup in X.*
(ii) *The Euler characteristic $\chi(X/P)$ is not divisible by p.*
(iii) *(P, α) is a p-compact toral subgroup of maximal order.*

Proof. The implication (ii)⇒(i) is [DW2, Proposition 2.14]. Conversely, if (P, α) is a Sylow subgroup for X, then (P, α) is conjugate to $\mathcal{N}_p(T)$, and hence $\chi(X/P) = \chi(X/\mathcal{N}_p(T)$, which is relatively prime to p, by [M2, Theorem 1.2].

Next we prove (i)⇒(iii). Let (P, α) be a Sylow subgroup of X, and let $Q \leq_\beta X$ be any other p-compact toral subgroup. Then, by definition, there is a monomorphism $f\colon BQ \longrightarrow BP$, and by Lemma A.7 $|Q| \leq |P|$, so P is a subgroup of maximal order.

Finally we show (iii)⇒(i). Let (P, α) be a p-compact toral subgroup of X of maximal order, and let (Q, β) be a Sylow subgroup. Then there is a monomorphism $f\colon P \longrightarrow Q$, such that $\beta \circ f \simeq \alpha$. But, by the previous argument, $|P| \leq |Q|$, and so by maximality $|P| = |Q|$, and f is an isomorphism. This shows that (P, α) is also a Sylow subgroup. □

Extensions. Our discussion requires the concept of "extensions" of p-compact groups. Before we can make sense of this, we need the following preliminary lemma.

Lemma A.11. *Let $F \xrightarrow{j} E \xrightarrow{q} B$ be a fibration of connected spaces, and assume that the evaluation map $\mathrm{Map}(F, B)_c \longrightarrow B$ is a homotopy equivalence, where $F \xrightarrow{c} B$ is the constant map. Then there is a fibration,*

$$\mathrm{Map}_*(F, F)_{\{j\}} \xrightarrow{j_\#} \mathrm{Map}(F, E)_j \xrightarrow{ev} E,$$

where $\{j\}$ denotes the union of components of the mapping space which are mapped to the component of j via $j_\#$.

Proof. Consider the diagram

$$\begin{array}{ccccc}
\mathrm{Map}(F,F) & \xrightarrow{j_\#} & \mathrm{Map}(F,E)_{\{c\}} & \xrightarrow{q_\#} & \mathrm{Map}(F,B)_c \\
\downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev}\,\simeq \\
F & \xrightarrow{j} & E & \xrightarrow{q} & B
\end{array}$$

Here $\{c\}$ denotes all components which are mapped to the component of the constant map by $q_\#$. The right vertical arrow is a equivalence by assumption, and both rows are fibrations. This shows that $\mathrm{Map}_*(F, F) \xrightarrow{j_\#} \mathrm{Map}_*(F, E)_{\{c\}}$ is a homotopy equivalence.

Since $j \in \{c\}$, the space $\mathrm{Map}(F,E)_j$ is a component of $\mathrm{Map}(F,E)_{\{c\}}$. Hence restriction defines a fibration

$$\mathrm{Map}(F,F)_{\{j\}} \longrightarrow \mathrm{Map}(F,E)_j \longrightarrow \mathrm{Map}(F,B)_c,$$

where the fibre consists of those components of the full mapping space over the component of j. Restricting the top row of the diagram to this fibration, and taking fibres on the vertical maps now gives a diagram whose central column is the required fibration. □

Corollary A.12. *Let*

$$BY \xrightarrow{\alpha} BX \xrightarrow{\pi} BZ$$

be a fibration, where all spaces are classifying spaces of p-compact groups. Then the natural map $\mathcal{N}_X(Y,\alpha) \longrightarrow X$ is a homotopy equivalence of loop spaces.

Proof. Since $\mathrm{Map}_*(BY,BX)_c$ is contractible, Lemma A.11 applies. Consequently, the homotopy fibre of the map

$$\mathrm{Map}_*(BY,BY)_{\{\alpha\}} \longrightarrow B\mathcal{C}_X(Y,\alpha) \overset{\mathrm{def}}{=} \mathrm{Map}(BY,BX)_\alpha$$

is homotopy equivalent to X as a loop space. But by definition, this homotopy fibre is $\mathcal{N}_X(Y,\alpha)$. □

Our construction of the normalizer space allows us to define what it means for a subgroup to be normal.

Definition A.13. *A subgroup $Y \leq_\alpha X$ of a p-compact group X is said to be normal if the natural map $\eta_\alpha \colon \mathcal{N}_X(Y) \longrightarrow X$ defined in Proposition 2.5 is an isomorphism of p-compact groups.*

Corollary A.12 thus provides a justification to regarding fibrations, where all spaces involved are classifying spaces of p-compact groups, as extensions of p-compact groups. Such a fibration defines Y as a normal subgroup of X with quotient group Z. We say in that case that X is an extension of Z by Y.

Centralizers and Centers. Let X be a p-compact group and let $Y \leq_\alpha X$ be a subgroup. The centralizer of (Y,α) in X is defined to be the loop space of the space

$$B\mathcal{C}_X(Y,\alpha) \overset{\mathrm{def}}{=} \mathrm{Map}(BY,BX)_\alpha.$$

Proposition A.14. *Let $Y \leq_\alpha X$ be a p-compact toral subgroup of a p-compact group X. Then $\mathcal{C}_X(Y,\alpha)$ is a p-compact group.*

Proof. [DW1, Proposition 5.1, and Proposition 6.1] □

Next we define the center of a p-compact group.

