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# POSTNIKOV PIECES AND $B\mathbb{Z}/p$-HOMOTOPY THEORY

NATÀLIA CASTELLANA, JUAN A. CRESPO, AND JÉRÔME SCHERER

Abstract. We present a constructive method to compute the cellularization with respect to $B^m\mathbb{Z}/p$ for any integer $m \geq 1$ of a large class of $H$-spaces, namely all those which have a finite number of non-trivial $B^m\mathbb{Z}/p$-homotopy groups (the pointed mapping space $\mathrm{map}_*(B^m\mathbb{Z}/p, X)$ is a Postnikov piece). We prove in particular that the $B^m\mathbb{Z}/p$-cellularization of an $H$-space having a finite number of $B^m\mathbb{Z}/p$-homotopy groups is a $p$-torsion Postnikov piece. Along the way, we characterize the $B\mathbb{Z}/p^r$-cellular classifying spaces of nilpotent groups.

## Introduction

The notion of $A$-homotopy theory was introduced by Dror Farjoun [9] for an arbitrary connected space $A$. Here $A$ and its suspensions play the role of the spheres in classical homotopy theory and so the $A$-homotopy groups of a space $X$ are defined to be the homotopy classes of pointed maps $[\Sigma^i A, X]$. The analogue to weakly contractible spaces are those spaces for which all $A$-homotopy groups are trivial. This means that the pointed mapping space $\mathrm{map}_*(A, X)$ is contractible, i.e. $X$ is an $A$-null space. On the other hand, the classical notion of $CW$-complex is replaced by the one of $A$-cellular space. Such spaces can be constructed from $A$ by means of pointed homotopy colimits.

Thanks to work of Bousfield [2] and Dror Farjoun [9] there is a functorial way to study $X$ through the eyes of $A$. The nullification $P_A X$ is the biggest quotient of $X$ which is $A$-null and $CW_A X$ is the best $A$-cellular approximation of the space $X$. Roughly speaking, $CW_A X$ contains all the transcendent information of the mapping space $\mathrm{map}_*(A, X)$, since the latter is equivalent to $\mathrm{map}_*(A, CW_A X)$. Hence, explicit computation of the cellularization would give access to information about $\mathrm{map}_*(A, X)$. The importance of mapping spaces (in the case $A = B\mathbb{Z}/p$) is well established thank to Miller's solution to the Sullivan conjecture [17] and later work.

While many computations of $P_A X$ are present in the literature, very few computations of $CW_A X$ are avalaible. For instance, Chachólski describes a strategy to compute the cellularization $CW_A X$ in [7]. His method has been successfully applied in some cases (cellularization with respect to Moore spaces [21], $B\mathbb{Z}/p$-cellularization of classifying spaces of finite groups [10]), but it is in general difficult to apply.

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An alternative way to compute $CW_A X$ is the following. The nullification map $l : X \to P_A X$ provides an equivalence $CW_A X \simeq CW_A \overline{P}_A X$, where, as usual, $\overline{P}_A X$ denotes the homotopy fiber of $l$. This equivalence gives a strategy when $\overline{P}_A X$ is known. Assume for example that $X$ is $A$-null. Then $\overline{P}_A X$ is contractible and thus, so is $CW_A X$. From the $A$-homotopy point of view, the next case in which the $A$-cellularization should be accessible is when $X$ has only a finite number of $A$-homotopy groups, that is, some iterated loop space $\Omega^n X$ is $A$-null. Natural examples of spaces satisfying this condition are the $n$-connected covers of $A$-null spaces.

Let us specialize to $H$-spaces and $A = B^m\mathbb{Z}/p$. Bousfield has determined in [2] the fiber of the nullification map $X \to P_{B^m\mathbb{Z}/p} X$ when $\Omega^n X$ is $B^m\mathbb{Z}/p$-null. He shows that, for such an $H$-space, $\overline{P}_{B^m\mathbb{Z}/p} X$ is a $p$-torsion Postnikov piece $F$, whose homotopy groups are concentrated in dimensions from $m$ to $m+n-1$. As $F$ is also an $H$-space (because $l$ is an $H$-map), we call it an *H-Postnikov piece*. The cellularization of $X$ (which is again an $H$-space because $CW_A$ preserves $H$-structures) therefore coincides with that of a Postnikov piece. In Section 3, we explain how to compute the cellularization of Postnikov pieces and this enables us to obtain our main result.

**Theorem 5.3.** *Let $X$ be a connected $H$-space such that $\Omega^n X$ is $B^m\mathbb{Z}/p$-null. Then*

$$CW_{B^m\mathbb{Z}/p} X \simeq F \times K(W, m),$$

*where $F$ is a $p$-torsion $H$-Postnikov piece with homotopy groups concentrated in dimensions from $m+1$ to $m+n-1$ and $W$ is an elementary abelian $p$-group.*

Thus, when $X$ is an $H$-space with only a finite number of $B^m\mathbb{Z}/p$-homotopy groups, the cellularization $CW_{B^m\mathbb{Z}/p} X$ is a $p$-torsion $H$-Postnikov piece. This is not true in general if we do not assume $X$ to be an $H$-space. For instance, the $B\mathbb{Z}/p$-cellularization of $B\Sigma_3$ is a space with infinitely many non-trivial homotopy groups [11]. Also, it is not true for an arbitrary space $A$ that the $A$-cellularization of an $H$-space having a finite number of $A$-homotopy group is always a Postnikov piece. This fails, for example, when $A$ is the product of the $K(\mathbb{Z}/p, p)$'s, where $p$ runs over the set of all primes, but it could be true for any $n$-supported $p$-torsion space $A$ (in the terminology of [2]).

In our previous work [6], we analyzed a large class of $H$-spaces which fits into the present framework. Namely, if the mod $p$ cohomology of an $H$-space $X$ is finitely generated as an algebra over the Steenrod algebra, then there must exist an integer $n$ such that $\Omega^n X$ is $B\mathbb{Z}/p$-null. Hence, we obtain the following.

**Proposition 4.2.** *Let $X$ be a connected $H$-space such that $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then*

$$CW_{B\mathbb{Z}/p} X \simeq F \times K(W, 1),$$

*where $F$ is a 1-connected $p$-torsion $H$-Postnikov piece and $W$ is an elementary abelian $p$-group. Moreover, there exists an integer $k$ such that $CW_{B^m\mathbb{Z}/p} X \simeq *$ for any $m \geq k$.*

Our results allow explicit computations which we exemplify by computing in Proposition 4.3 the $B\mathbb{Z}/p$-cellularization of the $n$-connected cover of any finite $H$-space, as well as the $B^m\mathbb{Z}/p$-cellularizations of the classifying spaces for real and complex vector bundles $BU$, $BO$, and their connected covers $BSU$, $BSO$, $BSpin$, and $BString$-see Proposition 5.6.

