RECTIFIABILITY VIA A SQUARE FUNCTION AND PREISS' THEOREM

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ABSTRACT. Let E be a set in $\mathbb{R}^d$ with finite n-dimensional Hausdorff measure $\mathcal{H}^n$ such that $\liminf_{r\to 0} r^{-n}\mathcal{H}^n(B(x,r)\cap E) > 0$ for $\mathcal{H}^n$-a.e. $x \in E$. In this paper it is shown that E is n-rectifiable if and only if

$$\int_0^1 \left| \frac{\mathcal{H}^n(B(x,r)\cap E)}{r^n} - \frac{\mathcal{H}^n(B(x,2r)\cap E)}{(2r)^n} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E,$$

and also if and only if

$$\lim_{r\to 0} \left(\frac{\mathcal{H}^n(B(x,r)\cap E)}{r^n} - \frac{\mathcal{H}^n(B(x,2r)\cap E)}{(2r)^n} \right) = 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E.$$

Other more general results involving Radon measures are also proved.

1. Introduction

A set $E \subset \mathbb{R}^d$ is called n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$, $i = 1, 2, \ldots$, such that

$$\mathcal{H}^n\left(\mathbb{R}^d \setminus \bigcup_i f_i(\mathbb{R}^n)\right) = 0, \tag{1.1}$$

where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. On the other hand, using Mattila's definition [Ma, Definition 16.6] we say that a a Radon measure μ on $\mathbb{R}^d$ is called n-rectifiable if μ vanishes out of a rectifiable set $E \subset \mathbb{R}^d$ and moreover μ is absolutely continuous with respect to $\mathcal{H}^n|_E$.

One of the main objectives of geometric measure theory consists in characterizing n-rectifiable sets and measures in different ways. For instance, there exist characterizations in terms of the existence of approximate tangent n-planes, in terms of the existence of densities, or in terms of the size of projections. These results stem from the works for the case $n = 1$ in the plane by Besicovitch at beginning of the last century and have been extended to the whole range of dimensions in the space by different authors. See for example the book by Mattila [Ma] for more details about this beautiful theory.

Preiss' theorem [Pr] is one of the great landmarks of geometric measure theory. This asserts that a Radon measure μ on $\mathbb{R}^d$ is n-rectifiable if and only if the density

$$\Theta^n(x,\mu) = \lim_{r\to 0} \frac{\mu(B(x,r))}{r^n} \tag{1.2}$$

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exists and is positive for μ-a.e. $x \in \mathbb{R}^d$. In [DKT] and [PTT] the authors proved that the rate of convergence of the density ratio to its limit yields additional information over the regularity of the support of the measure. In this paper we will prove two variants of Preiss' result. One can be considered as a square function version of Preiss' theorem. The other shows that the condition on the existence of the limit (1.2) can be weakened considerably. To state our results we need first to introduce some additional notation.

Given a Radon measure μ and $x \in \mathbb{R}^d$ we denote

$$\Theta^{n,*}(x,\mu) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^n}, \qquad \Theta^n_*(x,\mu) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^n}.$$

These are the upper and lower n-dimensional densities of μ at x. If they coincide, they are denoted by $\Theta^n(x,\mu)$. In the case when $\mu = \mathcal{H}^n|_E$ for some set $E \subset \mathbb{R}^d$, we also write $\Theta^{n,*}(x,E)$, $\Theta^n_*(x,E)$, $\Theta^n(x,E)$ instead of $\Theta^{n,*}(x,\mathcal{H}^n|_E)$, $\Theta^n_*(x,\mathcal{H}^n|_E)$, $\Theta^n(x,\mathcal{H}^n|_E)$, respectively.

The main result of this paper reads as follows.

Theorem 1.1. *Let μ be a Radon measure in $\mathbb{R}^d$ such that $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. The following are equivalent:*

(a) *μ is n-rectifiable.*

(b) $\displaystyle\int_0^1 \left| \frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r} < \infty$ *for μ-a.e. $x \in \mathbb{R}^d$.*

(c) $\displaystyle\lim_{r\to 0} \left(\frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n} \right) = 0$ *for μ-a.e. $x \in \mathbb{R}^d$.*

The following is an immediate consequence.

Corollary 1.2. *Let $E \subset \mathbb{R}^d$ be a Borel set with $\mathcal{H}^n(E) < \infty$ such that $\Theta^n_*(x,E) > 0$ for $\mathcal{H}^n$-a.e. $x \in E$. The following are equivalent:*

(i) *E is n-rectifiable.*

(ii) $\displaystyle\int_0^1 \left| \frac{\mathcal{H}^n(B(x,r)\cap E)}{r^n} - \frac{\mathcal{H}^n(B(x,2r)\cap E)}{(2r)^n} \right|^2 \frac{dr}{r} < \infty$ *for $\mathcal{H}^n$-a.e. $x \in E$.*

(iii) $\displaystyle\lim_{r\to 0} \left(\frac{\mathcal{H}^n(B(x,r)\cap E)}{r^n} - \frac{\mathcal{H}^n(B(x,2r)\cap E)}{(2r)^n} \right) = 0$ *for $\mathcal{H}^n$-a.e. $x \in E$.*

Notice that the fact that $\mathcal{H}^n(E) < \infty$ implies that the condition $\Theta^{n,*}(x,E) < \infty$ is satisfied for $\mathcal{H}^n$-a.e. $x \in E$ (see [Ma, Theorem 6.2], for example).

Some remarks are in order. First we mention that the equivalence (a)$\Leftrightarrow$(b) in Theorem 1.1 is a pointwise version of a related result which characterizes the so called uniform rectifiability, which was recently obtained in [CGLT]. The implication whose proof requires more effort in this paper is (a)$\Rightarrow$(b). To prove this we will introduce a square function operator and we will show that it is bounded from the space of finite real measures on $\mathbb{R}^d$ to $L^{1,\infty}(\mu)$, by using Calderón-Zygmund techniques. On the other hand, we will obtain (b)$\Rightarrow$(a) by combining some of the results from [CGLT] with others from Preiss regarding uniform measures, by using "tangent measure technology". The implication (c)⇒(a) follows by similar arguments. Notice, by the way, that the statement in (c) looks much weaker than the μ-a.e. existence of the limit (1.2). So (c)⇒(a) can be considered as a strengthening of Preiss' theorem.

The implication (c)⇒(a) in Theorem 1.1 does not hold if one replaces the assumption that $\Theta^n_*(x,\mu)>0$ by $\Theta^{n,*}(x,\mu)>0$ μ-a.e. Indeed, Preiss has constructed in [Pr, 5.8-5.9] a measure μ in the plane which is purely 1-unrectifiable (i.e. vanishes on any 1-rectifiable set and so (a) fails for μ) such that, for μ-a.e. $x\in\mathbb{R}^2$, $0<\Theta^{1,*}(x,\mu)<\infty$ and all tangent measures at x are 1-flat (see Section 6 for the definition of tangent and flat measures). It is easily seen that this fact implies that μ satisfies (c).

On the other hand, we do not know if the assumption that $0<\Theta^{n,*}(x,\mu)<\infty$ μ-a.e. suffices for the validity of the implication (b)⇒(a) in Theorem 1.1. If this were true, then Corollary 1.2 would assert that, given $E\subset\mathbb{R}^d$ with $\mathcal{H}^n(E)<\infty$, E being n-rectifiable is equivalent to (ii).