Definition A.15. *A subgroup $Z \leq_\alpha X$ is central if $\mathcal{C}_X(Z) \cong X$, or in other words if $ev\colon \mathrm{Map}(BZ,BX)_\alpha \to BX$ is a homotopy equivalence. A central subgroup of X is said to be the center of X, if every other central subgroup factors through it. A p-compact group X is said to be centerfree if it has no nontrivial central subgroup.*

For a p-compact group X, Dwyer and Wilkerson showed that

$$\mathcal{Z}(X) = \Omega\,\mathrm{Map}(BX,BX)_{id}$$

is an abelian p-compact toral group and has the property that the evaluation map $B\mathcal{Z}(X) \longrightarrow BX$ is, up to homotopy, a final object among all central monomorphisms into X [DW2, Theorems 1.2, 1.3]. Thus, whenever we say "the center of X", we mean the subgroup $(\mathcal{Z}(X), ev)$. Notice that X is centerfree if and only if $\mathcal{Z}(X)$ is weakly contractible.

The following lemma is useful in reducing certain claims to the centerfree case.

Lemma A.16. *Every p-compact group X is an extension of a centerfree p-compact group by a p-compact toral group.*

Proof. For a p-compact group X we denote by X_0 the component of the identity element and by $\pi = \pi_0(X)$ the group of components. The canonical fibration

$$BX_0 \longrightarrow BX \longrightarrow B\pi$$

is classified by a map $B\pi \xrightarrow{\alpha} B\,\mathrm{Aut}(BX_0)$. Let $Z_0 = \mathcal{Z}(X_0)$ be the center, and let $X_0' = X_0/Z_0$ be the centerfree quotient of X_0 (see [DW2, Theorem 6.3]). Every self equivalence h of BX_0 preserves the center, and thus there is a group homomorphism

$$\pi_0(\mathrm{Aut}(BX_0)) \longrightarrow \pi_0(\mathrm{Aut}(BX_0')).$$

But,

$$\mathrm{Aut}_1(BX_0') \overset{\mathrm{def}}{=} \mathrm{Map}(BX_0', BX_0')_{id} \overset{\mathrm{def}}{=} B\mathcal{Z}(X_0'),$$

and X_0' is centerfree. This shows that $\mathrm{Aut}_1(BX_0')$ is trivial, and hence that $\mathrm{Aut}(BX_0')$ is aspherical. Therefore, the canonical map

$$B\,\mathrm{Aut}(BX_0') \longrightarrow B\pi_0(\mathrm{Aut}(BX_0'))$$

is a homotopy equivalence, and the composite

$$B\pi \xrightarrow{\alpha} B\,\mathrm{Aut}(BX_0) \longrightarrow B\pi_0(\mathrm{Aut}(BX_0)) \longrightarrow$$
$$\longrightarrow B\pi_0(\mathrm{Aut}(BX_0')) \simeq B\,\mathrm{Aut}(BX_0')$$

gives rise to an extension X' of π by X_0', and a commutative diagram of extensions of p-compact groups

$$\begin{array}{ccccc} BX_0 & \longrightarrow & BX & \longrightarrow & B\pi \\ \downarrow & & \downarrow & & \downarrow{=} \\ BX_0' & \longrightarrow & BX' & \longrightarrow & B\pi \end{array} .$$

If X' is centerfree, we are done, as we have presented X as an extension of a centerfree p-compact group by Z_0, and abelian p-compact group (in particular p-compact toral). Otherwise, let $Z' = \mathcal{Z}(X')$ be the center of X'. Since X_0' is centerfree, the composite $BZ' \longrightarrow BX' \longrightarrow B\pi$ is a monomorphism, and Z' is a central subgroup of π. Taking the quotients of X' and π by Z' gives a commutative diagram of extensions of p-compact groups

$$\begin{array}{ccccc} BX_0' & \longrightarrow & BX' & \longrightarrow & B\pi \\ {=}\downarrow & & \downarrow & & \downarrow \\ BX_0' & \longrightarrow & BX_1 & \longrightarrow & B\pi_1 \end{array} ,$$

where $|\pi_1| \lneq |\pi|$. Notice that the homotopy fibre of the homomorphism $BX \longrightarrow BX_1$ is an extension of Z', which is a finite abelian p-group, by Z_0 which is an abelian p-compact group. Thus the homotopy fibre is the classifying space of a p-compact toral group.

If X_1 is centerfree, the proof is complete. Otherwise, divide X_1 by its center, which by the same argument as above is also a central subgroup of π_1, to obtain an extension

X_2 of a finite p-group π_2, with $|\pi_2| \lneq |\pi_1|$, by X_0', and such that the homotopy fibre of the projection $BX \longrightarrow BX_2$ is the classifying space of a p-compact toral group. Applying this process repeatedly yields in finitely many steps (since π is finite) a p-compact group quotient Y of X, such that Y is centerfree, and the homotopy fibre of the projection $BX \longrightarrow BY$ is the classifying space of a p-compact toral group. □

Next, we consider the maximal central elementary abelian subgroup of a p-compact group.