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## 1. A double filtration of the category of spaces

As mentioned in the Introduction, the condition that $\Omega^n X$ be $B^m\mathbb{Z}/p$-null will enable us to compute the $B^m\mathbb{Z}/p$-cellularization of $H$-spaces. This section is devoted to giving a picture of how such spaces are related for different choices of $m$ and $n$.

First of all, we present a lemma which collects various facts that are needed in the rest of the paper.

**Lemma 1.1.** *Let $X$ be a connected space and $m > 0$. Then,*

1. *If $X$ is $B^m\mathbb{Z}/p$-null, then $\Omega^n X$ is $B^m\mathbb{Z}/p$-null for all $n \geq 1$.*
2. *If $X$ is $B^m\mathbb{Z}/p$-null, then it is $B^{m+s}\mathbb{Z}/p$-null for all $s \geq 0$.*
3. *If $\Omega X$ is $B^m\mathbb{Z}/p$-null, then $X$ is $B^{m+s}\mathbb{Z}/p$-null for all $s \geq 1$.*

*Proof.* For (1), simply apply $\mathrm{map}_*(B\mathbb{Z}/p, -)$ to the path fibration $\Omega X \to * \to X$.

Statement (2) is given by Dwyer's version of Zabrodsky's lemma [8, Prop. 3.4] applied to the universal fibration $B^m\mathbb{Z}/p \to * \to B^{m+1}\mathbb{Z}/p$.

Finally, (3) is proven like (2), using Zabrodsky's lemma in its connected version [8, Prop. 3.5] (see also Lemma 2.3). Recall that if $\Omega X$ is $B^m\mathbb{Z}/p$-null, then the component $\mathrm{map}(B^m\mathbb{Z}/p, X)_c$ of the constant map is weakly equivalent to $X$. □

Of course, the converses of the previous results are not true. For the first statement, take the classifying space of a discrete group at $m = 1$. For the second and third, consider $X = BU$. It is a $B^2\mathbb{Z}/p$-null space (see Example 1.4), but neither $BU$ nor $\Omega BU$ are $B\mathbb{Z}/p$-null. Observe that in fact $\Omega^n BU$ is never $B\mathbb{Z}/p$-null. The next result shows that this is the general situation. That is, if a connected space $X$ is $B^{m+1}\mathbb{Z}/p$-null, then either $\Omega X$ is $B^m\mathbb{Z}/p$-null or none of the iterated loop spaces $\Omega^n X$ is $B^m\mathbb{Z}/p$-null for $n \geq 1$.

**Theorem 1.2.** *Let $X$ be a $B^{m+1}\mathbb{Z}/p$-null space such that $\Omega^k X$ is $B^m\mathbb{Z}/p$-null for some $k > 0$. Then $\Omega X$ is $B^m\mathbb{Z}/p$-null.*

*Proof.* It is enough to prove the result for $k = 2$. Consider the fibration

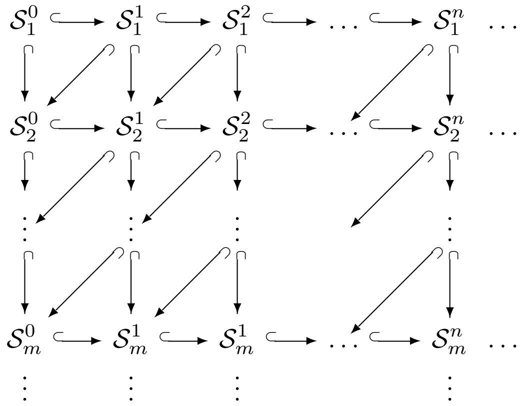
$$K(Q, m+1) \longrightarrow P_{\Sigma^2 B^m\mathbb{Z}/p}X \simeq X \longrightarrow P_{\Sigma B^m\mathbb{Z}/p}X,$$

where the fiber is a $p$-torsion Eilenberg-Mac Lane space by Bousfield's description of the fiber of the $\Sigma B^m\mathbb{Z}/p$-nullification [2, Theorem 7.2]. The base space is $B^{m+1}\mathbb{Z}/p$-null by Lemma 1.1.(3) and so is the total space, by assumption. Thus, the pointed mapping space $\mathrm{map}_*\big(B^{m+1}\mathbb{Z}/p, K(Q, m+1)\big)$ must be contractible as well, i.e. $Q = 0$. □

The previous analysis leads to a double filtration of the category of spaces. Let $n \geq 0$ and $m \geq 1$. We introduce the notation

$$\mathcal{S}^n_m = \{X | \ \Omega^n X \text{ is } B^m\mathbb{Z}/p\text{-null}\}.$$

then Lemma 1.1 yields a diagram of inclusions:



**Example 1.3.** We give examples of spaces in every stage of the filtration.

(1) $\mathcal{S}_1^0$ are the spaces that are $B\mathbb{Z}/p$-null. This contains in particular any finite space (by Miller's theorem [17, Thm. A]), and, for a nilpotent space $X$ (of finite type with finite fundamental group), to be $B\mathbb{Z}/p$-null is equivalent to its cohomology $H^*(X;\mathbb{F}_p)$ being locally finite by [22, Corollary 8.6.2].
(2) If $X\langle n\rangle$ denotes the $n$-connected cover of a space $X$, then the homotopy fiber of $\Omega^{n-1}X\langle n\rangle \to \Omega^{n-1}X$ is a discrete space. Hence, if $X \in \mathcal{S}_m^0$, then $X\langle n\rangle \in \mathcal{S}_m^{n-1}$.
(3) Observe that $\mathcal{S}_m^n \subset \mathcal{S}_{m+k}^{n-k}$ for all $0 \leq k \leq n$.
(4) The previous examples provide spaces in every stage of the double filtration. Consider a finite space. It is automatically $B\mathbb{Z}/p$-null. Its $n$-connected cover $X\langle n\rangle$ lies in $\mathcal{S}_1^{n-1}$, hence also in $\mathcal{S}_{k+1}^{n-k-1}$ for all $0 \leq k \leq n$.

The next example provides a number of spaces living in $\mathcal{S}_m^0$ which do not come from the first row of the filtration. Of course their connected covers will be *new* examples of spaces living in $\mathcal{S}_m^n$.