Finally it is worth pointing that there are other characterizations of n-rectifiability involving square functions in the literature. Most of them involve the so called β coefficients of Peter Jones [Jo]. See for example Pajot's theorem [Pa, Theorem 26] for a result in the spirit of Theorem 1.1, with β coefficients instead of densities.

In this paper the letter c stands for some constant which may change its value at different occurrences. The notation $A\lesssim B$ means that there is some fixed constant c such that $A\leq c\,B$, with c as above. Also, $A\approx B$ is equivalent to $A\lesssim B\lesssim A$.

2. Preliminaries

2.1. AD-regularity and uniformly rectifiable measures.

A measure μ is called n-AD-regular if there exists some constant $c_0>0$ such that

$$c_0^{-1}r^n\leq\mu(B(x,r))\leq c_0\,r^n\quad\text{ for all } x\in\operatorname{supp}(\mu)\text{ and } 0<r\leq\operatorname{diam}(\operatorname{supp}(\mu)).$$

A measure μ is uniformly n-rectifiable if it is n-AD-regular and there exist $\theta,M>0$ such that for all $x\in\operatorname{supp}(\mu)$ and all $r>0$ there is a Lipschitz mapping g from the ball $B_n(0,r)$ in $\mathbb{R}^n$ to $\mathbb{R}^d$ with $\operatorname{Lip}(g)\leq M$ such that

$$\mu(B(x,r)\cap g(B_n(0,r)))\geq\theta r^n.$$

In the case $n=1$, μ is uniformly 1-rectifiable if and only if $\operatorname{supp}(\mu)$ is contained in a rectifiable curve Γ in $\mathbb{R}^d$ such that the arc length measure on Γ is 1-AD-regular. A set $E\subset\mathbb{R}^d$ is called uniformly n-rectifiable if $\mathcal{H}^n|_E$ is uniformly n-rectifiable.

The notion of uniform rectifiability was introduced by David and Semmes [DaS1], [DaS2]. In these works they showed that a big class of singular singular integrals with odd kernel is bounded in $L^2(\mu)$ if μ is uniformly rectifiable. See [NToV] for a recent related result in the converse direction involving the n-dimensional Riesz transforms.

2.2. Dyadic cubes.

In the case when μ is an n-AD-regular measure in $\mathbb{R}^d$ we will use the David lattice $\mathcal{D}$ of "cubes" associated with μ (see [Da, Appendix 1], for example). Suppose for simplicity that $\mu(\mathbb{R}^d)=\infty$. In this case, $\mathcal{D}=\bigcup_{j\in\mathbb{Z}}\mathcal{D}_j$ and each set $Q\in\mathcal{D}_j$, which is called a cube, is a Borel subset of $\operatorname{supp}(\mu)$ which satisfies $\mu(Q)\approx 2^{-jn}$ and $\operatorname{diam}(Q)\approx 2^{-j}$. In fact, we will assume that

$$c^{-1}2^{-j}\leq\operatorname{diam}(Q)\leq 2^{-j}.$$

We set $\ell(Q) := 2^{-j}$. For $R \in \mathcal{D}$, we denote by $\mathcal{D}(R)$ the family of all cubes $Q \in \mathcal{D}$ which are contained in R. In the case when $\mu(\mathbb{R}^d) < \infty$ and $\operatorname{diam}(\operatorname{supp}(\mu)) \approx 2^{-j_0}$, then $\mathcal{D} = \bigcup_{j \geq j_0} \mathcal{D}_j$. The other properties of the lattice $\mathcal{D}$ are the same as in the previous case.

If $Q \in \mathcal{D}_j$, we write $J(Q) = j$. That is, $J(Q)$ is the generation of $\mathcal{D}$ to which Q belongs. On the other hand, we say that $Q, Q' \in \mathcal{D}$ are neighbors if they belong to the same generation (which is equivalent to saying $\ell(Q) = \ell(Q')$) and moreover there exist $x \in Q$ and $x' \in Q'$ such that $|x - x'| \leq \ell(Q)$. We denote the collection of neighbors of Q by $\mathcal{N}(Q)$. Notice that $Q \in \mathcal{N}(Q)$.

For $Q \in \mathcal{D}$, we denote by $\mathcal{C}h(Q)$ the family of dyadic cubes contained in R with side length equal to $\ell(Q)/2$. These are the so called children of Q.

We will call "true cubes" the usual cubes in $\mathbb{R}^d$, to distinguish them from the cubes from $\mathcal{D}$.

2.3. **The dyadic martingale.** Suppose again that μ is n-AD-regular, and let $\mathcal{D}$ be the associated dyadic lattice. Given $f \in L^1_{loc}(\mu)$ and $Q \in \mathcal{D}$, we denote by $m_Q f$ the mean of f on Q with respect to μ. That is,

$$m_Q f = \frac{1}{\mu(Q)} \int_Q f \, d\mu.$$

Then we define

$$\Delta_Q f = \sum_{P \in \mathcal{C}h(Q)} \chi_P \, (m_P f - m_Q f).$$

The functions $\Delta_Q f$, $Q \in \mathcal{D}$, are orthogonal, and it is well known that

$$\|f\|^2_{L^2(\mu)} = \sum_{Q \in \mathcal{D}} \|\Delta_Q f\|^2_{L^2(\mu)}.$$

For every $Q \in \mathcal{D}$, we also have

$$f \, \chi_Q = (m_Q f) \, \chi_Q + \sum_{P \in \mathcal{D}(Q)} \Delta_P f, \tag{2.1}$$

with the sum converging in $L^2(\mu)$.

3. The square function operator

To prove Theorem 1.1 we will first show (a)$\Rightarrow$(b). To this end, it is convenient to introduce the following operator T. Given a real Radon measure ν on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we set

$$T\nu(x) = \left(\int_0^\infty \left| \frac{\nu(B(x,r))}{r^n} - \frac{\nu(B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r} \right)^{1/2}.$$

Notice that T is a sublinear operator. For a positive Borel measure μ on $\mathbb{R}^d$ and a given function $f \in L^1_{loc}(\mu)$, we also write

$$T_\mu f(x) = T(f\mu)(x).$$

That is,

$$T_\mu f(x) = \left(\int_0^\infty \left| \frac{(f\mu)(B(x,r))}{r^n} - \frac{(f\mu)(B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r} \right)^{1/2},$$

where $(f\mu)(A) = \int_A f\,d\mu$ for any set $A \subset \mathbb{R}^d$.

It is easy to check that to prove that $T\mu(x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$ if μ is n-rectifiable we may assume that μ is compactly supported and thus finite. We will show that if Γ is an n-dimensional Lipschitz graph in $\mathbb{R}^d$ (and more generally, a uniformly n-rectifiable set), then T is bounded from the space of finite real Borel measures on $\mathbb{R}^d$, denoted by $M(\mathbb{R}^d)$, to $L^{1,\infty}(\mathcal{H}^n|_\Gamma)$. That is, there exists some constant $c > 0$ such that

$$\mathcal{H}^n\big(\{x \in \Gamma : T\nu(x) > \lambda\}\big) \le c\,\frac{\|\nu\|}{\lambda} \quad \text{for all } \nu \in M(\mathbb{R}^d) \text{ and } \lambda > 0. \tag{3.1}$$

To see that $T\mu(x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$, notice that there exists a countable union of possibly rotated n-dimensional Lipschitz graphs Γ_i such that μ is absolutely continuous with respect to $\mathcal{H}^n|_{\bigcup_i \Gamma_i}$ (indeed, in the definition (1.1) of n-rectifiable sets, one can replace the sets $f_i(\mathbb{R}^n)$ by possibly rotated n-dimensional Lipscthiz graphs Γ_i). Thus, it suffices to show that $T\mu(x) < \infty$ for $\mathcal{H}^n$-a.e. $x \in \Gamma_i$. This is an immediate consequence of the estimate (3.1) applied to the particular case $\Gamma = \Gamma_i$, $\nu = \mu$, recalling that we assume $\|\mu\| < \infty$.