Lemma A.17. *Any p-compact group X admits a maximal central elementary abelian subgroup $\mathsf{E}(X)$.*

Proof. Since $\mathcal{Z}(X)$ is an abelian p-compact toral group, it admits a maximal elementary abelian p-subgroup of $\mathsf{E}(X) \leq_\iota \mathcal{Z}(X)$, and since the evaluation map

$$ev\colon \operatorname{Map}(B\mathsf{E}(X), BX)_\iota \longrightarrow BX$$

is an equivalence, its homotopy fibre is weakly contractible. This shows that $\mathsf{E}(X)$ is unique up to homotopy. i.e., that if $B\mathsf{E}(X) \xrightarrow{f} BX$ is any other map such that $\operatorname{Map}(B\mathsf{E}(X), BX)_f \xrightarrow{\simeq} BX$, then $f \simeq \iota$. The subgroup $(\mathsf{E}(X), \iota)$ is clearly maximal in the sense that if $F \leq_\beta X$ is any other central elementary abelian p-subgroup of X, then β factors up to homotopy through $B\mathsf{E}(X)$. □

Remark A.18. *Notice that the symbol $\mathsf{E}(X)$ is used elsewhere to denote a functor. The reader should not be confused by this abuse of notation. If X is a p-compact toral group and $\check{X}$ is a discrete approximation, then the algebraic center of $\check{X}$ is a discrete approximation for the center of X. With this setup, it is possible to define $\mathsf{E}(X)$ canonically. We choose to use the same symbol here to emphasize that it is practically the same construction we discuss here, but without specifying discrete approximations.*

Remark A.19. *Let $Y \leq_\alpha X$ be a subgroup, let $\mathsf{E}(Y) \leq \mathcal{Z}(Y)$ denote a maximal central elementary abelian subgroup and consider the maps δ and ϵ given by the composites*

$$\delta \stackrel{\text{def}}{=} \big(BY \times B\mathsf{E}(Y) \longrightarrow BY \times B\mathcal{Z}(Y) \xrightarrow{mult} BY \xrightarrow{\alpha} BX\big)$$

and

$$\epsilon \stackrel{\text{def}}{=} \big(\mathsf{E}(Y) \longrightarrow \mathcal{Z}(Y) \longrightarrow Y \xrightarrow{\alpha} X\big).$$

Then the map $ad(\delta)\colon BY \longrightarrow \operatorname{Map}(B\mathsf{E}(Y), BX)_\epsilon$ makes the following diagram homotopy commutative

$$\begin{array}{ccc} BY & \xrightarrow{ad(\delta)} & \operatorname{Map}(B\mathsf{E}(Y), BX)_\epsilon \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle ev} & \\ & BX & \end{array}$$

Thus $Y \leq \mathcal{C}_X(\mathsf{E}(Y)) \leq X$ is a factorization of $Y \leq_\alpha X$.

Weyl Groups. The aim of the following discussion is to show that if $P, Q \leq X$ are p-compact toral subgroups, then the morphism sets between them as objects of $\mathcal{O}(X)$ are finite. In particular, the Weyl group of p-compact toral subgroup of X is always a finite group. The following lemma will be used in the proof of this statement.

Proposition A.20. *Let X be a p-compact group, and $Q, P \leq X$ be p-compact toral subgroups of X. Then there exists a finite p-subgroup K of P, such that $(X/Q)^{hP} \simeq (X/Q)^{hK}$.*

Proof. Let $\check{P} = \operatorname{colim} P_n$ be a discrete approximation of P. Let T_n be the kernel of the composite $P_n \longrightarrow \check{P} \longrightarrow \pi$, where $\pi \stackrel{\text{def}}{=} \pi_0(P)$. Then $\check{T} \stackrel{\text{def}}{=} \operatorname{colim}_n T_n$ is a discrete approximation for T, and for a sufficiently large n, the map $P_n \longrightarrow \pi$ is an epimorphism. For each n one has a map

$$(X/Q)^{hP} \simeq ((X/Q)^{hT})^{h\pi} \longrightarrow ((X/Q)^{hT_n})^{h\pi} \simeq (X/Q)^{hP_n}.$$

Hence, if we show that for n sufficiently large the map $(X/Q)^{hT_n} \longrightarrow (X/Q)^{hT}$ is an equivalence, then the lemma holds for all p-compact toral groups.

Thus assume $P = T$, a p-compact torus, with a discrete approximation $\check{T} = \operatorname{colim} T_n$. Let ι_n be the restriction of $\iota_T\colon BT \longrightarrow BX$ to BT_n. Then, there is a homotopy commutative diagram of fibrations

$$\begin{array}{ccc}
(X/Q)^{hT_{n+1}} & \longrightarrow & (X/Q)^{hT_n} \\
\downarrow & & \downarrow \\
\operatorname{Map}(BT_{n+1}, BQ)_{\{\iota_{n+1}\}} & \xrightarrow{inc^\sharp} & \operatorname{Map}(BT_n, BQ)_{\{\iota_n\}} \\
{\scriptstyle \iota_{Q\sharp}}\downarrow & & {\scriptstyle \iota_{Q\sharp}}\downarrow \\
\operatorname{Map}(BT_{n+1}, BX)_{\iota_{n+1}} & \xrightarrow{inc^\sharp} & \operatorname{Map}(BP_n, BQ)_{\iota_n}
\end{array} \tag{9}$$

where $\{\iota_k\}$ is the set of homotopy classes of maps $BT_k \xrightarrow{f} BQ$, such that $\iota_Q \circ f \simeq \iota_k$. Notice that the sets $\{\iota_k\}$ are finite sets, since by [DW1, Theorem 4.6, and Theorem 5.8], $(X/Q)^{hT_k}$ is $\mathbb{F}_p$-finite for all k.