**Example 1.4.** Let $E_*$ be a homology theory. If $\tilde{E}^i\big(K(\mathbb{Z}/p,m)\big) = 0$ for all $i$, then the spaces $E^i$ representing the corresponding homology theory are $B^m\mathbb{Z}/p$-null. If $\tilde{E}^j\big(K(\mathbb{Z}/p,m-1)\big) \neq 0$ for some $j$, then $E^j$ is not $B^{m-1}\mathbb{Z}/p$-null. In particular, if $E_*$ is periodic, it follows that the spaces $E^i$ are $B^m\mathbb{Z}/p$-null for all $i$, but none of their iterated loops are $B^{m-1}\mathbb{Z}/p$-null.

A first example of such behavior is obtained from complex K-theory: $BU$ is $B^2\mathbb{Z}/p$-null, but $BU$ and $U$ are not $B\mathbb{Z}/p$-null (see [18]). Note that real and quaternionic $K$-theory enjoy the same properties.

For every $m$, examples of homology theories following this pattern are given by $p$-torsion homology theories of type III-$m$ as described in [1]. The $m$th Morava $K$-theory $K(m)_*$ for $p$ odd is an example of such behavior with respect to Eilenberg-Mac Lane spaces. The spaces representing $K(m)_*$ are $B^{m+1}\mathbb{Z}/p$-null, but none of their iterated loops are $B^m\mathbb{Z}/p$-null.

Our aim is to provide tools to compute the $B^m\mathbb{Z}/p$-cellularization of any $H$-space lying in the $m$th row of the above diagram. The key point is the following result of Bousfield [2], which determines the fiber of the nullification map.

**Proposition 1.5.** *Let $n \geq 0$ and let $X$ be a connected $H$-space such that $\Omega^n X$ is $B^m\mathbb{Z}/p$-null. Then there is an $H$-fibration*

$$F \longrightarrow X \longrightarrow P_{B^m\mathbb{Z}/p}X,$$

*where $F$ is a $p$-torsion $H$-Postnikov piece whose homotopy groups are concentrated in dimensions from $m$ to $m+n-1$.* □

Therefore, since $F \to X$ is a $B^m\mathbb{Z}/p$-cellular equivalence, we only need to compute the cellularization of a Postnikov piece (which will end up being a Postnikov piece again; see Theorem 3.6). Actually, even more is true.

**Proposition 1.6.** *Let $X$ be a connected space such that $CW_{B^m\mathbb{Z}/p}X$ is a Postnikov piece. Then there exists an integer $n$ such that $\Omega^n X$ is $B^m\mathbb{Z}/p$-null.*

*Proof.* Let us loop once the Chachólski fibration $CW_{B^m\mathbb{Z}/p}X \to X \to P_{\Sigma B^m\mathbb{Z}/p}C$ (see [7, Theorem 20.5]). Since $\Omega P_{\Sigma B^m\mathbb{Z}/p}C$ is equivalent to $P_{B^m\mathbb{Z}/p}\Omega C$ by [9, Theorem 3.A.1], we get a fibration over a $B^m\mathbb{Z}/p$-null base space

$$\Omega CW_{B^m\mathbb{Z}/p}X \longrightarrow \Omega X \longrightarrow P_{B^m\mathbb{Z}/p}\Omega C.$$

Now there exists an integer $n$ such that $\Omega^n CW_{B^m\mathbb{Z}/p}X$ is discrete, thus $B^m\mathbb{Z}/p$-null. Therefore, so is $\Omega^n X$. □

## 2. Cellularization of fibrations over $BG$

In general, it is very difficult to compute the cellularization of the total space of a fibration. In this section, we explain how to deal with this problem when the base space is the classifying space of a discrete group. The first step applies to any group. In the second step - see Proposition 2.4 below, we specialize to nilpotent groups.

**Proposition 2.1.** *Let $r \geq 1$ and let $F \longrightarrow E \xrightarrow{\pi} BG$ be a fibration, where $G$ is a discrete group. Let $S$ be the (normal) subgroup generated by all elements $g \in G$ of order $p^i$ for some $i \leq r$ such that the inclusion $B\langle g\rangle \to BG$ lifts to $E$. Then the pullback of the fibration along $BS \to BG$*

$$\begin{array}{ccccc} E' & \xrightarrow{f} & E & \xrightarrow{p} & B(G/S) \\ \downarrow & & \downarrow \pi & & \| \\ BS & \longrightarrow & BG & \xrightarrow{p'} & B(G/S) \end{array}$$

*induces a $B\mathbb{Z}/p^r$-cellular equivalence $f : E' \to E$ on the total space level.*

*Proof.* We have to show that $f$ induces a homotopy equivalence on pointed mapping spaces $\mathrm{map}_*(B\mathbb{Z}/p^r, -)$. The top fibration in the diagram yields a fibration

$$\mathrm{map}_*(B\mathbb{Z}/p^r, E') \xrightarrow{f_*} \mathrm{map}_*(B\mathbb{Z}/p^r, E) \xrightarrow{p_*} \mathrm{map}_*\big(B\mathbb{Z}/p^r, B(G/S)\big).$$

Since the base is homotopically discrete, we only need to check that all components of the total space are sent by $p_*$ to the component of the constant. Thus consider a map $h : B\mathbb{Z}/p^r \to E$. The composite $p \circ h$ is homotopy equivalent to a map induced by a group homomorphism $\alpha\colon \mathbb{Z}/p^r \to G$ whose image $\alpha(1) = g$ is in $S$ by construction. Therefore $p \circ h = p' \circ \pi \circ h$ is null-homotopic. □

*Remark* 2.2. If the fibration in the above proposition is an $H$-fibration (in particular if $G$ is abelian), the set of elements $g$ for which there is a lift to the total space forms a subgroup of $G$. The central extension $Z(D_8) \hookrightarrow D_8 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ of the dihedral group $D_8$ provides an example where the subgroup $S$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$, but the element in $S$ represented by an element of order 4 in $D_8$ does not admit a lift.

The next lemma is a variation of Dwyer's version of Zabrodsky's Lemma in [8].