The first step to prove the key estimate (3.1) consists in showing that T_μ is bounded in $L^2(\mu)$ if μ is a uniformly n-rectifiable measure. This is shown in the next section. Later, by means of a suitable Calderón-Zygmund decomposition we will prove that T is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$, which in particular yields the estimate (3.1) just by choosing $\mu = \mathcal{H}^n|_\Gamma$.

4. T_μ is bounded in $L^2(\mu)$ if μ is uniformly rectifiable

The objective of this section consists in proving the following.

Theorem 4.1. *Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Then T_μ is bounded in $L^2(\mu)$.*

The following theorem, which is one main results from [CGLT], will be a fundamental ingredient of the proof of Theorem 4.1.

Theorem 4.2. *Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Then there exists a constant c such that for any ball $B \subset \mathbb{R}^d$ with radius R,*

$$\int_0^R \int_{x\in B} \left| \frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n} \right|^2 d\mu(x)\,\frac{dr}{r} \le c\,R^n,$$

with c depending only on n, d and the constants involved in the AD-regular and uniformly n-rectifiable character of μ.

We are ready now to prove Theorem 4.1. The proof below is somewhat similar in spirit to some of the arguments in [MT].

Proof of Theorem 4.1. For $k \in \mathbb{Z}$, $f \in L^2(\mu)$, $x \in \mathbb{R}^d$, we denote

$$T_{\mu,k}f(x) = \left(\int_{2^{-k-2}}^{2^{-k-1}} \left| \frac{(f\mu)(B(x,r))}{r^n} - \frac{(f\mu)(B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r} \right)^{1/2},$$

so that

$$T_\mu f(x)^2 = \sum_{k\in\mathbb{Z}} T_{\mu,k} f(x)^2.$$

Consider the family $\mathcal{D}$ of dyadic cubes associated with μ, as described in Section 2, and write

$$\int |T_\mu f|^2\,d\mu = \sum_{k\in\mathbb{Z}} \int |T_{\mu,k} f|^2\,d\mu = \sum_{Q\in\mathcal{D}} \int |T_Q f|^2\,d\mu,$$

where

$$T_Q f = \chi_Q T_{\mu,J(Q)} f.$$

Recall that $J(Q)$ stands for integer k such that $Q\in\mathcal{D}_k$. Observe that if $x\in Q$ and $r\leq \ell(Q)/2$, then

$$B(x,2r)\cap \operatorname{supp}(\mu)\subset \bigcup_{P\in\mathcal{N}(Q)} P.$$

So denoting $\widetilde{Q}=\bigcup_{P\in\mathcal{N}(Q)} P$ we deduce that

$$T_Q f = T_Q(\chi_{\widetilde{Q}} f).$$

By the martingale decomposition (2.1), we have

$$\begin{aligned}\chi_{\widetilde{Q}} f &= \sum_{R\in\mathcal{N}(Q)} \Big(\chi_R\, m_R f + \sum_{P\in\mathcal{D}(R)} \Delta_P f\Big)\\ &= \chi_{\widetilde{Q}}\, m_Q f + \sum_{R\in\mathcal{N}(Q)} \chi_R\,(m_R f - m_Q f) + \sum_{R\in\mathcal{N}(Q)}\sum_{P\in\mathcal{D}(R)} \Delta_P f.\end{aligned}$$

Therefore,

$$\begin{aligned}T_Q f &\leq |m_Q f|\, T_Q \chi_{\widetilde{Q}} + \sum_{R\in\mathcal{N}(Q)} |m_R f - m_Q f|\, T_Q\chi_R + \sum_{R\in\mathcal{N}(Q)} T_Q\bigg(\sum_{P\in\mathcal{D}(R)} \Delta_P f\bigg)\\ &=: A_Q f + B_Q f + C_Q f.\end{aligned}\tag{4.1}$$

So we have

$$\int |T_\mu f|^2\,d\mu \lesssim \sum_{Q\in\mathcal{D}}\int |A_Q f|^2\,d\mu + \sum_{Q\in\mathcal{D}}\int |B_Q f|^2\,d\mu + \sum_{Q\in\mathcal{D}}\int |C_Q f|^2\,d\mu.\tag{4.2}$$

To estimate the first sum on the right side of the preceding inequality notice that for any $S\in\mathcal{D}$

$$\begin{aligned}\sum_{Q\in\mathcal{D}(S)}\int |T_Q\chi_{\widetilde{Q}}|^2\,d\mu &= \sum_{Q\in\mathcal{D}(S)}\int |T_Q 1|^2\,d\mu\\ &= \int_0^{\ell(S)/2}\int_{x\in S}\left|\frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n}\right|^2 d\mu(x)\,\frac{dr}{r} \leq c\,\mu(S),\end{aligned}$$

by Theorem 4.2. Then, by the Carleson imbedding theorem, we deduce that

$$\sum_{Q\in\mathcal{D}}\int |A_Q f|^2\,d\mu = \sum_{Q\in\mathcal{D}} |m_Q f|^2 \int |T_Q\chi_{\widetilde{Q}}|^2\,d\mu \leq c\,\|f\|_{L^2(\mu)}^2.$$

To estimate the second sum on the right side of (4.2) we denote

$$b_Q(f) = \sum_{R\in\mathcal{N}(Q)} |m_R f - m_Q f|,$$

and then we write

$$B_Q f = \sum_{R\in\mathcal{N}(Q)} |m_R f - m_Q f|\, T_Q\chi_R \leq c\, b_Q(f)\,\chi_Q,$$

just using the trivial estimate $T_Q\chi_R \lesssim \chi_Q$. Thus

$$\sum_{Q\in\mathcal{D}} \int |B_Q f|^2\, d\mu \leq c \sum_{Q\in\mathcal{D}} b_Q(f)^2\, \mu(Q).$$

As shown in Proposition 5.9 of [MT], the last sum is bounded by $c\,\|f\|_{L^2(\mu)}^2$.

So it only remains to show that

$$\sum_{Q\in\mathcal{D}} \int |C_Q f|^2\, d\mu \leq c\,\|f\|_{L^2(\mu)}^2. \tag{4.3}$$

To this end we set

$$|C_Q f|^2 \lesssim \sum_{R\in\mathcal{N}(Q)} \left| T_Q\biggl(\sum_{P\in\mathcal{D}(R)} \Delta_P f\biggr)\right|^2, \tag{4.4}$$

taking into account that number of neighbors of Q is uniformly bounded.