Next we show that, for n sufficiently large, the map between the total spaces in the diagram above induces a monomorphism on the sets of components. This is equivalent to the claim that, if $f, g\colon BT_{n+1} \longrightarrow BQ$ are two maps whose homotopy classes are contained in $\{\iota_{n+1}\}$, and such that their restriction to BT_n are homotopic, then $f \simeq g$. In other words, if f and g are as above and both render the diagram

$$\begin{array}{ccc}
BT_n & \xrightarrow{\iota_n} & BQ \\
{\scriptstyle j}\downarrow & \nearrow^{f}_{g} & \downarrow{\scriptstyle \iota_Q} \\
BT_{n+1} & \xrightarrow[\iota_{n+1}]{} & BX
\end{array}$$

homotopy commutative, then $f \simeq g$.

Apply the functor $\operatorname{Map}(T_n, -)$ to the diagram above. Since , T_n and T_{n+1} are abelian, this gives a homotopy commutative diagram

$$\begin{array}{ccc}
BT_n & \xrightarrow{\iota_{n\sharp}} & BC_Q(T_n) \\
{\scriptstyle j_\sharp}\downarrow & \nearrow^{f_\#}_{g_\#} & \downarrow{\scriptstyle \iota_{Q\sharp}} \\
BT_{n+1} & \xrightarrow[h_{n+1\sharp}]{} & BC_X(T_n)
\end{array}.$$

For n sufficiently large, the map $BC_X(T_{n+1}) \longrightarrow BC_X(T_n)$ is an equivalence by [DW1, Propsition 6.18], and hence the compsite

$$BT_{n+1} \longrightarrow BC_X(T_{n+1}) \longrightarrow BC_X(T_n)$$

is central. By [DW2, Lemma 6.5] it now follows that $f_\sharp \simeq g_\sharp$ and hence that $f \simeq g$.

We have thus shown that, for n sufficiently large, the map between total spaces in Diagram (9) above induces a monomorphism on path components. Hence the sequence of path components stabilizes, and is equal to the set $\{\iota_T\}$ which is finite. By [DW1, Propsition 6.18] again, the map between total spaces in the diagram is a homotopy equivalence for n sufficiently large. Hence for such an n, the induced map on homotopy fibres is a homotopy equivalence, and the result follows. □

Corollary A.21. *For any p-compact toral subgroups $P, Q \leq_\alpha X$, $(X/Q)^{hP}$ is $\mathbb{F}_p$-finite and $\chi((X/Q)^{hP}) \equiv \chi(X/Q)$ mod p. In particular $W_X(P)$ is a finite group.*

Proof. By Proposition A.20, $(X/Q)^{hP} \simeq (X/P)^{hK}$, for some finite p-group K. Hence, by [DW1, Proposition 5.8], for every subgroup $L \leq K$, each component of $(X/Q)^{hL}$ is p-complete. Hence by [DW1, Theorem 4.6] $(X/Q)^{hP} \simeq (X/Q)^{hK}$ is $\mathbb{F}_p$-finite and $\chi((X/Q)^{hP}) \equiv \chi(X/Q)$ mod p. The last statement follows at once. □

Normal Refinement. The main aim of the next two lemmas is to show that any monomorphism between p-compact toral groups can be refined to a (generally infinite) normal sequence with finite p-group quotients (Lemma A.23).

Lemma A.22. *Let P and Q be p-compact toral groups and let $\check{P} \leq_{\iota_P} P$ and $\check{Q} \leq_{\iota_Q} Q$ be discrete approximations. Then the map*

$$\iota_{Q\#} \colon \operatorname{Map}(B\check{P}, B\check{Q}) \longrightarrow \operatorname{Map}(B\check{P}, BQ)$$

is a mod-p equivalence.

Proof. The homotopy fibre of ι_Q is an Eilenberg-MacLane space of type $K(V,1)$, where V is a $\mathbb{Q}_p^\wedge$-vector space. For each homotopy class of maps $B\check{P} \xrightarrow{\alpha} BQ$, there is a fibration

$$BV^{h\check{P}} \longrightarrow \operatorname{Map}(B\check{P}, B\check{Q})_{\bar{\alpha}} \longrightarrow \operatorname{Map}(B\check{P}, BQ)_\alpha.$$

Since $\check{P}$ is discrete and V is a rational vector space, it follows that $BV^{h\check{P}}$ is again a space of type $K(U,1)$, where U is the invariant subspace in V under the $\check{P}$ action. In particular $BV^{h\check{P}}$ is connected and mod-p acyclic and so $\bar{\alpha}$ consists of a single component and

$$\operatorname{Map}(B\check{P}, B\check{Q})_{\bar{\alpha}} \xrightarrow{\iota_{Q\#}} \operatorname{Map}(B\check{P}, BQ)_\alpha$$

is a mod-p equivalence. □

Lemma A.23. *Let Q be a p-compact toral group and let $P \leq_\alpha Q$ be a subgroup. Then there is a sequence*

$$P = P_0 \leq P_1 \leq P_2 \leq \cdots \leq P_k \leq P_{k+1} \leq \cdots Q_0 \leq Q_1 \leq \cdots \leq Q_{n-1} \leq Q_n = Q$$

such that $BQ_0 = \operatorname{hocolim}_i BP_i$, and each group in the sequence is a normal subgroup of finite index in the following one.