**Lemma 2.3.** *Let $F \longrightarrow E \xrightarrow{f} B$ be a fibration over a connected base, and let $A$ be a connected space such that $\Omega A$ is $F$-null. Then any map $g : E \to A$ which is homotopic to the constant map when restricted to $F$ factors through a map $h : B \to A$ up to unpointed homotopy and, moreover, $g$ is pointed null-homotopic if and only if $h$ is so.*

*Proof.* Since $\Omega A$ is $F$-null, we see that the component $\mathrm{map}_*(F, A)_c$ of the constant map is contractible and therefore, the evaluation at the base point $\mathrm{map}(F, A)_c \to A$ is an equivalence. By [8, Proposition 3.5], $f$ induces a homotopy equivalence

$$\mathrm{map}(B, A) \simeq \mathrm{map}(E, A)_{[F]}.$$

where $\mathrm{map}(E, A)_{[F]}$ denotes the space of maps $E \to A$ which are homotopic to the constant map when restricted to $F$.

Looking at the component of the constant map, we see that $\mathrm{map}(B, A)_c \simeq \mathrm{map}(E, A)_c$. Since any map homotopic to the constant map is also homotopic by a pointed homotopy, the result follows. □

**Proposition 2.4.** *Let $r \geq 1$ and let $F \xrightarrow{i} E \xrightarrow{\pi} BG$ be a fibration, where $G$ is a nilpotent group generated by elements of order $p^i$ with $i \leq r$. Assume that for each of these generators $x \in G$, the inclusion $B\langle x\rangle \to BG$ lifts to $E$. If $F$ is $B\mathbb{Z}/p^r$-cellular, then so is $E$.*

*Proof.* In [7], Chachólski describes the cellularization $CW_{B\mathbb{Z}/p^r}E$ as the homotopy fiber of the composite

$$f : E \longrightarrow C \longrightarrow P_{\Sigma B\mathbb{Z}/p^r}C,$$

where $C$ is the homotopy cofiber of the evaluation map $\bigvee_{[B\mathbb{Z}/p^r,E]} B\mathbb{Z}/p^r \to E$. This tells us that $E$ is cellular if the map $f$ is null-homotopic. Observe that if $f$ is null-homotopic, then the fiber inclusion $CW_{B\mathbb{Z}/p^r}E \to E$ has a section and therefore, $E$ is cellular, since it is a retract of a cellular space ([9, 2.D.1.5]).

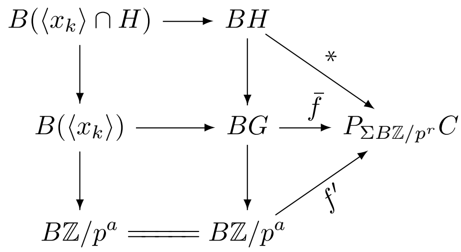
As the existence of an unpointed homotopy to the constant map implies the existence of a pointed one, we work now in the category of unpointed spaces. We remark that for any map $g : Z \to E$ from a $B\mathbb{Z}/p^r$-cellular space $Z$, the composite $f \circ g$ is null-homotopic, since $g$ factors through the cellularization of $E$. In particular, the composite $f \circ i$ is null-homotopic. By Lemma 2.3, there exists a map $\bar{f} : BG \to P_{\Sigma B\mathbb{Z}/p^r}C$ such that $\bar{f} \circ \pi \simeq f$ and, moreover, $f$ is null-homotopic if and only if $\bar{f}$ is so.

We first assume that $G$ is a finite group and show by induction on the order of $G$ that $\bar{f}$ is null-homotopic. If $|G| = p$, the existence of a section $s : BG \to E$ implies that $f \circ s = \bar{f}$ is null-homotopic since $BG = B\mathbb{Z}/p$ is cellular.

Let $\{x_1, \ldots, x_k\}$ be a minimal set of generators which admit a lift. Let $H \trianglelefteq G$ be the normal subgroup generated by $x_1, \ldots, x_{k-1}$ and their conjugates by powers of $x_k$. There is a short exact sequence

$$H \longrightarrow G \longrightarrow \mathbb{Z}/p^a,$$

where the quotient group is generated by the image of the generator $x_k$. Consider the fibration $F \to E' \to BH$ obtained by pulling back along $BH \to BG$, and denote by $h : E' \to E$ the induced map between the total spaces. Since $H$ satisfies the assumptions of the proposition, the induction hypothesis tells us that $E'$ is cellular and therefore, $f \circ h$ is null-homotopic. This implies that the restriction of $\bar{f}$ to $BH$ is null-homotopic. Consider the following diagram:




By Lemma 2.3, it is enough to show that $f'$ is null-homotopic. Again, applying Lemma 2.3 to the fibration on the left shows that $f'$ is null-homotopic since $\bar{f}$ restricted to $\langle x_k \rangle$ is so. Therefore, $\bar{f}$ is null-homotopic.

Assume now that $G$ is not finite. Any subgroup of $G$ generated by a finite number of elements of order a power of $p$ has a finite abelianization, and must therefore be itself finite by [20, Theorem 2.26]. Thus, $G$ is locally finite, i.e. $G$ is a filtered colimit of finite nilpotent groups generated by elements of order $p^i$ for $i \leq r$. Likewise, $BG$ is a filtered homotopy colimit of classifying spaces of finite groups (generated by finite subsets of the set of generators) which satisfy the hypotheses of the proposition. The total space $E$ can be obtained as a pointed filtered colimit of the total spaces obtained by pulling back the fibration. By the case when $G$ is finite, these total spaces are all cellular and therefore, so is $E$. □

Sometimes the existence of the "local" sections defined for every generator permits the construction of a global section of the fibration. By a result of Chachólski [7, Theorem 4.7], the total space of such a split fibration is cellular since $F$ and $BG$ are so. This is the case for an $H$-fibration and $E$ is then weakly equivalent to the product $F \times BG$.

A straightforward consequence of the above proposition (in the case when the fibration is the identity on $BG$) is the following characterization of the $B\mathbb{Z}/p^r$-cellular classifying spaces. For $r = 1$, we obtain R. Flores' result [10, Theorem 4.14].

**Corollary 2.5.** *Let $r \geq 1$ and let $G$ be a nilpotent group generated by elements of order $p^i$ with $i \leq r$. Then $BG$ is $B\mathbb{Z}/p^r$-cellular.* □

**Example 2.6.** The quaternion group $Q_8$ of order 8 is generated by elements of order 4. Therefore, $BQ_8$ is $B\mathbb{Z}/4$-cellular. We do not know an explicit way to construct $BQ_8$ as a pointed homotopy colimit of a diagram whose values are copies of $B\mathbb{Z}/4$.

We can now state the main result of this section. It provides a constructive description of the cellularization of the total space of certain fibrations over classifying spaces of nilpotent groups.