For $x \in Q$ and $R \in \mathcal{N}(Q)$ we have

$$\begin{aligned} T_Q\biggl(\sum_{P\in\mathcal{D}(R)} \Delta_P f\biggr)(x)^2 \\ = \int_{\ell(Q)/4}^{\ell(Q)/2} \left| \frac{\sum_{P\in\mathcal{D}(R)} (\Delta_P f\,\mu)(B(x,r))}{r^n} - \frac{\sum_{P\in\mathcal{D}(R)} (\Delta_P f\,\mu)(B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r}. \end{aligned} \tag{4.5}$$

Recall now that $\int \Delta_P f\, d\mu = 0$, and so $(\Delta_P f\,\mu)(B(x,r)) = 0$ unless both $P\cap B(x,r) \neq \varnothing$ and $P \cap B(x,r)^c \neq \varnothing$, and analogously replacing r by $2r$. So if we denote

$$\begin{aligned} J_{R,r}(x) = \bigl\{P \in \mathcal{D}(R) : P\cap B(x,r) \neq \varnothing \text{ and } P\cap B(x,r)^c \neq \varnothing\bigr\} \\ \cup \bigl\{P \in \mathcal{D}(R) : P\cap B(x,2r) \neq \varnothing \text{ and } P\cap B(x,2r)^c \neq \varnothing\bigr\}, \end{aligned}$$

then we have

$$\left| \frac{\sum_{P\in\mathcal{D}(R)} (\Delta_P f\,\mu)(B(x,r))}{r^n} - \frac{\sum_{P\in\mathcal{D}(R)} (\Delta_P f\,\mu)(B(x,2r))}{(2r)^n} \right| \leq \frac{c}{\ell(Q)^n} \sum_{P\in J_{R,r}(x)} \|\Delta_P f\|_{L^1(\mu)}.$$

By Cauchy-Schwarz, the right hand side is bounded by

$$\frac{c}{\ell(Q)^n}\Bigg(\sum_{P\in J_{R,r}(x)}\ell(P)^{n-1/2}\Bigg)^{1/2}\Bigg(\sum_{P\in J_{R,r}(x)}\frac{1}{\ell(P)^{n-1/2}}\,\|\Delta_P f\|^2_{L^1(\mu)}\Bigg)^{1/2}$$
$$\leq \frac{c}{\ell(Q)^n}\Bigg(\sum_{P\in J_{R,r}(x)}\ell(P)^{n-1/2}\Bigg)^{1/2}\Bigg(\sum_{P\in \mathcal{D}(R)}\ell(P)^{1/2}\,\|\Delta_P f\|^2_{L^2(\mu)}\Bigg)^{1/2}.$$

Plugging this estimate into (4.5) we obtain

$$T_Q\Bigg(\sum_{P\in\mathcal{D}(R)}\Delta_P f\Bigg)(x)^2 \leq \frac{c}{\ell(Q)^{2n}}\Bigg(\sum_{P\in\mathcal{D}(R)}\ell(P)^{1/2}\,\|\Delta_P f\|^2_{L^2(\mu)}\Bigg)\int_{\ell(Q)/4}^{\ell(Q)/2}\sum_{P\in J_{R,r}(x)}\ell(P)^{n-1/2}\,\frac{dr}{r}. \tag{4.6}$$

To estimate the last integral, notice if $P\in J_{R,r}(x)$, then either $\partial B(x,r)$ or $\partial B(x,2r)$ intersect the convex hull of P, which we denote by $\mathrm{conv}(P)$. By Fubini then we get

$$\int_{\ell(Q)/4}^{\ell(Q)/2}\sum_{P\in J_{R,r}(x)}\ell(P)^{n-1/2}\,\frac{dr}{r}$$
$$\leq \frac{c}{\ell(Q)}\sum_{P\in\mathcal{D}(R)}\ell(P)^{n-1/2}\,\Big|\big\{r>0:\big[(\partial B(x,r))\cup(\partial B(x,2r))\big]\cap \mathrm{conv}(P)\neq\varnothing\big\}\Big|.$$

Since $\mathrm{diam}(P)\approx \ell(P)$, for any fixed x we have

$$\Big|\big\{r>0:(\partial B(x,r))\cap\mathrm{conv}(P)\neq\varnothing\big\}\Big|\lesssim \ell(P), \tag{4.7}$$

and analogously replacing $\partial B(x,r)$ by $\partial B(x,2r)$. So we obtain

$$\int_{\ell(Q)/4}^{\ell(Q)/2}\sum_{P\in J_{R,r}(x)}\ell(P)^{n-1/2}\,\frac{dr}{r}\leq \frac{c}{\ell(Q)}\sum_{P\in\mathcal{D}(R)}\ell(P)^{n+1/2}\leq c\,\frac{\ell(Q)^{n+1/2}}{\ell(Q)} = c\,\ell(Q)^{n-1/2}.$$

Together with (4.6) this gives us that, for all $x\in\mathbb{R}^d$,

$$T_Q\Bigg(\sum_{P\in\mathcal{D}(R)}\Delta_P f\Bigg)(x)^2\leq \frac{c}{\ell(Q)^{n+1/2}}\sum_{P\in\mathcal{D}(R)}\ell(P)^{1/2}\,\|\Delta_P f\|^2_{L^2(\mu)}\,\chi_Q(x).$$

From the last estimate and (4.4) we deduce that

$$\sum_{Q\in\mathcal{D}}\int|C_Q f|^2\,d\mu \leq c\sum_{Q\in\mathcal{D}}\sum_{R\in\mathcal{N}(Q)}\int\Bigg|T_Q\Bigg(\sum_{P\in\mathcal{D}(R)}\Delta_P f\Bigg)\Bigg|^2 d\mu$$
$$\leq c\sum_{Q\in\mathcal{D}}\sum_{R\in\mathcal{N}(Q)}\frac{1}{\ell(Q)^{1/2}}\sum_{P\in\mathcal{D}(R)}\ell(P)^{1/2}\,\|\Delta_P f\|^2_{L^2(\mu)}$$
$$= c\sum_{P\in\mathcal{D}}\|\Delta_P f\|^2_{L^2(\mu)}\sum_{Q\in\mathcal{D}}\sum_{R\in\mathcal{N}(Q):R\supset P}\frac{\ell(P)^{1/2}}{\ell(Q)^{1/2}}.$$

Since

$$\sum_{Q\in\mathcal{D}}\sum_{R\in\mathcal{N}(Q):R\supset P}\frac{\ell(P)^{1/2}}{\ell(Q)^{1/2}}\lesssim 1,$$

(4.3) follows, and the proof of the theorem is concluded. □

5. T is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$ if μ is uniformly rectifiable

In this section we will prove the following.

Theorem 5.1. *Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Then T is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$. That is, there exists some constant c such that*

$$\mu\big(\{x\in\mathbb{R}^d:\,T\nu(x)>\lambda\}\big)\leq c\,\frac{\|\nu\|}{\lambda}\quad\text{for all }\nu\in M(\mathbb{R}^d)\text{ and }\lambda>0. \tag{5.1}$$

As shown in Section 3, this result implies that if μ is an n-rectifiable measure in $\mathbb{R}^d$, then

$$\int_0^1\left|\frac{\mu(B(x,r))}{r^n}-\frac{\mu(B(x,2r))}{(2r)^n}\right|^2\frac{dr}{r}<\infty\quad\text{for }\mu\text{-a.e. }x\in\mathbb{R}^d.$$

Before proving Theorem 5.1 we state the Calderón-Zygmund decomposition we need.