Proof. Without loss of generality we may assume that α is a homomorphism (a pointed map). Consider first the case where P is of finite index in Q (in particular, P and Q have the same cohomological dimension). Let $P_0 \overset{\text{def}}{=} P$, $\alpha_0 \overset{\text{def}}{=} \alpha$, $P_1 \overset{\text{def}}{=} \mathcal{N}_Q(P_0, \alpha_0)$ and let $\alpha_1 \colon BP_1 \longrightarrow BQ$ denote η_{α_0} as defined in Proposition 2.5. Assuming P_k and $\alpha_k \colon BP_k \longrightarrow BQ$ have been defined, let

$$P_{k+1} \overset{\text{def}}{=} \mathcal{N}_Q(P_k, \alpha_k),$$

and let $\alpha_{k+1} \stackrel{\text{def}}{=} \eta_{\alpha_k}$. Then by Proposition 2.7(ii), (P_k, α_k) is a subgroup of Q, $P_k \lhd P_{k+1}$ for each $k \geq 0$ and $\mathcal{W}_{P_{k+1}}(P_k)$ is a non-trivial p-compact toral group. In particular $P_k \lneqq P_{k+1}$. An easy inductive argument also shows that

$$[Q\colon P] = (\prod_{i=0}^{k-1}[P_{i+1}\colon P_i]) \cdot [Q\colon P_k]$$

for each $k \geq 0$. Since $[Q\colon P]$ is finite, there is some $n \geq 0$, such that $P_n = Q$. This proves the claim in the finite index case. Notice also that in this case the sequence is finite

For the general case, let T_Q denote the identity component of Q and let π_Q denote its group of components. Let $\pi' \leq \pi_Q$ be the image of the composite $BP \to BQ \to B\pi_Q$ on fundamental groups. Define BQ' to be the pull-back space in the square

$$\begin{array}{ccc} BQ' & \longrightarrow & BQ \\ \Big\downarrow & & \Big\downarrow \\ B\pi' & \longrightarrow & B\pi_Q \end{array} .$$

Then $Q' \leq Q$ is a subgroup of Q of finite index $[Q\colon Q'] = [\pi_Q\colon \pi']$ by construction. The map $\alpha\colon BP \longrightarrow BQ$ factors through BQ' via a map, which by Lemma A.2 is a monomorphism, and by construction induces an epimorphism on fundamental groups. The map $BQ' \longrightarrow BQ$ resulting from this construction can be refined into a finite sequence of normal subgroups by the previous paragraph, and thus to complete the proof we only need to consider the case where α induces an epimorphism on groups of components.

Thus assume that the composite $BP \xrightarrow{\alpha} BQ \longrightarrow B\pi_Q$ induces an epimorphism on fundamental groups. Hence, its homotopy fibre is the classifying space of a subgroup $A \leq P$, which maps injectively to BT_Q. Thus we get a diagram of fibrations,

$$\begin{array}{ccccc} BA & \longrightarrow & BP & \longrightarrow & B\pi_Q \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow{=} \\ BT_Q & \longrightarrow & BQ & \longrightarrow & B\pi_Q. \end{array}$$

By Lemma A.22 if one picks discrete approximations $\check{P}, \check{Q}$ for P and Q respectively, there is a unique monomorphism $\check{\alpha}\colon \check{P} \longrightarrow \check{Q}$, such that $(B\check{\alpha})^\wedge_p \simeq \alpha$. Furthermore, the projection from $\check{Q}$ to the group of components π_Q is determined uniquely, and by taking kernels one gets discrete approximations for T_Q and A. Thus the left square in the diagram above turns into a square of discrete p-toral groups and homomorphisms between them

$$\begin{array}{ccc} \check{A} & \longrightarrow & \check{P} \\ \Big\downarrow & & \Big\downarrow \\ \check{T}_Q & \longrightarrow & \check{Q}. \end{array}$$

Now, let $\check{P}'_i$, $i \geq 0$, denote the subgroup of $\check{Q}$ generated by $\check{P}$ and all elements of order p^i in $\check{T}_Q$. Then $\check{P}'_0 = \check{P}$, $\bigcup_i \check{P}'_i = \check{Q}$, and $[\check{P}_{i+1}\colon \check{P}_i]$ is finite. Apply classifying spaces and p-completion to the resulting sequence to get a sequence

$$P = P'_0 \leq P'_1 \leq \cdots \leq P'_k \leq P'_{k+1} \leq \cdots Q,$$

such that each subgroup has finite index in the next one. Now apply the procedure for the finite index case to each pair (P'_k, P'_{k+1}) to complete the proof. □

Some categorical constructions. Standard references for this material is [HV] and [T]. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between small categories. For an object $d \in \mathcal{D}$, the "*undercategory*" $d{\downarrow}F$ is the category with objects given by pairs (c, α), where c is an object in $\mathcal{C}$, and $\alpha\colon F(c) \longrightarrow d$ is a morphism in $\mathcal{D}$. A morphism $(c, \alpha) \longrightarrow (c', \alpha')$ in $d{\downarrow}F$ is a morphism $\varphi\colon c \longrightarrow c'$ in $\mathcal{C}$, such that $\alpha' \circ F(\varphi) = \alpha$. The functor F is said to be "*right cofinal* if the nerve of the undercategory $d{\downarrow}F$ is contractible for every object d in $\mathcal{D}$. The "*overcategory*" $F{\downarrow}d$ is defined by analogy, and F is said to be "*left cofinal*" if $F{\downarrow}d$ is contractible for every $d \in \mathcal{D}$. If $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is right cofinal and $\phi\colon \mathcal{D} \longrightarrow \mathsf{Top}$ is any functor then the induced map

$$\operatorname*{hocolim}_{\mathcal{C}} F^*\phi \longrightarrow \operatorname*{hocolim}_{\mathcal{D}} \phi$$

is a homotopy equivalence.

Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between small categories, and let $\mathcal{C} \xrightarrow{\phi} \mathsf{Top}$ be any functor. The "*left homotopy Kan extension*" of ϕ along F is the functor $L_F(\phi)\colon \mathcal{D} \longrightarrow \mathsf{Top}$ defined on objects by

$$L_F(\phi)(d) \stackrel{\text{def}}{=} \operatorname*{hocolim}_{F\downarrow d} \phi \circ \iota,$$

where $\iota\colon F{\downarrow}d \longrightarrow \mathcal{C}$ is the obvious functor taking (c, α) to c. The left homotopy Kan extension has the property that there is a natural homotopy equivalence

$$\operatorname*{hocolim}_{\mathcal{D}} L_F(\phi) \simeq \operatorname*{hocolim}_{\mathcal{C}} F.$$

Let $\mathcal{C}at$ denote the category of small categories and functors between them. Given a functor $F\colon \mathcal{C} \longrightarrow \mathcal{C}at$, one obtains a functor $|F|\colon \mathcal{C} \longrightarrow \mathsf{Top}$ by composing F with the nerve followed by geometric realization functor $|-|$. The "*Grothendieck category*" $Gr(\mathcal{C}, F)$ is the category whose objects are pairs (c, x), where c is an object in $\mathcal{C}$, and x an object in $F(c)$. A morphism $(c, x) \longrightarrow (c', x')$ in $Gr(\mathcal{C}, F)$ is a pair (f, g), where $f\colon c \longrightarrow c'$ is a morphism in $\mathcal{C}$, and $g\colon F(f)(x) \longrightarrow x'$ is a morphism in $F(c')$. Composition of morphism is defined by $(f, g) \circ (f', g') = (f \circ f', g \circ F(f)(g'))$. The Grothendieck category has the property that the realization of its nerve is homotopy equivalent to $\operatorname{hocolim}_{\mathcal{C}} |F|$, where $|F|$ denotes the functor F followed by the geometric realization of the nerve.

Appendix B. Subgroup decompositions for classifying spaces of compact Lie groups

The main theorem of this paper is the existence of a subgroup homology decomposition for p-compact groups with respect to the collection of all their radical subgroups. The first such decomposition was constructed for classifying space of compact Lie groups by Jackowski, McClure and Oliver [JMO]. In this appendix we show that our main theorem is in fact a generalization of the Jackowski-McClure-Oliver result. By this we mean that the orbit category of radical subgroups, as defined in [JMO] is equivalent to the orbit category of radical subgroups constructed in this paper from the homotopy type of the respective p-completed classifying space. Furthermore, the decomposition functor constructed in this paper, and the one used in [JMO] also coincide up to homotopy, as we explain below. We start by recalling the basic construction from [JMO].

For a compact Lie group G, the orbit category $\mathcal{O}_p(G)$ is a category whose objects are G-orbits G/P, where $P \leq G$ is a p-toral subgroup, and whose morphisms are G-maps $G/P \longrightarrow G/Q$. The morphism set $\mathrm{Mor}_{\mathcal{O}_p(G)}(G/P, G/Q)$ can be identified with the fixed point set $(G/Q)^P$. We call this category "*the group theoretic orbit category of all p-toral subgroups of G*". Let $\mathcal{O}^r(G) \subset \mathcal{O}_p(G)$ denote the full subcategory whose objects are orbits G/P, where $P \leq G$ is a p-toral p-radical subgroup of G, i.e. those subgroups P whose Weyl group $W_G(P) \overset{\text{def}}{=} N_G(P)/P$ is finite and p-reduced. There is a functor

$$\Phi_{\mathsf{g}} \colon \mathcal{O}^r(G) \longrightarrow \mathsf{Top},$$

which takes an orbit G/P to the homotopy orbit space $(G/P)_{hG} \overset{\text{def}}{=} G/P \times_G EG$, and a G-map $G/P \longrightarrow G/Q$ to the induced map. Furthermore, the obvious natural transformation from the forgetful functor $\mathcal{O}^r(G) \longrightarrow G\text{-}\mathsf{Top}$ to the constant functor with value a point induces a natural transformation $\xi \colon \Phi_{\mathsf{g}} \longrightarrow 1_{BG}$. Thus one gets a map

$$\xi_* \colon \underset{\mathcal{O}^r(G)}{\mathrm{hocolim}}\, \Phi_{\mathsf{g}} \longrightarrow BG,$$

which by [JMO] is a mod-p equivalence.

If G is a compact Lie group and $\pi_0(G)$ is a finite p-group, then $G^\wedge_p$ is a p-compact group with classifying space $B(G^\wedge_p) \simeq (BG)^\wedge_p$. The respective orbit category, as defined in this paper, is called "*the homotopy theoretic orbit category of all p-toral subgroups of G*". Our aim is to show that there the group theoretic orbit category of p-radical subgroups of G is equivalent to the orbit category of radical subgroups of $G^\wedge_p$. We will also observe that this claim fails if one does not restrict to p-radical subgroups.

Let

$$\varphi_G \colon \mathcal{O}^r(G) \longrightarrow \mathcal{O}_p(G^\wedge_p)$$

be the functor taking an object G/Q to the p-compact toral subgroup $(Q^\wedge_p, \iota_Q)$, where $\iota_Q \colon BQ^\wedge_p \to BG^\wedge_p$ is the p-completion of the map

$$BQ \simeq (G/Q)_{hG} \longrightarrow *_{hG} = BG.$$

For a morphism $G/Q \xrightarrow{a} G/Q'$ in $\mathcal{O}^r(G)$, $\varphi_G(a)$ is defined to be the homotopy class of the induced map.