**Theorem 2.7.** *Let $G$ be a nilpotent group and let $F \longrightarrow E \longrightarrow BG$ be a fibration with $B\mathbb{Z}/p^r$-cellular fiber $F$. Then the cellularization of $E$ is the total space of a fibration $F \longrightarrow CW_{B\mathbb{Z}/p^r}E \longrightarrow BS$ where $S \lhd G$ is the (normal) subgroup generated by the $p$-torsion elements $g$ of order $p^i$ with $i \leq r$, such that the inclusion $B\langle g\rangle \to BG$ lifts to $E$.*

*Proof.* By Proposition 2.1, pulling back along $BS \to BG$ yields a cellular equivalence $f$ in the following square:

$$\begin{array}{ccc} E_S & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ BS & \longrightarrow & BG. \end{array}$$

By Proposition 2.4, the total space $E_S$ is cellular and therefore $E_S \simeq CW_{B\mathbb{Z}/p^r}E$. □

**Corollary 2.8.** *Let $G$ be a nilpotent group and let $S \lhd G$ be the (normal) subgroup generated by the $p$-torsion elements $g$ of order $p^i$ with $i \leq r$. Then $CW_{B\mathbb{Z}/p^r}BG \simeq BS$. Moreover, when $G$ is finitely generated, $S$ is a finite $p$-group.*

*Proof.* We only need to show that $S$ is a finite $p$-group. Notice that the abelianization of $S$ is $p$-torsion. Thus, $S$ is also a torsion group (see [23, Cor. 3.13]). Moreover, since $G$ is finitely generated, $S$ is finite, by [23, 3.10]. □

In fact, Theorem 2.7 also holds when the base space is an Eilenberg-Mac Lane space $K(G, n)$.

**Proposition 2.9.** *Let $n$ be an integer $\geq 2$ and let $G$ be a finitely generated abelian group of exponent dividing $p^r$. Consider a fibration $F \xrightarrow{i} E \xrightarrow{\pi} K(G, n)$ such that, for each generator $x \in G$, the inclusion $K(\langle x\rangle, n) \to K(G, n)$ lifts to $E$. If $F$ is $B\mathbb{Z}/p^r$-cellular, then so is $E$.* □

## 3. Cellularization of nilpotent Postnikov pieces

In this section, we compute the cellularization with respect to $B\mathbb{Z}/p^r$ of nilpotent Postnikov pieces. The main difficulty lies in the fundamental group, so it will be no surprise that these results hold as well for cellularization with respect to $B^m\mathbb{Z}/p^r$ with $m \geq 2$. We will often use the following closure property [9, Theorem 2.D.11].

**Proposition 3.1.** *Let $F \to E \to B$ be a fibration where $F$ and $E$ are $A$-cellular. Then so is $B$.* □

**Example 3.2.** [9, Corollary 3.C.10] The Eilenberg-Mac Lane space $K(\mathbb{Z}/p^k, n)$ is $B\mathbb{Z}/p^r$-cellular for any integer $k$ and any $n \geq 2$.

The construction of the cellularization is performed by looking first at the universal cover of the Postnikov piece. We start with the basic building blocks, the Eilenberg-Mac Lane spaces. For the structure results on infinite abelian groups, we refer the reader to Fuchs' book [12].

**Lemma 3.3.** *An Eilenberg-Mac Lane space $K(A, m)$, with $m \geq 2$, is $B\mathbb{Z}/p^r$-cellular if and only if $A$ is a $p$-torsion abelian group.*

*Proof.* It is clear that $A$ must be $p$-torsion. Thus, assume that $A$ is a $p$-torsion group. If $A$ is bounded, it is isomorphic to a direct sum of cyclic groups. Since cellularization commutes with finite products, $K(A, m)$ is $B\mathbb{Z}/p^r$-cellular when $A$ is a finite direct sum of cyclic groups. By taking a (possibly transfinite) telescope of $B\mathbb{Z}/p^r$-cellular spaces, we obtain that $K(A, m)$ is $B\mathbb{Z}/p^r$-cellular for any bounded group.

In general, $A$ splits as a direct sum of a divisible group $D$ and a reduced one $T$. A $p$-torsion divisible group is a direct sum of copies of $\mathbb{Z}/p^\infty$, which is a union of bounded groups. Thus, $K(D, m)$ is cellular. Now $T$ has a basic subgroup $P < T$, which is a direct sum of cyclic groups, and the quotient $T/P$ is divisible. So $K(T, m)$ is the total space of a fibration

$$K(P, m) \longrightarrow K(T, m) \longrightarrow K(D, m).$$

When $m \geq 3$, we are done because of the closure property Proposition 3.1. If $m = 2$, we have to refine the analysis of the fibration because $K(D, m-1)$ is not cellular. However, since $D$ is a union of bounded groups $D[p^k]$, the space $K(T, 2)$ is the telescope of total spaces $X_k$ of fibrations with cellular fiber $K(P, 2)$ and base $K(D[p^k], 2)$. We claim that these total spaces are cellular (and thus, so is $K(T, 2)$) and proceed by induction on the bound. Consider the subgroup $D[p^k] < D[p^{k+1}]$ whose quotient is a direct sum of cyclic groups $\mathbb{Z}/p$. Therefore, $X_{k+1}$ is the base space in a fibration

$$K(\oplus\mathbb{Z}/p, 1) \longrightarrow X_k \longrightarrow X_{k+1},$$

where the fiber and total space are cellular. We are done. □

We are now ready to prove that any simply connected $p$-torsion Postnikov piece is a $B\mathbb{Z}/p^r$-cellular space.

**Proposition 3.4.** *A simply connected Postnikov piece is $B\mathbb{Z}/p^r$-cellular if and only if it is $p$-torsion.*

*Proof.* Let $X$ be a simply connected $p$-torsion Postnikov piece. For some integer $m$, the $m$-connected cover $X\langle m\rangle$ is an Eilenberg-Mac Lane space, which is cellular by Lemma 3.3. Consider the principal fibration

$$K(\pi_m X, m-1) \longrightarrow X\langle m\rangle \longrightarrow X\langle m-1\rangle.$$

If $m \geq 3$, both $X\langle m\rangle$ and $K(\pi_m X, m-1)$ are cellular. It follows that $X\langle m-1\rangle$ is cellular by the closure property Proposition 3.1. An iteration of the same argument shows that $X\langle 2\rangle$ is cellular.