Lemma 5.2. *Let μ be an n-AD-regular measure. For every $\nu\in M(\mathbb{R}^d)$ with compact support and every $\lambda>2^{d+1}\|\nu\|/\|\mu\|$ we have:*

(a) *There exists a finite or countable collection of cubes $\{Q_j\}_{j\in J}$ with bounded overlap (that is, $\sum_j\chi_{Q_j}\leq c$) and a function $f\in L^1(\mu)$ such that, for each $j\in J$,*

$$|\nu|(Q_j)>2^{-d-1}\lambda\,\mu(2Q_j), \tag{5.2}$$

$$|\nu|(\eta Q_j)\leq 2^{-d-1}\lambda\,\mu(2\eta Q_j)\quad\text{for every }\eta>2, \tag{5.3}$$

and moreover,

$$\nu=f\mu\quad\text{in }\mathbb{R}^d\setminus\textstyle\bigcup_{j\in J}Q_j,\quad\text{with }|f|\leq\lambda\ \mu\text{-a.e.} \tag{5.4}$$

(b) *For each $j\in J$, let $R_j=6Q_j$ and denote $w_j=\chi_{Q_j}\left(\sum_k\chi_{Q_k}\right)^{-1}$. There exists a family of functions $\{b_j\}_{j\in J}$ with $\operatorname{supp}b_j\subset R_j$, each one with constant sign, such that*

$$\int b_j\,d\mu=\int w_j\,d\nu, \tag{5.5}$$

$$\|b_j\|_{L^\infty(\mu)}\,\mu(R_j)\leq c\,|\nu|(Q_j), \tag{5.6}$$

$$\sum_{j\in J}|b_j|\leq c\,\lambda. \tag{5.7}$$

Let us remark that the cubes in the preceding lemma are not cubes from $\mathcal{D}$, but true cubes. Observe also that, in particular, that (5.3) implies that $4\bar{Q}_j\cap\operatorname{supp}(\mu)\neq\varnothing$ and thus $\mu(R_j)\approx\ell(R_j)^n$.

For the proof of the lemma the reader can see Lemma 2.14 of [To], where this is shown in the more general situation where μ need not be doubling.

Proof of Theorem 5.1. Suppose first that $\nu \in M(\mathbb{R}^d)$ has compact support. Clearly, we may assume that $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$.

For such $\lambda > 0$, consider Q_j, R_j, w_j, b_j, for $j \in J$, and f as in Lemma 5.2. Then write $\nu = g\,\mu + \beta$, where

$$g\,\mu = \chi_{\mathbb{R}^d \setminus \bigcup_{j\in J} Q_j}\,\nu + \sum_{j\in J} b_j\,\mu$$

and

$$\beta = \sum_{j\in J} \beta_j := \sum_{j\in J} (w_j\,\nu - b_j\,\mu).$$

Observe that $\|g\|_{L^\infty(\mu)} \leq c\,\lambda$ and, for each $j \in J$,

$$\operatorname{supp}(\beta_j) \subset R_j \quad \text{and} \quad \beta_j(R_j) = 0.$$

So β_j is a real measure with zero mean.

By (5.2) we have

$$\mu\Big(\bigcup_j 2Q_j\Big) \leq \frac{c}{\lambda} \sum_j |\nu|(Q_j) \leq \frac{c}{\lambda}\,\|\nu\|.$$

So we only have to check that

$$\mu\Big(\Big\{x \in \mathbb{R}^d \setminus \bigcup_j 2Q_j : T\nu(x) > \lambda\Big\}\Big) \leq \frac{c}{\lambda}\,\|\nu\|. \tag{5.8}$$

Taking into account that T_μ is bounded in $L^2(\mu)$, using the fact that $\|g\|_{L^\infty(\mu)} \leq c\lambda$ we derive

$$\mu\Big(\Big\{x \in \mathbb{R}^d \setminus \bigcup_j 2Q_j : T_\mu g(x) > \lambda/2\Big\}\Big) \leq \frac{c}{\lambda^2} \int |g|^2\,d\mu \leq \frac{c}{\lambda} \int |g|\,d\mu,$$

using that $\|g\|_{L^\infty(\mu)} \leq c\,\lambda$. Also, by the definition of g and (5.6) we get

$$\int |g|\,d\mu \leq |\nu|\Big(\mathbb{R}^d \setminus \bigcup_{j\in J} Q_j\Big) + \sum_{j\in J} \int |b_j|\,d\mu \leq \|\nu\| + c \sum_{j\in J} |\nu|(Q_j) \leq c\,\|\nu\|.$$

Thus

$$\mu\Big(\Big\{x \in \mathbb{R}^d \setminus \bigcup_j 2Q_j : T_\mu g(x) > \lambda/2\Big\}\Big) \leq \frac{c}{\lambda}\,\|\nu\|. \tag{5.9}$$

Let us turn attention to $T\beta$ now. We set

(5.10)
$$\begin{aligned}
\mu\Big(\Big\{x \in \mathbb{R}^d \setminus \bigcup_j 2Q_j : T\beta(x) > \lambda/2\Big\}\Big) &\leq \frac{2}{\lambda} \int_{\mathbb{R}^d \setminus \bigcup_j 2Q_j} T\beta\,d\mu \\
&\leq \frac{2}{\lambda} \sum_{j\in J} \int_{\mathbb{R}^d \setminus 2R_j} T\beta_j\,d\mu + \frac{2}{\lambda} \sum_{j\in J} \int_{2R_j \setminus 2Q_j} T\beta_j\,d\mu.
\end{aligned}$$

First we will estimate the first sum on the right side. To this end, notice that since β_j has zero mean and is supported on R_j, for each $x \in \mathbb{R}^d \setminus 2R_j$ and $r > 0$ we deduce that

$\beta_j(B(x,r)) = 0$ unless $\partial B(x,r) \cap R_j \neq \varnothing$. Of course, the analogous statement holds for $\beta_j(B(x,2r))$. Thus

$$|T\beta_j(x)|^2 \leq \int_0^\infty \left(\frac{|\beta_j(B(x,r))|}{r^n} + \frac{|\beta_j(B(x,2r))|}{(2r)^n} \right)^2 \frac{dr}{r}$$
$$\leq 2\int_{\{r:\partial B(x,r)\cap R_j\neq\varnothing\}} \left(\frac{|\beta_j|(R_j)}{r^n} \right)^2 \frac{dr}{r} + 2\int_{\{r:\partial B(x,2r)\cap R_j\neq\varnothing\}} \left(\frac{|\beta_j|(R_j)}{(2r)^n} \right)^2 \frac{dr}{r}.$$

Observe that if $\partial B(x,r) \cap R_j \neq \varnothing$ or $\partial B(x,2r) \cap R_j \neq \varnothing$, then $r \approx |x - x_j|$ where x_j stands for the center of R_j (and of Q_j), because $x \notin 2R_j$. Therefore by (4.7)

(5.11)
$$|T\beta_j(x)|^2 \leq c\,\frac{|\beta_j|(R_j)^2}{|x-x_j|^{2n+1}} \left(\int_{\{r:\partial B(x,r)\cap R_j\neq\varnothing\}} dr + \int_{\{r:\partial B(x,2r)\cap R_j\neq\varnothing\}} dr \right) \leq c\,\frac{|\beta_j|(R_j)^2\,\ell(R_j)}{|x-x_j|^{2n+1}},$$

where we also took into account that for any fixed $x \in \mathbb{R}^d$ we have

$$|\{r > 0 : \partial B(x,r) \cap R_j \neq \varnothing\}| \leq \operatorname{diam}(R_j) \leq c\,\ell(R_j),$$

and, analogously replacing r by $2r$. From the estimate (5.11), using also the upper n-AD-regularity of μ we infer that

(5.12)
$$\int_{\mathbb{R}^d\setminus 2R_j} |T\beta_j|\,d\mu \leq c\,|\beta_j|(R_j)\,\ell(R_j)^{1/2} \int_{\mathbb{R}^d\setminus 2R_j} \frac{1}{|x-x_j|^{n+1/2}}\,d\mu(x) \leq c\,|\beta_j|(R_j) \leq c\,|\nu|(Q_j).$$