Proposition B.1. *Let G be a compact Lie group such that $\pi_0(G)$ is a finite p-group. Then, the functor φ_G takes values in $\mathcal{O}^r(G^\wedge_p)$ and*

$$\varphi_G \colon \mathcal{O}^r(G) \longrightarrow \mathcal{O}^r(G^\wedge_p)$$

is an equivalence of categories.

Let $\Phi \colon \mathcal{O}^r(G^\wedge_p) \longrightarrow \mathsf{Top}$ be any subgroup decomposition functor, and consider the composite functor $\Phi \circ \varphi_G$ on $\mathcal{O}^r(G)$. By Proposition B.1, $\mathcal{O}^r(G)$ can be identified with $\mathcal{O}^r(G^\wedge_p)$, and the functors $(\Phi_{\mathsf{g}})^\wedge_p$ and $\Phi \circ \varphi_G$ are clearly subgroup approximation functors on it. By Proposition 1.5, these two functors are naturally homotopy equivalent. Hence, one obtains a homotopy equivalence

$$\underset{\mathcal{O}^r(G)}{\mathrm{hocolim}} (\Phi_{\mathsf{g}})^\wedge_p \simeq \underset{\mathcal{O}^r(G)}{\mathrm{hocolim}}\, \Phi \circ \varphi_G.$$

This shows that our decomposition, restricted to the class of p-compact groups which arise as the p-completed classifying spaces of appropriate Lie groups, coincides with the Jackowski-McClure-Oliver decomposition, up to p-completion.

The following two lemmas are needed for the proof of Proposition B.1. Recall that a p-toral group is an extension of a finite p-group π by a torus $T = (S^1)^n$ for some $n \geq 0$. A space X is said to be p-good if the completion map $X \longrightarrow X^\wedge_p$ is a mod-p equivalence.

Lemma B.2. *Let Q be a p-toral compact Lie group and K a p-good finite Q-complex. Then, $(K^Q)^\wedge_p \simeq (K^\wedge_p)^{h(Q^\wedge_p)}$.*

Proof. By the generalized Sullivan conjecture for p-toral compact Lie groups [N], the map $K^Q \longrightarrow (K^\wedge_p)^{hQ}$ is a mod-p equivalence. Hence, we have to show that $(K^\wedge_p)^{hQ} \simeq (K^\wedge_p)^{h(Q^\wedge_p)}$. As homotopy fixed point sets, these spaces are given as the fibre of the left and right vertical arrows in the diagram

$$\begin{array}{ccccc}
\mathrm{Map}(BQ, (K^\wedge_p)_{hQ})_{\{id\}} & \xrightarrow{l_*} & \mathrm{Map}(BQ, (K^\wedge_p)_{h(Q^\wedge_p)})_{\{l\}} & \underset{\simeq}{\xleftarrow{l^*}} & \mathrm{Map}(BQ^\wedge_p, (K^\wedge_p)_{h(Q^\wedge_p)})_{\{id\}} \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
\mathrm{Map}(BQ, BQ)_{id} & \xrightarrow{l_*} & \mathrm{Map}(BQ, BQ^\wedge_p)_l & \underset{\simeq}{\xleftarrow{l^*}} & \mathrm{Map}(BQ^\wedge_p, BQ^\wedge_p)_{id},
\end{array}$$

where l denotes the completion map $BQ \longrightarrow BQ^\wedge_p$, and the map π denotes, in each case, the map induced by the projection from the homotopy orbit space to the respective classifying space. Since $(K^\wedge_p)_{h(Q^\wedge_p)}$ and $BQ^\wedge_p$ are p-complete, both arrows marked l^* are homotopy equivalences, and so the homotopy fibres of the right and middle vertical arrows are equivalent. The left square arises by applying the functor $\mathrm{Map}(BQ, -)$ to a pull-back diagram and is therefore itself a pull-back diagram. This shows that homotopy fibres of all vertical arrows homotopy equivalent and finishes the proof. □

Lemma B.3. *Let G be a compact Lie group such that $\pi_0(G)$ is a finite p-group. Let (Q, β) be a p-compact toral subgroup of $G^\wedge_p$, which is either finite or radical in $BG^\wedge_p$. Then there exists a p-toral subgroup $P \leq G$ and a mod-p equivalence $h \colon BP \longrightarrow BQ$ such that the diagram*

$$\begin{array}{ccc}
BP & \xrightarrow{B\iota_P} & BG \\
\downarrow{\scriptstyle h} & & \downarrow \\
BQ & \xrightarrow{\beta} & BG^\wedge_p
\end{array}$$

commutes up to homotopy, where $P \xrightarrow{\iota_P} G$ is the inclusion.

Proof. If Q is a finite p-group, then $\mathrm{Map}(BQ, BG^\wedge_p) \simeq \mathrm{Map}(BQ, BG)^\wedge_p$ (see for instance [BL, Proposition 2.1]. In particular both sides have the same path components, and the components in the right hand side are given by $\mathrm{Rep}(Q, G) \overset{\mathrm{def}}{=} \mathrm{Hom}(Q, G)/\sim$, where the equivalence relation is given by conjugation in G. Thus let $Q \xrightarrow{\varphi} G$ be a homomorphism, such that $B\varphi \simeq \beta$, let $P \overset{\mathrm{def}}{=} \mathrm{Im}\,\varphi$, and let $BP \xrightarrow{h} BQ$ the map induced by the inverse of φ (considered as a map $Q \to P$). Then the statement holds for P and h.