Thus, let us look at the fibration $X\langle 2\rangle \to X \to K(\pi_2 X, 2)$. The discussion in the proof of Lemma 3.3 also applies to the $p$-torsion group $\pi_2 X$. If this is a bounded group, say of exponent $p^k$, an induction on the bound shows that $X$ is actually the base space of a fibration where the total space is cellular, because its second homotopy group is of exponent $p^{k-1}$, and the fiber is cellular because it is of the form $K(V, 1)$, with $V$ a $p$-torsion abelian group of exponent $\leq p^r$. Then the closure property Proposition 3.1 ensures that $X$ is cellular.

If $\pi_2 X$ is divisible, $X$ is a telescope of cellular spaces, hence cellular. If it is reduced, taking a basic subgroup $B < \pi_2 X$ yields a diagram of fibrations

$$\begin{array}{ccccc} X\langle 2\rangle & \longrightarrow & Y & \longrightarrow & K(B,2) \\ \| & & \downarrow & & \downarrow \\ X\langle 2\rangle & \longrightarrow & X & \longrightarrow & K(\pi_2 X,2) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & K(D,2) & = & K(D,2), \end{array}$$

which exhibits $X$ as the total space of a fibration over $K(D,2)$ with $D$ divisible and a $B\mathbb{Z}/p^r$-cellular fiber. Therefore, by writing $D$ as a union of bounded groups as in the proof of Lemma 3.3, one obtains $X$ as a telescope of cellular spaces. Thus, $X$ is $B\mathbb{Z}/p^r$-cellular as well. □

*Remark* 3.5. The proof of the proposition holds in the more general setting where $X$ is a $p$-torsion space such that $X\langle m\rangle$ is $B\mathbb{Z}/p^r$-cellular for some $m \geq 2$. The proposition corresponds to the case when some $m$-connected cover $X\langle m\rangle$ is contractible.

Recall from [13, Corollary 2.12] that a connected space is nilpotent if and only if its Postnikov system admits a principal refinement

$$\cdots \longrightarrow X_s \longrightarrow X_{s-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0.$$

This means that each map $X_{s+1} \to X_s$ in the tower is a principal fibration with fiber $K(A_s, i_s - 1)$ for some increasing sequence of integers $i_s \geq 2$. We are only interested in finite Postnikov pieces, i.e. nilpotent spaces that can be constructed in a finite number of steps by taking homotopy fibers of $k$-invariants $X_s \to K(A_s, i_s)$.

The key step in the study of the cellularization of a nilpotent finite Postnikov piece is the analysis of principal fibrations (given in our case by the $k$-invariants).

**Theorem 3.6.** *Let $X$ be a $p$-torsion nilpotent Postnikov piece. Then there exists a fibration*

$$X\langle 1\rangle \longrightarrow CW_{B\mathbb{Z}/p^r}X \longrightarrow BS,$$

*where $S$ is the (normal) subgroup of $\pi_1 X$ generated by the elements $g$ of order $p^i$ with $i \leq r$, such that the inclusion $B\langle g\rangle \to B\pi_1 X$ admits a lift to $X$.*

*Proof.* By Proposition 3.4, the universal cover $X\langle 1\rangle$ is cellular and there is a fibration $X\langle 1\rangle \to X \to BG$, where $G = \pi_1 X$ is nilpotent. The result follows then from Theorem 2.7. □

## 4. Cellularization of $H$-spaces

In this section, we will use the computations of the cellularization of $p$-torsion nilpotent Postnikov systems to determine $CW_{B\mathbb{Z}/p}X$ when $X$ is an $H$-space. We prove:

**Theorem 4.1.** *Let $X$ be a connected $H$-space such that $\Omega^n X$ is $B\mathbb{Z}/p$-null. Then*

$$CW_{B\mathbb{Z}/p}X \simeq Y \times K(W,1),$$

*where $Y$ is a simply connected $p$-torsion $H$-Postnikov piece with homotopy groups concentrated in dimensions $\leq n$ and $W$ is an elementary abelian $p$-group.*

*Proof.* The fibration in Bousfield's result Proposition 1.5 yields a cellular equivalence between a connected $p$-torsion $H$-Postnikov piece $F$ and $X$. Theorem 3.6 thus applies. Moreover, since $F$ is an $H$-space as well, the subgroup $S$ is abelian and generated by elements of order $p$. Therefore, the $H$-fibration $F\langle 1\rangle \to CW_{B\mathbb{Z}/p}F \to K(W,1)$ admits a section (summing up the local sections) and the cellularization splits as a product. □

This result applies for $H$-spaces satisfying certain finiteness conditions.

**Proposition 4.2.** *Let $X$ be a connected $H$-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then*

$$CW_{B\mathbb{Z}/p}X \simeq F \times K(W,1),$$

*where $F$ is a 1-connected $p$-torsion $H$-Postnikov piece and $W$ is an elementary abelian $p$-group. Moreover, there exists an integer $k$ such that $CW_{B^m\mathbb{Z}/p}X \simeq *$ for any $m \geq k$.*

*Proof.* In [6], we prove that if $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra, then $\Omega^n X$ is $B\mathbb{Z}/p$-null for some $n \geq 0$. Hence, Theorem 4.1 applies and we obtain the desired result. In addition, Lemma 1.1 shows that $X$ is $B^{n+s+1}\mathbb{Z}/p$-null for any $s \geq 0$, which implies the second part of the result. □

The technique we propose in this paper is not only a nice theoretical tool which provides a general statement about what the $B\mathbb{Z}/p$-cellularization of $H$-spaces looks like. Our next result shows that one can actually identify precisely this new space when dealing with connected covers of finite $H$-spaces. Recall that by Miller's theorem [17, Thm. A], any finite $H$-space $X$ is $B\mathbb{Z}/p$-null and hence, $CW_{B\mathbb{Z}/p}X \simeq *$. The universal cover of $X$ is still finite and thus, $CW_{B\mathbb{Z}/p}(X\langle 1\rangle)$ is contractible as well. We can therefore assume that $X$ is 1-connected. The computation of the cellularization of the 3-connected cover is already implicit in [4].

**Proposition 4.3.** *Let $X$ be a simply connected finite $H$-space and let $k$ denote the rank of the free abelian group $\pi_3 X$. Then $CW_{B\mathbb{Z}/p}(X\langle 3\rangle) \simeq K(\oplus_k \mathbb{Z}/p, 1)$. For $n \geq 4$, up to $p$-completion, the universal cover of $CW_{B\mathbb{Z}/p}(X\langle n\rangle)$ is weakly equivalent to the 2-connected cover of $\Omega(X[n])$.*

*Proof.* By Browder's famous result [5, Theorem 6.11], $X$ is even 2-connected and its third homotopy group $\pi_3 X$ is free abelian (of rank $k$) by Hubbuck and Kane's theorem [14]. This means we have a fibration

$$K(\oplus_k \mathbb{Z}_{p^\infty}, 1) \longrightarrow X\langle 3\rangle \longrightarrow P_{B\mathbb{Z}/p}X\langle 3\rangle,$$

which shows that $CW_{B\mathbb{Z}/p}X\langle 3\rangle \simeq K(\oplus_k \mathbb{Z}/p, 1)$.