To estimate the last term in (5.10) we set

$$\int_{2R_j\setminus 2Q_j} T\beta_j\,d\mu \leq c\,\ell(R_j)^{n/2} \left(\int_{2R_j\setminus 2Q_j} |T\beta_j|^2\,d\mu \right)^{1/2}$$
$$\leq c\,\ell(R_j)^{n/2} \left[\left(\int_{2R_j\setminus 2Q_j} |T(w_j\nu)|^2\,d\mu \right)^{1/2} + \left(\int |T(b_j\mu)|^2\,d\mu \right)^{1/2} \right].$$

By the $L^2(\mu)$ boundedness of T_μ and the condition (5.6) we have

(5.13)
$$\ell(R_j)^{n/2} \left(\int |T(b_j\mu)|^2\,d\mu \right)^{1/2} \leq c\,\ell(Q_j)^{n/2}\|b_j\|_{L^2(\mu)} \leq c\,\ell(Q_j)^n\,\|b_j\|_{L^\infty(\mu)} \leq c\,|\nu|(Q_j).$$

On the other hand, for $x \in 2R_j \setminus 2Q_j$, since $\operatorname{dist}(x, \operatorname{supp}(\nu|_{Q_j})) \geq \ell(Q_j)/2$, we have

$$|T(w_j\nu)(x)|^2 \leq c\int_{r\geq \ell(Q_j)/4} \frac{|\nu|(Q_j)^2}{r^{2n}}\,\frac{dr}{r} \leq c\,\frac{|\nu|(Q_j)^2}{\ell(Q_j)^{2n}}.$$

Therefore,

$$\ell(R_j)^{n/2} \left(\int_{2R_j\setminus 2Q_j} |T(w_j\nu)|^2\,d\mu \right)^{1/2} \leq c\,\ell(R_j)^{n/2}\,\frac{|\nu|(Q_j)}{\ell(Q_j)^n}\,\mu(2R_j)^{1/2} \leq c\,|\nu|(Q_j).$$

From this estimate and (5.13) we deduce that

$$\int_{2R_j\setminus 2Q_j} T\beta_j\,d\mu \leq c\,|\nu|(Q_j).$$

Together with (5.10) and (5.12) this yields

$$\mu\Big(\Big\{x\in\mathbb{R}^d\setminus\bigcup_j 2Q_j : T\beta(x)>\lambda/2\Big\}\Big)\leq \frac{c}{\lambda}\sum_{j\in J}|\nu|(Q_j)\leq\frac{c}{\lambda}\,\|\nu\|,$$

which finishes the proof of (5.8).

Suppose now that ν is not compactly supported. Let N_0, N_1 be two big positive integers, with $N_1\geq 2N_0$. Denote $\nu_{N_1}=\chi_{B(0,N_1)}\,\nu$. It is easy to check that for $x\in B(0,N_0)$

$$T(\chi_{\mathbb{R}^d\setminus B(0,N_1)}\nu)(x)\leq c\,\frac{|\nu|(\mathbb{R}^d\setminus B(0,N_1))}{N_1-N_0}\leq c\,\frac{\|\nu\|}{N_1-N_0}\leq\frac{\lambda}{2},$$

assuming N_1 big enough so that $N_1-N_0>2c\|\nu\|/\lambda$. Thus, for such N_1, since ν_{N_1} has compact support,

$$\begin{aligned}\mu\big(\{x\in B(0,N_0):|T\nu(x)|>\lambda\}\big)&\leq\mu\big(\{x\in B(0,N_0):|T\nu_{N_1}(x)|>\lambda/2\}\big)\\&\leq c\,\frac{\|\nu_{N_1}\|}{\lambda}\leq c\,\frac{\|\nu\|}{\lambda}.\end{aligned}$$

Since N_0 is arbitrary and this estimate is uniform on N_0, (5.1) follows in full generality. □

6. Finiteness of the square function implies n-rectifiability

In this section we will prove the implication (b)⇒(a) from Theorem 1.1. We have to show that if μ is a Radon measure on $\mathbb{R}^d$ such that, for μ-a.e. $x\in\mathbb{R}^d$, $0<\Theta^n_*(x,\mu)\leq\Theta^{n,*}(x,\mu)<\infty$ and

$$\int_0^1\left|\frac{\mu(B(x,r))}{r^n}-\frac{\mu(B(x,2r))}{(2r)^n}\right|^2\frac{dr}{r}<\infty,\tag{6.1}$$

then μ is n-rectifiable. Without loss of generality, to prove this result we can assume μ to be compactly supported, and thus finite.

As in [CGLT], we denote

$$\Delta_\mu(x,r):=\frac{\mu(B(x,r))}{r^n}-\frac{\mu(B(x,2r))}{(2r)^n}.$$

Notice that for any $r>0$, $|\Delta_\mu(x,r)|\leq\|\mu\|/r^n$. Thus,

$$\int_1^\infty\Delta_\mu(x,r)^2\,\frac{dr}{r}\leq\|\mu\|^2\int_1^\infty\frac{dr}{r^{2n+1}}\leq c\,\|\mu\|^2.$$

Therefore, (6.1) is equivalent to

$$\int_0^\infty\Delta_\mu(x,r)^2\,\frac{dr}{r}<\infty.\tag{6.2}$$

We consider the auxiliary function $\varphi(x)=e^{-|x|^2}$ and, for $r>0$, we set

$$\varphi_r(x)=\frac{1}{r^n}\,\varphi\left(\frac{x}{r}\right).$$

Then we define

$$\Delta_{\mu,\varphi}(x,r):=\varphi_r*\mu-\varphi_{2r}*\mu(x)=\int\big(\varphi_r(x-y)-\varphi_{2r}(x-y)\big)\,d\mu(y).$$

In the proof of Corollary 3.12 from [CGLT], it is shown that $\Delta_{\mu,\varphi}(x,r)$ can be written as a suitable convex combination of $\Delta_\mu(x,s)$, $s>0$, and that then

$$\int_0^\infty \Delta_{\mu,\varphi}(x,r)^2\,\frac{dr}{r} \leq c\int_0^\infty \Delta_\mu(x,r)^2\,\frac{dr}{r}.$$

So (6.2) implies that

$$\int_0^\infty \Delta_{\mu,\varphi}(x,r)^2\,\frac{dr}{r} < \infty. \tag{6.3}$$

Lemma 6.1. *Let μ be a finite Borel measure in $\mathbb{R}^d$ and $x\in\mathbb{R}^d$ such that $\Theta^{n,*}(x,\mu)<\infty$. If*

$$\int_0^\infty \Delta_{\mu,\varphi}(x,r)^2\,\frac{dr}{r} < \infty,$$

then

$$\lim_{r\to 0}\Delta_{\mu,\varphi}(x,r)=0.$$

Proof. Denote $I_k=[2^{-k},\,2^{-k+1}]$ and

$$\lambda_k=\frac{1}{|I_k|}\int_{I_k}\Delta_{\mu,\varphi}(x,r)^2\,dr.$$

Since $\sum_{k\in\mathbb{Z}}\lambda_k<\infty$, we have

$$\lim_{k\to\infty}\lambda_k=0.$$

By Chebichev, we get

$$\big|\{r\in I_k:\,|\Delta_{\mu,\varphi}(x,r)|>\lambda_k^{1/3}\}\big| \leq \frac{1}{\lambda_k^{2/3}}\int_{I_k}\Delta_{\mu,\varphi}(x,r)^2\,dr \leq \frac{\lambda_k\,|I_k|}{\lambda_k^{2/3}} = \lambda_k^{1/3}\,|I_k|.$$