Let G be a compact Lie group with $\pi_0(G)$ a finite p-group. Let $O_p(G)$ denote the maximal normal p-toral subgroup of G, and let $\overline{G}$ denote the quotient group $G/O_p(G)$. (Notice that $O_p(G)$ exists, since it can be taken to be the intersection of all Sylow subgroups of G) Then $\overline{G}$ contains no normal p-toral subgroup, and in particular its center contains no such subgroup. On the other hand, since $Z(\overline{G})$ is an abelian compact Lie group, it is isomorphic to a product of a torus and a finite abelian group. Hence,

$Z(\overline{G})$ is a finite group of order prime to p. Notice also that $\pi_0(\overline{G})$ is a finite p-group, since $\overline{G}$ is a quotient group of G. Thus $\overline{G}^\wedge_p$ is a p-compact group, and since $Z(\overline{G})$ is a finite group of order prime to p, $\overline{G}^\wedge_p$ is centerfree. By Proposition 5.4, there is a 1–1 correspondence between isomorphism classes of p-compact toral radical subgroups of $G^\wedge_p$ and those of $\overline{G}^\wedge_p$ (considered as objects of the orbit category in both cases). Hence, it suffices to prove the claim for compact Lie groups whose center is a finite group of order prime to p.

The claim is obvious if G is p-toral, since the only radical subgroup of $G^\wedge_p$ in that case is $G^\wedge_p$ itself. The proof now proceeds by induction on the order. Let G be an arbitrary compact Lie group, such that $\pi_0(G)$ is a finite p-group. Assume the lemma holds for all compact Lie groups H, satisfying the same condition, and such that $|H| \lneqq |G|$. We must show that it holds for G.

By the discussion above, we may assume that $Z(G)$ is finite of order prime to p. Thus $G^\wedge_p$ is a centerfree p-compact group. Let $Q \leq_\beta G^\wedge_p$ be a radical p-compact toral subgroup, and let $\mathsf{E}(Q)$ be the maximal central elementary abelian subgroup in Q. Since $\mathsf{E}(Q)$ is a finite p-group, we may assume by the discussion above that $\mathsf{E}(Q)$ is a p-subgroup of G. Since

$$C_G(\mathsf{E}(Q))^\wedge_p \simeq \Omega(\mathrm{Map}(B\mathsf{E}(Q), BG)_{inc})^\wedge_p \simeq \Omega(\mathrm{Map}(B\mathsf{E}(Q), BG^\wedge_p)_{inc}) = \mathcal{C}_{G^\wedge_p}(\mathsf{E}(Q)),$$

and since $Q \leq \mathcal{C}_{G^\wedge_p}(\mathsf{E}(Q))$ is radical there by Lemma 4.3, it suffices to prove the claim for $C_G(\mathsf{E}(Q))$. But since $Z(G)$ is finite of order prime to p, $|C_G(\mathsf{E}(Q))| \lneqq |G|$ and the claim follows from the induction hypothesis. □

Proof of Proposition B.1. Fix a compact Lie group G with $\pi_0(G)$ a finite p-group. For radical p-toral subgroups $Q, Q' \leq G$, the fixed point set $(G/Q')^Q$ is finite or empty. Hence there are homotopy equivalences

$$(G/Q')^Q \simeq (G/Q')^{Q^\wedge_p} \simeq ((G/Q')^\wedge_p)^{h(Q^\wedge_p)},$$

where the second equivalence follows from Lemma B.2. This shows that $Q \leq G$ is radical if and only if $Q^\wedge_p \leq G^\wedge_p$ is radical, and so the functor φ_G takes values in the category $\mathcal{O}^r(G^\wedge_p)$. Furthermore, since the morphisms in the respective categories are the path components of the left and right hand sides of the spaces in the equation above, φ_G induces an isomorphism on morphism sets. It is also clear that φ_G is an injection on isomorphism classes of objects, and by Lemma B.3, it is also an epimorphism on the isomorphism classes of objects. Thus φ_G is an equivalence of categories as stated. □

We end this appendix with the observation that Lemma B.3 (and hence our argument in the proof of Lemma B.1) fails if one does not require that the subgroup $Q \leq_\beta G^\wedge_p$ is either finite or radical.

Remark B.4. Let $G \overset{\text{def}}{=} S^1 \times S^1$, let $\alpha, \beta \in \mathbb{Z}^\wedge_p$ be units, and let $f\colon (BS^1)^\wedge_p \longrightarrow BG^\wedge_p$ be a map induced by the monomorphism $\mathbb{Z}^\wedge_p \xrightarrow{(\alpha,\beta)} \mathbb{Z}^\wedge_p \times \mathbb{Z}^\wedge_p$, sending 1 to (α, β). If Lemma B.3 held with respect to this setup, it would mean that there is a map $g\colon BS^1 \longrightarrow BG$ and a mod-p equivalence $h\colon BS^1 \longrightarrow (BS^1)^\wedge_p$, such that $g^\wedge_p \simeq f \circ h^\wedge_p$. But g must be induced by a monomorphism $\mathbb{Z} \xrightarrow{(a,b)} \mathbb{Z} \times \mathbb{Z}$, sending 1 to the pair (a, b) for some integers a, b, whereas h is induced by multiplication by some p-adic unit u. An easy calculation now shows that $\frac{\alpha}{\beta} = \frac{a}{b}$, and since the right hand side is a rational number, there are clearly choices of α and β, where this equation cannot hold. Thus the lemma fails in this case.

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