We deal now with the higher connected covers. Consider the following commutative diagram of fibrations

$$\begin{array}{ccccc}
F & = & F & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\Omega X[n] & \longrightarrow & X\langle n\rangle & \longrightarrow & X \\
\downarrow & & \downarrow & & \| \\
P_{B\mathbb{Z}/p}(\Omega X[n]) & \longrightarrow & P_{B\mathbb{Z}/p}(X\langle n\rangle) & \longrightarrow & X
\end{array}$$

where $F$ is a $p$-torsion Postnikov piece by [2, Thm 7.2] and the fiber inclusions are all $B\mathbb{Z}/p$-cellular equivalences, because the base spaces are $B\mathbb{Z}/p$-null. Therefore,

$$CW_{B\mathbb{Z}/p}(X\langle n\rangle) \simeq CW_{B\mathbb{Z}/p}F \simeq F\langle 1\rangle \times K(W,1).$$

We wish to identify $F\langle 1\rangle$. Since the fibrations in the diagram are nilpotent, by [3, II.4.8] they remain fibrations after $p$-completion. By Neisendorfer's theorem [19], the map $P_{B\mathbb{Z}/p}(X\langle n\rangle) \to X$ is an equivalence up to $p$-completion, which means that $P_{B\mathbb{Z}/p}\big(\Omega(X[n])\big)^\wedge_p \simeq *$. Thus $F^\wedge_p \simeq \big(\Omega(X[n])\big)^\wedge_p$. Notice that $\Omega(X[n])$ is simply connected and its second homotopy group is free by the above mentioned theorem of Hubbuck and Kane (which corresponds up to $p$-completion to the direct sum of $k$ copies of the Prüfer group $\mathbb{Z}/p^\infty$ in $\pi_1 F$). Hence, $F\langle 1\rangle$ coincides with $\big(\Omega(X[n])\big)\langle 2\rangle$ up to $p$-completion. □

To illustrate this result, we compute the $B\mathbb{Z}/2$-cellularization of the successive connected covers of $S^3$. The only delicate point is the identification of the fundamental group.

**Example 4.4.** Recall that $S^3$ is $B\mathbb{Z}/2$-null since it is a finite space. Thus, the cellularization $CW_{B\mathbb{Z}/2}S^3$ is contractible. Next, the fibration

$$K(\mathbb{Z}_{2^\infty},1) \to S^3\langle 3\rangle \to P_{B\mathbb{Z}/2}(S^3\langle 3\rangle)$$

shows that $CW_{B\mathbb{Z}/2}(S^3\langle 3\rangle) \simeq K(\mathbb{Z}/2,1)$. Finally, since $S^3[4]$ does not split as a product (the $k$-invariant is not trivial), we see that $CW_{B\mathbb{Z}/2}(S^3\langle 4\rangle) \simeq K(\mathbb{Z}/2,3)$. Likewise, for any integer $n \geq 4$, we have that $CW_{B\mathbb{Z}/2}(S^3\langle n\rangle)$ is weakly equivalent to the 2-completion of the 2-connected cover of $\Omega(S^3[n])$. The same phenomenon occurs at odd primes.

## 5. Cellularization with respect to $B^m\mathbb{Z}/p$

All the techniques developed for fibrations over $BG$ apply to fibrations over $K(G,n)$ when $n>1$ and we get the following results.

**Lemma 5.1.** *Let $m \geq 2$ and let $X$ be a connected space. Then $CW_{B^m\mathbb{Z}/p^r}X$ is weakly equivalent to $CW_{B^m\mathbb{Z}/p^r}(X\langle n-1\rangle)$.*

*Proof.* Consider the fibrations $X\langle i\rangle \longrightarrow X\langle i-1\rangle \longrightarrow K(\pi_i X, i)$. For $i<m$, the base space is $B^m\mathbb{Z}/p^r$-null and so $CW_{B^m\mathbb{Z}/p^r}(X\langle i\rangle) \simeq CW_{B^m\mathbb{Z}/p^r}(X\langle i-1\rangle)$. □

**Proposition 5.2.** *Let $m \geq 2$ and let $X$ be a $p$-torsion nilpotent Postnikov piece. Then there exists a fibration*

$$X\langle m\rangle \longrightarrow CW_{B^m\mathbb{Z}/p^r}X \longrightarrow K(W,m),$$

*where $W$ is a $p$-torsion subgroup of $\pi_m X$ of exponent dividing $p^r$.* □

**Theorem 5.3.** *Let $X$ be a connected $H$-space such that $\Omega^n X$ is $B^m\mathbb{Z}/p$-null. Then*

$$CW_{B^m\mathbb{Z}/p}X \simeq F \times K(W,m),$$

*where $F$ is a $p$-torsion $H$-Postnikov piece with homotopy groups concentrated in dimensions from $m+1$ to $m+n-1$, and $W$ is an elementary abelian $p$-group.* □

**Example 5.4.** Let $X$ denote "Milgram's space" (see [16]) the homotopy fiber of $Sq^2 : K(\mathbb{Z}/2,2) \to K(\mathbb{Z}/2,4)$. This is an infinite loop space. By Proposition 3.4, we know it is already $B\mathbb{Z}/2$-cellular. Since the $k$-invariant is not trivial, we see that $CW_{B^2\mathbb{Z}/2}X \simeq CW_{B^3\mathbb{Z}/2}X \simeq K(\mathbb{Z}/2,3)$.

Finally, we compute the cellularization of the (infinite loop) space $BU$ and its 2-connected cover $BSU$ with respect to Eilenberg-Mac Lane spaces $B^m\mathbb{Z}/p$. By Bott periodicity, this actually tells us the answer for all connected covers of $BU$.

**Example 5.5.** First of all, recall from Example 1.4 that $BU$ is $B^2\mathbb{Z}/p$-null since $\widetilde{K}^*(B^2\mathbb{Z}/p) = 0$ and its iterated loops are never $B\mathbb{Z}/p$-null. Therefore, the cellularization $CW_{B^m\mathbb{Z}/p}BU$ is contractible if $m \geq 2$. Since $BU \simeq BSU \times BS^1$, the same holds for $BSU$.