Thus, assuming $\lambda_k<1$, for any $r\in I_k$ there exists some $r'\in I_k$ satisfying

$$|r-r'|\leq \lambda_k^{1/3}\,|I_k| \quad\text{and}\quad |\Delta_{\mu,\varphi}(x,r')|\leq\lambda_k^{1/3}. \tag{6.4}$$

Now we wish to estimate the difference between $\Delta_{\mu,\varphi}(x,r)$ and $\Delta_{\mu,\varphi}(x,r')$, for $r,r'\in I_k$. By the mean value theorem, we have

$$\big|\Delta_{\mu,\varphi}(x,r)-\Delta_{\mu,\varphi}(x,r')\big|\leq|r-r'|\sup_{s\in I_k}|\partial_s\Delta_{\mu,\varphi}(x,s)|.$$

Notice that

$$|\partial_s\Delta_{\mu,\varphi}(x,s)|\leq\big|\partial_s(\varphi_s*\mu(x))\big|+\big|\partial_s(\varphi_{2s}*\mu(x))\big|.$$

We have

$$\begin{aligned}\big|\partial_s(\varphi_s*\mu(x))\big| &= \left|\int\partial_s\left(\frac{1}{s^n}\,e^{-|x-y|^2/s^2}\right)\,d\mu(y)\right| \\ &\leq \int\left(\frac{n}{s^{n+1}}+\frac{2}{s^{n+1}}\,\frac{|x-y|^2}{s^2}\right)e^{-|x-y|^2/s^2}\,d\mu(y) \\ &= \frac{1}{s}\int\frac{1}{s^n}\left(n+\frac{2\,|x-y|^2}{s^2}\right)e^{-|x-y|^2/s^2}\,d\mu(y).\end{aligned} \tag{6.5}$$

Denote

$$b_x=\sup_{r>0}\frac{\mu(B(x,r))}{r^n}.$$

Observe that $b_x < \infty$ because $\Theta^{n,*}(x,\mu) < \infty$ and $\|\mu\| < \infty$. Using the fast decay of the function inside the integral on the right side of (6.5) and splitting the domain of integration into annuli, it follows easily that

$$\int \frac{1}{s^n}\left(n + \frac{2\,|x-y|^2}{s^2}\right) e^{-|x-y|^2/s^2}\,d\mu(y) \lesssim b_x.$$

So for $s \in I_k$ we obtain

$$\left|\partial_s(\varphi_s * \mu(x))\right| \lesssim \frac{b_x}{s} \approx \frac{b_x}{|I_k|}.$$

An analogous estimate holds replacing s by $2s$, and thus $|\partial_s \Delta_{\mu,\varphi}(x,s)| \lesssim b_x\,|I_k|^{-1}$. Then for $r, r' \in I_k$ we get

$$\left|\Delta_{\mu,\varphi}(x,r) - \Delta_{\mu,\varphi}(x,r')\right| \lesssim \frac{b_x}{|I_k|}\,|r-r'|.$$

Let k be big enough so that $\lambda_k < 1$. Given any $r \in I_k$ we can take $r' \in I_k$ satisfying (6.4). From the last estimate we deduce that

$$\left|\Delta_{\mu,\varphi}(x,r)\right| \leq \left|\Delta_{\mu,\varphi}(x,r')\right| + \left|\Delta_{\mu,\varphi}(x,r) - \Delta_{\mu,\varphi}(x,r')\right| \lesssim \lambda_k^{1/3} + \frac{b_x}{|I_k|}\,|r-r'| \leq (1+b_x)\lambda_k^{1/3},$$

which tends to 0 as $k \to \infty$. Thus, the lemma follows. □

Our next objective consists in proving the following.

Proposition 6.2. *Let μ be a finite Borel measure in $\mathbb{R}^d$ such that, for μ-a.e. $x \in \mathbb{R}^d$, $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ and moreover*

$$\lim_{r\to 0} \Delta_{\mu,\varphi}(x,r) = 0.$$

Then μ is n-rectifiable.

It is clear that the preceding proposition together with Lemma 6.1 completes the proof of (b)⇒(a) from Theorem 1.1.

We will prove Proposition 6.2 by using the so called tangent measures. Given $x \in \mathbb{R}^d$ and $r > 0$, denote by $T_{x,r}$ the homothety that maps $B(x,r)$ to $B(0,1)$. That is,

$$T_{x,r}(y) = \frac{1}{r}\,(y-x).$$

Observe that the image measure of μ by $T_{x,r}$ satisfies

$$T_{x,r}\#\mu(A) = \mu(rA + x), \qquad \text{for } A \subset \mathbb{R}^d.$$

One says that ν is a tangent measure of μ at x if ν is a non-zero Radon measure and there are a sequences $\{r_k\}_k$, $\{c_k\}_k$ of positive numbers with $\lim_{k\to\infty} r_k = 0$ such that the measures $c_k\,T_{x,r_k}\#\mu$ converge weakly to ν as $k \to \infty$.

A measure ν in $\mathbb{R}^d$ is called n-flat if there exists an n-plane $L \subset \mathbb{R}^d$ and $c > 0$ such that $\nu = c\,\mathcal{H}^n|_L$. On the other hand, ν is called n-uniform if it is a non-zero measure and there exists $c > 0$ such that

$$\nu(B(x,r)) = c\,r^n \quad \text{for all } x \in \operatorname{supp}(\nu),\ r > 0.$$

By the so called Marstrand-Mattila rectifiability criterion (see Theorem 16.7 of [Ma]) if μ is a finite Borel measure in $\mathbb{R}^d$ such that, for μ-a.e. $x \in \mathbb{R}^d$, $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$, it turns out that μ is n-rectifiable if and only if, for μ-a.e. $x \in \mathbb{R}^d$, all tangent measures at

x are n-flat. We will apply this criterion to prove Proposition 6.2. The first step consists in showing that for μ-a.e. $x \in \mathbb{R}^d$ all tangent measures at x are n-uniform:

Lemma 6.3. *Let μ be a finite Borel measure in $\mathbb{R}^d$ such that, for μ-a.e. $x \in \mathbb{R}^d$, $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ and moreover*

$$\lim_{r\to 0} \Delta_{\mu,\varphi}(x,r) = 0. \tag{6.6}$$

Then all tangent measures of μ at x are n-uniform for μ-a.e. $x \in \mathbb{R}^d$.

Proof. We will show that for μ almost all $x \in \mathbb{R}^d$, any tangent measure ν at x is n-AD-regular and satisfies

$$\Delta_{\nu,\varphi}(x,r) = 0 \quad \text{for all } x \in \operatorname{supp}(\nu) \text{ and all } r > 0. \tag{6.7}$$

By Theorem 3.10 from [CGLT], this implies that ν is n-uniform.

The n-AD-regularity of any tangent measure ν at x, for μ-a.e. $x \in \mathbb{R}^d$, follows from the fact that $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ μ-a.e., as shown in Lemma 14.7(1) of [Ma].

We turn now our attention to (6.7). To prove this we follow the same approach of [Ma, Lemma 20.7] in connection with the existence principal values for singular integrals. For every $\varepsilon > 0$, by Egoroff's theorem we can find a compact set F with $\mu(\mathbb{R}^d \setminus F) < \varepsilon$ where the convergence (6.6) is uniform. Moreover, F can be chosen so that $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ for all $x \in F$.