We now compute the $B^m\mathbb{Z}/p$-cellularization of $BO$ and its connected covers $BSO$, $BSpin$, and $BString$.

**Proposition 5.6.** *Let $m \geq 2$. Then*

(1) $CW_{B^m\mathbb{Z}/p}BO \simeq CW_{B^m\mathbb{Z}/p}BSO \simeq CW_{B^m\mathbb{Z}/p}BSpin \simeq *$,
(2) $CW_{B^m\mathbb{Z}/p}BString \simeq *$ *if* $m > 2$,
(3) $CW_{B^2\mathbb{Z}/p}BString \simeq K(\mathbb{Z}/p, 2)$ *and* $\mathrm{map}_*(B^2\mathbb{Z}/p, BString) \simeq \mathbb{Z}/p$.

*Proof.* In [15], W. Meier proves that real and complex $K$-theory have the same acyclic spaces, hence $BO$ is also $B^2\mathbb{Z}/p$-null. Therefore, $CW_{B^m\mathbb{Z}/p}BO$ is contractible for any $m \geq 2$. The 2-connected cover of $BO$ is $BSO$ and there is a splitting $BO \simeq BSO \times B\mathbb{Z}/2$, so that $CW_{B^m\mathbb{Z}/p}BSO \simeq *$.

The 4-connected cover of $BO$ is $BSpin$. From the fibration

$$BSpin \longrightarrow BSO \xrightarrow{w_2} K(\mathbb{Z}/2, 2),$$

we infer that the homotopy fiber of $BSpin \to BSO$ is $B\mathbb{Z}/2$. Since $BSO$ and $B\mathbb{Z}/2$ are $B^2\mathbb{Z}/p$-null, so is $BSpin$. Therefore, $CW_{B^m\mathbb{Z}/p}BSpin$ is contractible.

Finally, the 8-connected cover of $BO$ is $BString$. It is the homotopy fiber of $BSpin \xrightarrow{p_1/4} K(\mathbb{Z}, 4)$, where $p_1$ denotes the first Pontrjagin class. Consider the fibration

$$K(\mathbb{Z}, 3) \longrightarrow BString \longrightarrow BSpin,$$

where the base space is $B^m\mathbb{Z}/p$-null for $m \geq 2$. Together with the exact sequence $\mathbb{Z} \to \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}/p^\infty$, this implies that

$$CW_{B^m\mathbb{Z}/p}BString \simeq CW_{B^m\mathbb{Z}/p}K(\mathbb{Z}, 3) \simeq CW_{B^m\mathbb{Z}/p}K(\mathbb{Z}/p^\infty, 2).$$

This is a contractible space unless $m = 2$, when we obtain $K(\mathbb{Z}/p, 2)$. The explicit description of the pointed mapping space $\mathrm{map}_*(B^2\mathbb{Z}/p, BString)$ follows. □

Observe that the iterated loops of the $m$-connected covers of $BO$ and $BU$ are never $B\mathbb{Z}/p$-null. Hence, their cellularizations with respect to $B\mathbb{Z}/p$ must have infinitely many non-vanishing homotopy groups by Proposition 1.6.

## References


[1] A. K. Bousfield, *Constructions of factorization systems in categories*, J. Pure Appl. Algebra **9** (1976/77), no. 2, 207–220.

[2] ———, *Localization and periodicity in unstable homotopy theory*, J. Amer. Math. Soc. **7** (1994), no. 4, 831–873.

[3] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Springer-Verlag, Berlin, 1972, Lecture Notes in Mathematics, Vol. 304.

[4] C. Broto and J. A. Crespo, *H-spaces with Noetherian mod two cohomology algebra*, Topology **38** (1999), no. 2, 353–386.

[5] W. Browder, *Torsion in H-spaces*, Ann. of Math. (2) **74** (1961), 24–51.

[6] N. Castellana, J. A. Crespo, and J. Scherer, *Deconstructing Hopf spaces*, preprint available at http://front.math.ucdavis.edu/math.AT/0404031, 2004.
[7] W. Chachólski, *On the functors $CW_A$ and $P_A$*, Duke Math. J. **84** (1996), no. 3, 599–631.
[8] W. G. Dwyer, *The centralizer decomposition of $BG$*, Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994), Progr. Math., vol. 136, Birkhäuser, Basel, 1996, pp. 167–184.
[9] E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Springer-Verlag, Berlin, 1996.
[10] R. Flores, *Nullification and cellularization of classifying spaces of finite groups*, to appear in Trans. Amer. Math. Soc.
[11] R. Flores and J. Scherer, *Cellularization of classifying spaces and fusion properties of finite groups*, preprint available at http://front.math.ucdavis.edu/math.AT/0501442, 2005.
[12] L. Fuchs, *Infinite abelian groups. Vol. I*, Pure and Applied Mathematics, Vol. 36, Academic Press, New York, 1970.
[13] P. Hilton, G. Mislin, and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Publishing Co., Amsterdam, 1975, North-Holland Mathematics Studies, No. 15, Notas de Matemática, No. 55. [Notes on Mathematics, No. 55].
[14] J. R. Hubbuck and R. Kane, *On $\pi_3$ of a finite $H$-space*, Trans. Amer. Math. Soc. **213** (1975), 99–105.
[15] W. Meier, *Complex and real $K$-theory and localization*, J. Pure Appl. Algebra **14** (1979), no. 1, 59–71.
[16] R. J. Milgram, *The structure over the Steenrod algebra of some 2-stage Postnikov systems*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 161–169.
[17] H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) **120** (1984), no. 1, 39–87.
[18] G. Mislin, *Localization with respect to $K$-theory*, J. Pure Appl. Algebra **10** (1977/78), no. 2, 201–213.
[19] J.A. Neisendorfer, *Localization and connected covers of finite complexes*, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 385–390.
[20] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups. Part 1*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 62.
[21] J. L. Rodríguez and J. Scherer, *Cellular approximations using Moore spaces*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 357–374.
[22] L Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994.
[23] R. B. Warfield, Jr., *Nilpotent groups*, Springer-Verlag, Berlin, 1976, Lecture Notes in Mathematics, Vol. 513.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
*E-mail address*: natalia@mat.uab.es

Departament de Economia i de Història Econòmica, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
*E-mail address*: JuanAlfonso.Crespo@uab.es

Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
*E-mail address*: jscherer@mat.uab.es