Let $x \in F$ be a μ-density point of F. That is,

$$\lim_{r\to 0} \frac{\mu(B(x,r)\setminus F)}{\mu(B(x,r))} = 0.$$

We claim that if ν is a tangent measure of μ at x, then (6.7) holds. To see this, take sequences $c_k, r_k > 0$ such that $c_k\, T_{x,r_k}\#\mu$ converges weakly to ν. By [Ma, Remark 14.4(1)] we may assume that $c_k = 1/\mu(B(x,r_k))$. Moreover, we can take a sequence of points $x_k \in F$ such that

$$z_k = \frac{x_k - x}{r_k} \to z \quad \text{as } k \to \infty.$$

This is also shown in the proof of [Ma, Lemma 14.7(1)].

Notice that

$$\int \big(\varphi_r(z-y) - \varphi_{2r}(z-y)\big)\, d\nu(y) = \lim_{k\to\infty} \frac{1}{\mu(B(x,r_k))} \int \big(\varphi_r(z_k-y) - \varphi_{2r}(z_k-y)\big)\, dT_{x,r_k}\#\mu(y).$$

This follows from the weak convergence of $\frac{1}{\mu(B(x,r_k))} T_{x,r_k}\#\mu$ to ν and the uniform convergence

$$\big(\varphi_r(z_k - \cdot) - \varphi_{2r}(z_k - \cdot)\big) \to \big(\varphi_r(z - \cdot) - \varphi_{2r}(z - \cdot)\big) \quad \text{as } k \to \infty,$$

taking also into account the fast decay at ∞ of φ_r and φ_{2r}. Then we have

$$\begin{aligned}\Delta_{\nu,\varphi}(x,r) &= \lim_{k\to\infty} \frac{1}{\mu(B(x,r_k))} \int \big(\varphi_r(z_k - y) - \varphi_{2r}(z_k - y)\big)\, dT_{x,r_k}\#\mu(y)\\ &= \lim_{k\to\infty} \frac{1}{\mu(B(x,r_k))} \int \big(\varphi_r(z_k - T_{x,r_k}(y)) - \varphi_{2r}(z_k - T_{x,r_k}(y))\big)\, d\mu(y)\\ &= \lim_{k\to\infty} \frac{1}{\mu(B(x,r_k))} \int \Big(\varphi_r\Big(\frac{x_k - y}{r_k}\Big) - \varphi_{2r}\Big(\frac{x_k - y}{r_k}\Big)\Big)\, d\mu(y)\\ &= \lim_{k\to\infty} \frac{r_k^n}{\mu(B(x,r_k))} \int \big(\varphi_{r\,r_k}(x_k - y) - \varphi_{2\,r\,r_k}(x_k - y)\big)\, d\mu(y).\end{aligned}$$

Recalling that $\Theta^n_*(x,\mu) > 0$ and using the uniform convergence of (6.6) in F, we infer that the last limit vanishes, and so $\Delta_{\nu,\varphi}(x,r) = 0$, as wished. □

Proposition 6.4. *Let μ be a Radon measure in $\mathbb{R}^d$ such that, for μ-a.e. $x \in \mathbb{R}^d$, $0 < \Theta^n_*(x,\mu) \le \Theta^{n,*}(x,\mu) < \infty$. If for μ-a.e. $x \in \mathbb{R}^d$ all tangent measures of μ at x are n-uniform, then μ is rectifiable.*

Clearly, this result in conjunction with Lemma 6.3 proves Proposition 6.2 and concludes the proof of (b)⇒(a) from Theorem 1.1.

Proof. As remarked above, by the Marstrand-Mattila criterion, it is enough to show that for μ-a.e. $x \in \mathbb{R}^d$ all tangent measures of μ at x are n-flat. To prove the proposition we will take into account that, for μ-a.e. $x \in \mathbb{R}^d$, if ν is tangent measure of μ at x, then any tangent measure σ of ν at any point $y \in \operatorname{supp}(\nu)$ is also a tangent measure of μ at x. See Theorem 14.16 from [Ma].

By a result of Kirchheim and Preiss [KiP] it follows that any n-uniform measure is supported on an n-dimensional real analytic variety. In particular, it turns out that it is n-rectifiable and thus has flat tangent measures at some points in its support. Thus we deduce that for μ-a.e. $x \in \mathbb{R}^d$ there exists flat tangent measures to μ at x.

To summarize, we know that for μ-a.e. $x \in \mathbb{R}^d$ all tangent measures at x are n-uniform and at least one of the tangent measures is n-flat. It is shown in [Pr] that this implies that all tangent measures at x are flat. Indeed this is one of the key ingredients of the proof of Preiss' theorem, which has been stated in (1.2). See also Theorem 6.10 of the nice monograph by De Lellis [DeL] for a very transparent argument. So the proposition follows. □

7. Proof of (c)⇔(a) from Theorem 1.1

Let μ be a Radon measure in $\mathbb{R}^d$ such that, for μ-a.e. $x \in \mathbb{R}^d$, $0 < \Theta^n_*(x,\mu) \le \Theta^{n,*}(x,\mu) < \infty$. We will show in this section that μ is n-rectifiable if and only if

$$\lim_{r\to 0} \left| \frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n} \right| = 0 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{7.1}$$

Notice first that if μ is n-rectifiable, then the density $\Theta^n(x,\mu)$ exists μ-a.e. and thus (7.1) holds.

To prove the converse implication we may assume that μ is compactly supported and thus finite. By Proposition 6.2 it is enough to show that

$$\lim_{r\to 0} \Delta_{\mu,\varphi}(x,r) = 0.$$

To this end, recall that $\Delta_{\mu,\varphi}(x,r)$ can be written as a convex combination of the terms of $\Delta_\mu(x,s)$, $s>0$. Indeed, as shown in the proof of Corollary 3.12 from [CGLT], we have

$$\Delta_{\mu,\varphi}(x,r) = \int_0^\infty \Delta_\mu(x,s)\, \widetilde{\varphi}_r(s)\, ds,$$

where

$$\widetilde{\varphi}_r(s) = \frac{2\, s^{n+1}}{r^{n+2}}\, e^{-s^2/r^2}.$$

For $\lambda \geq 0$, we split the integral as follows:

$$\big|\Delta_{\mu,\varphi}(x,r)\big| \leq \int_0^{\lambda r} \big|\Delta_\mu(x,s)\big|\, \widetilde{\varphi}_r(s)\, ds + \int_{\lambda r}^\infty \big|\Delta_\mu(x,s)\big|\, \widetilde{\varphi}_r(s)\, ds.$$

We have

$$\int_{\lambda r}^\infty \widetilde{\varphi}_r(s)\, ds = \int_{\lambda r}^\infty \frac{2\, s^{n+1}}{r^{n+2}}\, e^{-s^2/r^2}\, ds = \int_\lambda^\infty 2\, t^{n+1}\, e^{-t^2}\, dt.$$

Therefore, for $0<r<1$, choosing $\lambda = r^{-1/2}$ we get

$$\big|\Delta_{\mu,\varphi}(x,r)\big| \leq c \sup_{0<s\leq r^{1/2}} \big|\Delta_\mu(x,s)\big| + \sup_{s>0}\big|\Delta_\mu(x,s)\big| \int_{r^{-1/2}}^\infty 2\, t^{n+1}\, e^{-t^2}\, dt.$$

The first term on the right side tends to 0 as $r\to 0$ because of (7.1), and the second too because $\sup_{s>0}\big|\Delta_\mu(x,s)\big| < \infty$, taking into account that $\Theta^{n,*}(x,\mu)<\infty$ and $\|\mu\|<\infty$, by our assumption.

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